An intuitive approach to inertial forces and the centrifugal force paradox in general relativity

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Abstract. As the velocity of a rocket in a circular orbit near a black hole increases, the outwardly directed rocket thrust must increase to keep the rocket in its orbit. This feature might appear paradoxical from a Newtonian viewpoint, but we show that it follows naturally from the equivalence principle together with special relativity and a few general features of black holes. We also derive a general relativistic formalism of inertial forces for reference frames with acceleration and rotation. The resulting equation relates the real experienced forces to the time derivative of the speed and the spatial curvature of the particle trajectory relative to the reference frame. We show that an observer who follows the path taken by a free (geodesic) photon will experience a force perpendicular to the direction of motion that is independent of the observers velocity. We apply our approach to resolve the submarine paradox, which regards whether a submerged submarine in a balanced state of rest will sink or float when given a horizontal velocity if we take relativistic effects into account. We extend earlier treatments of this topic to include spherical oceans and show that for the case of the Earth the submarine floats upward if we take the curvature of the ocean into account.

I Introduction

Consider a rocket in a circular orbit outside the event horizon of a black hole.[1] If the orbit lies within the photon radius, the radius where free photons can move on circular orbits,[2] a greater outward rocket thrust is required to keep the rocket in orbit the faster the rocket moves. However, outside of the photon radius the outward thrust decreases as the orbital speed increases just as it would for a similar scenario in Newtonian mechanics (the thrust will be inward directed for sufficiently high speeds, see Fig. 1).

Analogous to the situation in Newtonian mechanics we can introduce in general relativity a gravitational force that is velocity independent. This force is fictitious (unlike in Newtonian mechanics). We can also introduce a velocity dependent (fictitious) centrifugal force that together with the gravitational force balances the real force from the jet engine of the rocket. By this definition, the

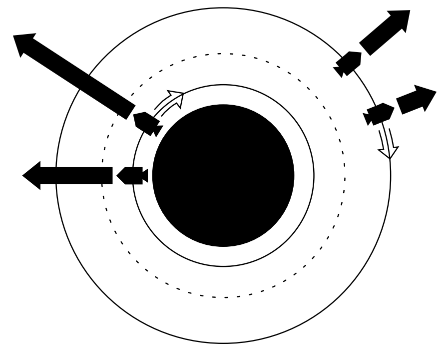

Figure 1: Rockets orbiting a static black hole. The solid arrows correspond to the force (the rocket thrust) necessary to keep the rocket in circular orbit. Inside of the photon radius (the dashed circle), the required force increases as the orbital velocity increases.

centrifugal force is directed inward inside of the photon radius and directed outward outside of the photon radius. This reversal of the direction of the fictitious centrifugal force is described by the formalism of optical geometry (see Appendix A) in which the phenomena has been discussed.[3, 4, 5, 6, 7, 8]

Our purpose is not to explain the velocity dependence of the rocket thrust by analogy with Newtonian theory, and we will use neither gravitational nor centrifugal forces. Instead we will use the basic principles of relativity to explain how the real force required to keep an object moving along a specified path depends on the velocity of the object.

We start by illustrating how the fact that the rocket thrust increases with increasing orbital speed (sufficiently close to the black hole) follows naturally from the equivalence principle (reviewed in Appendix B). We do so by first considering an idealized special relativistic scenario of a train moving relative to an (upward) accelerating platform.

We then consider a more general but still effectively two-dimensional discussion of forces perpendicular to the direction of motion for motion relative to an accelerated reference frame in special relativity. In a static spacetime, the reference frame connected to the static observers behaves locally like an accelerating reference frame in special relativity and the formalism can therefore be applied also to this case.

We then illustrate how to apply the formalism of this paper to the submarine paradox.[9] We ask whether a submarine with a density such that it is vertically balanced when at rest, will sink or float if given a horizontal velocity and relativistic effects are taken into account.

Next we generalize the formalism of forces and curvature of spatial paths to include three-dimensional cases, forces parallel to the direction of motion, and rotating reference frames. The acceleration and rotation of the reference frame will be shown (as in Newtonian mechanics) to introduce terms that can be interpreted as inertial (fictitious) forces. By using the equivalence principle, the formalism can be applied to arbitrary rigid reference frames in general relativity. We verify the results by comparing with Ref. 10.

Although this paper is primarily aimed at readers with a background in general relativity, the main part assumes only an elementary knowledge of special relativity together with a knowledge of a few basic concepts of general relativity. Some of the more important concepts such as curvature of a spatial path, spatial geometry, geodesics, and the equivalence principle are reviewed in Appendix B. Sections IX–XI are more specialized.

II The train and the platform

We consider the special relativistic description of a train moving relative to a platform with proper upward acceleration a.[11] The force required by a man on the train to hold an apple at a fixed height increases as the train speed increases (assuming nonzero acceleration of the platform) as illustrated in Fig. 2.

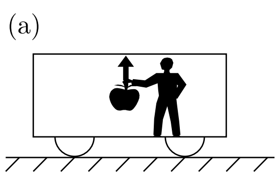

Figure 2: A train on a platform with a constant proper acceleration a upward. (a) The train is at rest; (b) the train is moving relative to the platform. The force required of a man on the train to keep an apple at a fixed height is higher when the train moves than when it is at rest relative to the platform.

To understand this effect we consider the accelerating train as observed from two inertial systems. The first system S is a system in which the platform is momentarily ($t = 0$) at rest. The second system S' is comoving with the train at the same moment. The two systems are related to each other by a Lorentz transformation of velocity v, where v is the velocity of the train relative to the platform along the x-axis.

Relative to S the apple moves to the right and accelerates upward with acceleration a. Consider now two physical events at the apple, one at $t = 0$ and one at $t = \delta t$, as observed from S. The time separation as observed in S' is (to lowest nonzero order in δt) given by $\delta t' = \delta t/\gamma$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$ and c is the velocity of light. In the following we will use c as the unit of velocity so that $v = 1$ for photons.[12] The height δh separating the two events as observed in S equals the corresponding separation $\delta h'$ relative to S'. If we denote the upward acceleration relative to S' by a', we have to lowest nonzero order in δt

$$\delta h = a\delta t^2/2 \tag{1a}$$

$$\delta h' = a'\delta t'^2/2 \tag{1b}$$

$$\delta h' = \delta h. \tag{1c}$$

From these equations follows that

$$a' = a\left(\frac{\delta t}{\delta t'}\right)^2 = a\gamma^2. \tag{2}$$

Thus the proper acceleration, that is, the acceleration as observed from an inertial system momentarily comoving with the apple, is greater than the acceleration of the platform by a factor of γ^2. The force required to keep the apple of rest mass m at a fixed height relative to the train is thus given by $F = m\gamma^2 a$.

To further clarify the main idea, we can also consider a similar scenario where there are two apples on a horizontal straight line which accelerates upward relative to an inertial system, as depicted in Fig. 3.

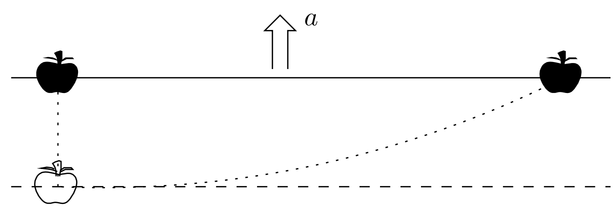

Figure 3: Two apples on an upward accelerating line (the solid line). The apples were initially at the position of the unfilled apple, one at rest and the other moving horizontally to the right. Both apples have to move up the same amount for a given coordinate time. But the one that moves horizontally has less proper time to do it. It must therefore experience a greater acceleration.

It follows from the equivalence principle (see Appendix B) that a flat platform on Earth (neglecting the Earth's rotation) behaves like a flat platform with proper upward acceleration g in special relativity. Hence for a sufficiently flat platform, the force required to hold an apple at rest inside a moving train on Earth would increase as the velocity of the train increases.

III The static black hole

Let us apply the reasoning of Sec. II to circular motion around a static black hole. A schematic of the exterior spatial geometry of an equatorial plane through a black hole is depicted in Fig. 4 (also see Appendix B).

A local static reference system outside of the black hole will behave as an accelerating reference frame (train platform) in special relativity (again see Appendix B). Locally, the scenario is thus identical to that in Sec. II, except that the path along which the object in question moves (a circle in the latter case) is not straight in general

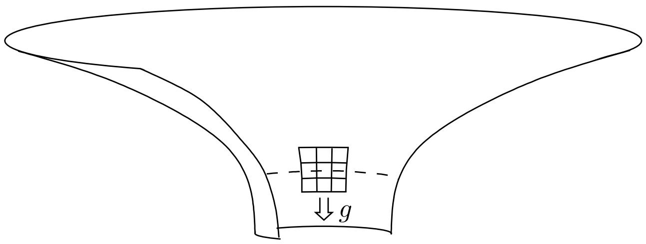

Figure 4: A freely falling frame (the grid) accelerating relative to the spatial geometry of a black hole.[13] We consider circular motion along the dashed line. The bottom edge of the depicted surface corresponds to the horizon. At this edge the embedding approaches a cylinder and the circle at the horizon is thus straight in the sense that it does not curve relative to the surface.

(although circles can in fact be straight, see Appendix B). Instead, the circular motion corresponds locally to letting the object in question follow a slightly curved path relative to the accelerating platform (see Fig. 5).

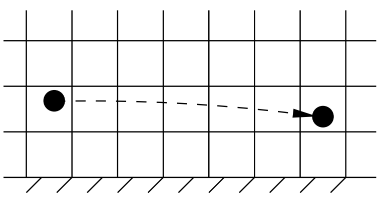

Figure 5: A zoom-in on the circular motion observed from a static system (with proper upward acceleration). The trajectory curves slightly downward, which will decrease the upward acceleration of the object relative to a freely falling system. In the limit that the acceleration of the freely falling frames is infinite, we can disregard the small curvature.

It is a well known property of Schwarzschild black holes that the proper acceleration of the local static reference frame goes to infinity as the radius approaches the radius of the event horizon. In other words the acceleration of a freely falling inertial frame (where special relativity holds, see Appendix B) which falls relative to the static reference frame, goes to infinity as the radius approaches the radius of the horizon. Furthermore we know that there is a maximum velocity $v = 1$ for material objects. Thus the perpendicular acceleration relative to the properly accelerated reference frame due to the curvature of the path remains finite (it is given by v^2/R) for non-zero R; R is non-zero for the circular motion in question. It follows that the acceleration $a_{\rm rel}$, relative to a freely falling frame, of an object in circular motion is dominated by the acceleration of the freely falling frame in the limit where the radius of the circle approaches the radius of the event horizon. Thus in this limit we can neglect the curvature of the path and from the reasoning in Sec. II we conclude that the force required to keep an object in a circular orbit (given by $F = m\gamma^2 a_{\rm rel}$) increases as the orbital velocity increases.

In brief, if an object moves it has less time (due to time dilation) to accelerate the necessary distance upward needed to remain at a fixed height (that is, fixed radius). Thus we can understand that that close to the horizon a greater outward force is needed to keep an object in orbit the faster the object moves.

IV A more quantitative analysis

To understand where the transition from a more Newtonian-like behavior occurs, we need a more detailed analysis. Because the reference frame connected to the static observers around the black hole locally behaves like an accelerating reference frame in special relativity, we first consider this special relativistic case.

Let $\mathbf{v}$ be the velocity of a particle relative to the accelerated reference frame and let $\mathbf{g}$ be the acceleration of an inertial frame, momentarily at rest relative to the reference frame, which falls relative to the the reference frame. Also assume that the direction of curvature $\hat{\mathbf{n}}$ (discussed in Appendix B) of the particle trajectory relative to the reference frame lies in the same plane as that spanned by $\mathbf{v}$ and $\mathbf{g}$. In this way we consider an effectively two-dimensional scenario as depicted in Fig. 6.

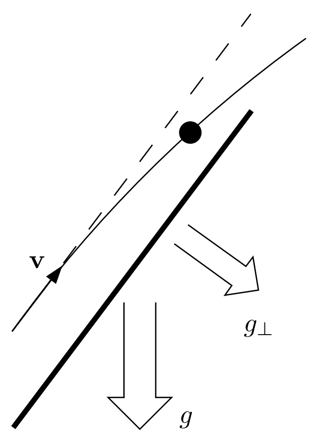

Figure 6: A particle moving along a trajectory of curvature R relative to the accelerating reference frame. The thick line is freely falling and is initially ($t = 0$) aligned with the dashed line. Concerning forces perpendicular to the direction of motion, only the perpendicular part of the acceleration $\mathbf{g}$ is relevant.

The perpendicular acceleration of a particle moving on a curve of radius R, as observed from the accelerated reference frame, is given by v^2/R. In other words, the proper spatial distance δs from the particle to a straight line fixed to the accelerated reference frame and aligned with the particle initial ($t = 0$) direction of motion is given by $\delta s = (\delta t)^2 v^2/(2R)$ to lowest nonzero order in δt; the latter is the local time as measured in the accelerated reference frame. Because the inertial (freely falling) system is initially at rest with respect to the accelerating reference frame, the time as measured by a grid of ideal clocks in the freely falling system is identical to time (to first order in δt) relative to the accelerating reference frame. The same goes for distances (length contraction does not kick in until the two frames have an appreciable relative velocity).

Consider a straight line fixed to the freely falling system that at $t = 0$ coincides with the previously mentioned line fixed to the reference frame. The separation between the two lines is (to lowest nonzero order in δt) given by $g_\perp(\delta t)^2/2$. Here $g_\perp$ is the part of the acceleration of the freely falling system that is perpendicular to the initial direction of motion, as observed from the accelerating reference frame. It follows that observed from the freely falling system, where special relativity holds, the particle will have an acceleration perpendicular to the direction of motion given by $g_\perp - v^2/R$. In analogy to our earlier reasoning, the perpendicular acceleration as observed in a system comoving with the particle is greater by a factor of γ^2, and the perpendicular force $F_\perp$ (as experienced in the particle's own system) is thus given by

$$F_\perp = m\gamma^2(g_\perp - v^2/R). \tag{3}$$

To clarify any sign ambiguities we rewrite Eq. (3) in terms of vectors:

$$\frac{\mathbf{F}_\perp}{m} = -\gamma^2\mathbf{g}_\perp + \gamma^2 v^2 \frac{\hat{\mathbf{n}}}{R}. \tag{4}$$

Equation (4) relates the perpendicular part of the force (as observed in the particle's own reference system) to the spatial curvature of the particle trajectory relative to a reference frame with proper acceleration $-\mathbf{g}$. Although Eq. (4) was derived for an effectively two-dimensional scenario, it holds also in three dimensions as we will see in Sec. IX. From the equivalence principle it follows that Eq. (4) applies also to motion around a black hole. For this case the curvature vector is defined relative to the spatial geometry connected to the static observers, as explained in Appendix B.

V Following the geodesic photon

Inspired by the reasoning of Abramowicz et al.,[3] we now consider motion along the spatial trajectory of a geodesic photon (a photon whose motion is determined by gravity alone, see Appendix B). For a geodesic particle we have $F_\perp = 0$, and thus according to Eq. (3), $g_\perp = v^2/R$. For a geodesic photon whose path curvature we denote by R_{phot}, we have thus $g_\perp = 1/R_{\text{phot}}$ (because $v = 1$ for photons). For a particle following the path of a such a photon (so $1/R = g_\perp$) we have according to Eq. (3), $F_\perp = m\gamma^2(g_\perp - v^2 g_\perp)$, which simplifies to $F_\perp = mg_\perp$. Thus, the perpendicular force required to make a particle follow the trajectory of a geodesic photon is independent of the velocity of the particle.

To make this fact more transparent, we consider the curvature vector of a geodesic photon given by Eq. (4) (set $\mathbf{F}_\perp = 0$ and $v = 1$)

$$\frac{\hat{\mathbf{n}}_{\text{phot}}}{R_{\text{phot}}} = \mathbf{g}_\perp. \tag{5}$$

We introduce $\hat{\mathbf{n}}_{\text{rel}}/R_{\text{rel}}$ as the curvature vector relative to the trajectory of a geodesic photon:

$$\frac{\hat{\mathbf{n}}_{\text{rel}}}{R_{\text{rel}}} = \frac{\hat{\mathbf{n}}}{R} - \frac{\hat{\mathbf{n}}_{\text{phot}}}{R_{\text{phot}}}. \tag{6}$$

This definition of $\hat{\mathbf{n}}_{\text{rel}}/R_{\text{rel}}$ gives how quickly a particle trajectory deviates from a geodesic photon trajectory in analogy to how $\hat{\mathbf{n}}/R$ gives how quickly the particle trajectory deviates from a straight line. We substitute Eqs. (5) and (6) in Eq. (3) and find

$$\frac{\mathbf{F}_\perp}{m} = -\mathbf{g}_\perp + \gamma^2 v^2 \frac{\hat{\mathbf{n}}_{\text{rel}}}{R_{\text{rel}}}. \tag{7}$$

Equation (7) also holds for more general three-dimensional scenarios as will be shown in Sec. IX.

Within the photon radius, a photon trajectory departs inward relative to a locally tangent circle. Thus the relative curvature direction $\hat{\mathbf{n}}_{\text{rel}}$ of a circle within the photon radius is directed outward. From Eq. (7) it then follows that the faster an object orbits the black hole, the greater the outward force must be. However, outside of the photon radius, a photon trajectory departs outward from a locally tangent circle. Thus the relative curvature of the circle is directed inward. It follows from Eq. (7) that outside of the photon radius, the outward force required to keep the rocket in orbit will decrease as the velocity increases. Thus we see that the effective centrifugal force reversal for circular motion occurs exactly at the photon radius.

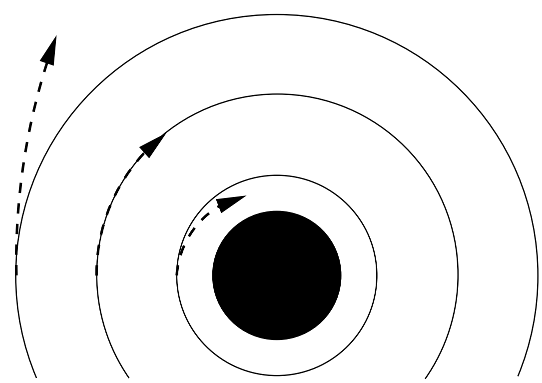

Figure 7: Trajectories of geodesic photons (dashed curves) relative to circles around a black hole. Inside the photon radius a circle curves outward relative to a locally tangent photon trajectory.

VI The difference between the given and the received force

Before considering a more general analysis, we will distinguish between two types of forces. The perpendicular force that we have discussed is the force as observed in a system comoving with the object in question. Consider now a situation where the observers connected to the reference frame in question (like the accelerating platform we have considered previously) provide the force that keeps the object on its path. How is this force, which we will refer to as the *given* force, related to the previously considered force, which we will refer to as the *received* force, that is, the force as observed in a system comoving with the object? For instance, we might be interested in the magnitude of the vertical force by the rail that is needed to support a train moving with a relativistic speed along the track. Unlike in Newtonian theory, this given force will be different from the force as observed in a system comoving with the train.

For forces perpendicular to the direction of motion, the relation between the given and the received force can be understood by considering a simple model of force exertion (a more formal derivation is given in Ref. 10). Assume that the force on the object is exerted by little particles bouncing elastically on the object. Every bounce gives the object a certain impulse (see Fig. 8).

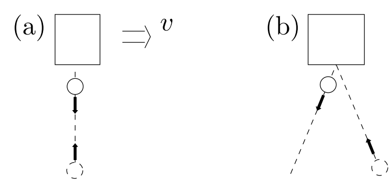

Figure 8: A simple model where small particles bounce elastically on the object. (a) The scenario as observed from a system where the impulse giving particle has no horizontal velocity. (b) The corresponding scenario as observed from a system comoving with the object.

If we give an object moving relative to an inertial system S an impulse perpendicular to the direction of motion, the object will in its own reference system receive the same impulse because a Lorentz transformation does not affect the perpendicular part of the momentum change. On the other hand, the proper time of the object runs slower by a factor of γ compared to local time in S. Hence the bouncing particles will bounce more frequently by a factor of γ as observed from the reference frame of the object. Because force equals transferred impulse per unit time, it follows that the received force, perpendicular to the direction of motion, is greater than the corresponding given force, by a factor of γ. We let $F_\perp$ denote the perpendicular received force and $F_{c\perp}$ the perpendicular given force (to conform with the notation of Ref. 10) and write

$$F_{c\perp} = F_\perp/\gamma. \tag{8}$$

Hence the given force is smaller than the received force by a factor of γ. Because the received force required to keep an object (like an entire train) moving along a straight horizontal line relative to a vertically accelerating reference frame is proportional to γ^2 (as discussed in Sec. II), it follows that the force required by the rail to support the train scales with a factor of γ.

In Sec. V we showed that the perpendicular received force is independent of the velocity for an object that follows the trajectory of a geodesic photon. Now we ask if there is a corresponding path for which the perpendicular given force is velocity independent. The analogue of Eq. (4) for the given force is

$$\frac{\mathbf{F}_{c\perp}}{m} = -\gamma \mathbf{g}_\perp + \gamma v^2 \frac{\hat{\mathbf{n}}}{R}. \tag{9}$$

We now require $\mathbf{F}_{c\perp}$ to be the same for an object moving with speed v as that of an object at rest. For $v = 0$, Eq. (9) gives $\mathbf{F}_{c\perp}/m = -\mathbf{g}_\perp$. We substitute this result into Eq. (9) and find

$$\frac{\hat{\mathbf{n}}}{R} = \frac{\mathbf{g}_\perp}{v^2}\Big(1 - \frac{1}{\gamma}\Big) = \mathbf{g}_\perp \frac{\gamma}{\gamma + 1}. \tag{10}$$

This curvature depends on the velocity.[14] Thus considering the given force (the force as observed from the accelerating reference frame), there is no path for which the perpendicular force is independent of the velocity.

VII The relativistic submarine

As an application of our discussion we consider a submarine submerged in water with a density such that it remains at rest.[9] If we take relativistic effects into account, but disregard the more subtle aspects of fluid dynamics such as viscosity and turbulence, will the submarine sink or float when it is given a horizontal velocity?

A A flat ocean in special relativity

We first consider a special relativistic scenario where the flat bottom of the ocean has a constant proper upward acceleration. In Ref. 9 accelerated (Rindler) coordinates are used to find out whether the submarine sinks or floats, after several pages of calculation.

By using our earlier reasoning, we can readily find the answer without any calculations. The received force needed to keep the submarine at the same depth increases by a factor of γ^2 as demonstrated in Sec. II. The given force needed to keep it at a constant depth thus increases by a factor of γ because it is smaller than the received force by a factor of γ as explained in Sec. VI. However, the submarine is length contracted, so the actual given force from the water pressure (or rather the differences of

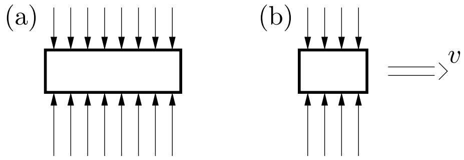

Figure 9: (a) An idealized (rectangular) submarine submerged in water at rest relative to the water. (b) As the submarine moves, it will be length contracted and thus the given force from the water will decrease by a factor of γ.

water pressure at the top and bottom of the submarine) will decrease by a factor of γ (see Fig. 9).

Therefore the given force decreases by a factor of γ, whereas it should increase by a factor of γ in order for the submarine to remain at a fixed depth. Thus the submarine will sink (see Fig. 10).

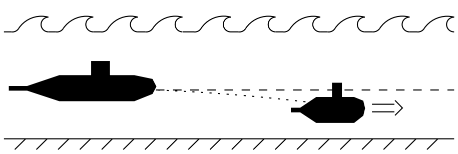

Figure 10: A submarine submerged in a balanced state of rest in a flat ocean with proper upward acceleration, will sink due to relativistic effects if it starts moving horizontally.

Now let us analyze the situation from the submarine. Due to length contraction the actual given force is decreased by a factor of γ, as we have argued. The received force is γ times the given force. Thus, the received force is independent of the velocity.[15] This force is not sufficient to keep the submarine at the same depth. The experienced force would have to increase by γ^2 for that. The submarine thus sinks.

To understand why the received force on the submarine is independent of the velocity, we can also look at the water at the molecular level. Assume that the water molecules are moving along columns fastened to the ocean bottom (a very crude model). Assume also that the particles elastically bounce back down the same column (without interfering with the up-moving water molecules) when they hit the hull of the submarine (and analogously for the water molecules on top of the submarine). The impulse given by a single molecule is the same as when the submarine was at rest. However, as observed from the moving submarine the columns of water molecules are length contracted by a factor of γ. There are thus more columns under the submarine (and above) in the submarine frame, when the submarine moves than when it is at rest. On the other hand, due to time dilation, how often a molecule from a single column hits the hull is decreased by a factor of γ (consider a clock fixed to the column just where the column intersects the submarine hull). The effects thus cancel. It follows that the received force on the submarine is independent of the velocity.

Although every single column of water yields a received force that is smaller than the force given by that column (by a factor of γ), there are γ^2 times more columns contributing to the net force as observed from the submarine frame, than observed from the rest frame of the water (see Fig. 11). Thus consistent with the reasoning of Sec. VI, the net received force is greater than the given force by a factor of γ.

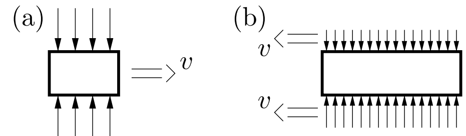

Figure 11: (a) Observed from the water system the submarine is length contracted by a factor of γ. (b) Observed from the submarine the water columns are length contracted and thus denser by a factor of γ.

B A real spherical ocean

It is easy to generalize the above discussion to apply to a submarine in the ocean of a spherical planet. Locally the scenario is almost identical to the one we have discussed, assuming that the submarine is small compared to the size of the planet. The question of whether the submarine floats or sinks amounts to whether the submarine, when given an azimuthal velocity, departs outward or inward from a circle locally tangent to the direction of motion. The answer follows from our previous discussion. As argued in Sec. A, the force as experienced in a system comoving with the submarine, is independent of the velocity for this case. It then follows from Eq. (7) that the submarine will have zero curvature relative to a geodesic photon and will thus follow the path of a geodesic photon.

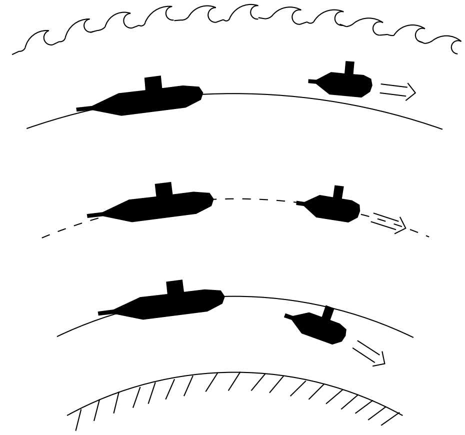

Figure 12: Submarines moving at different depths in the ocean of an imaginary very dense planet. The dashed line is the photon radius. The submarines outside the photon radius will float upward if they are given an azimuthal velocity; the opposite holds within the photon radius.

So, outside the photon radius (the radius where photons would move on circular orbits if there were no refraction effects from the water) the submarine will float upward, at the photon radius it will remain at the same depth, and inside the photon radius it will sink. The scenario is illustrated in Fig. 12.

Consider the Earth, which is not sufficiently dense to have a photon radius. If we take into account the Earth's curvature, it follows that when given a horizontal velocity, the submarine will not sink after all but rather float upward.

VIII The weight of a box with moving particles

As another application of our discussion, we consider the weight of an object whose internal components move. In general relativity, if we for instance heat an object, it will become heavier. In other words, a greater upward force is required to keep the object at rest (on Earth) when the object is warm (molecules moving faster) than when it is cold. Although not directly related to the main topic of this article (inertial forces), we can give a simple explanation.

Consider a black box containing two balls connected by a rod of negligible mass which is suspended in such a way that the balls can rotate in a horizontal plane. If they rotate, the upward force needed to keep a single ball in the horizontal plane as observed from the balls' reference system is $mg\gamma^2$, where m is the rest mass of the ball. The given force is smaller by a factor of γ and is hence given by $mg\gamma$. So the weight of the box is greater when the internal particles move than when they are at rest (see Fig. 13).

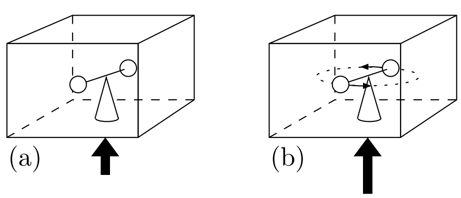

Figure 13: A black box (transparent for clarity) containing a pair of balls that (a) are at rest and (b) are moving. The force needed to hold the box at a fixed height on Earth is greater when the balls are moving than when they are at rest. The force is proportional to the total relativistic energy of the box.

For vertical or arbitrary motion, this type of reasoning is not as powerful, and we can instead make a more formal proof using four-vectors and conservation of four-momentum.

IX Generalizing to three dimensions

Consider a reference frame with a proper (upward) acceleration. Given the curvature and curvature direction of the path taken by a test particle relative to the reference frame, we want to express the perpendicular acceleration of the test particle relative to an inertial system S in which the reference frame is momentarily ($t = 0$) at rest. In Fig. 14 we illustrate how the trajectory will deviate from a straight line (directed along the particle initial direction of motion) which is fixed to S, and thus falls relative to the accelerated reference frame. From this deviation we can find the perpendicular acceleration relative to S, analogous to the two-dimensional discussion in Sec. IV.

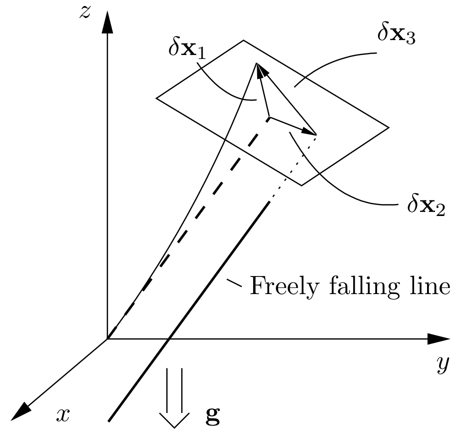

Figure 14: Deviations from a straight line relative to the (properly) accelerated reference system. The z-direction is chosen to be antiparallel to the local $\mathbf{g}$. The plane in which we study the deviations is perpendicular to the momentary direction of motion (the dashed line) and the three vectors lie in this plane. The solid curve is the particle trajectory as observed in the (properly) accelerated reference system. The thick line is a freely falling line that was aligned with the dashed line (and at rest relative to the reference frame) at the time when the particle was at the origin.

If we let δt denote a small time step and use the definitions introduced in Fig. 14, we have to lowest nonzero order in δt:

$$\delta\mathbf{x}_1 = \frac{\hat{\mathbf{n}}}{R}\frac{v^2\delta t^2}{2} \tag{11a}$$

$$\delta\mathbf{x}_2 = \mathbf{g}_\perp\frac{\delta t^2}{2} \tag{11b}$$

$$\delta\mathbf{x}_3 = \delta\mathbf{x}_1 - \delta\mathbf{x}_2, \tag{11c}$$

where R and $\hat{\mathbf{n}}$ are the curvature and curvature direction of the spatial trajectory relative to the accelerated reference frame. Let $\mathbf{a}_{\mathrm{rel}\perp}$ be the acceleration of the test particle perpendicular to the direction of motion relative to the freely falling frame. By using $\delta \mathbf{x}_3 = \mathbf{a}_{\mathrm{rel}\perp}\delta t^2/2$ and Eq. (11), we find

$$\mathbf{a}_{\mathrm{rel}\perp} = -\mathbf{g}_\perp + v^2\frac{\hat{\mathbf{n}}}{R}. \tag{12}$$

We denote the received perpendicular force by $\mathbf{F}_\perp$. According to our previous reasoning, we have $\mathbf{F}_\perp = m\gamma^2\mathbf{a}_{\mathrm{rel}\perp}$, and thus

$$\frac{1}{m\gamma^2}\mathbf{F}_\perp = -\mathbf{g}_\perp + v^2\frac{\hat{\mathbf{n}}}{R}. \tag{13}$$

Equation (13) relates the experienced perpendicular force and the curvature relative to the accelerating reference system. We note that the only difference from its Newtonian analogue is the factor of γ^2 on the left-hand side. In analogy to the two-dimensional discussion in Sec. V, we may introduce a curvature relative to that of a geodesic photon as

$$\frac{\hat{\mathbf{n}}_{\mathrm{rel}}}{R_{\mathrm{rel}}} = \frac{\hat{\mathbf{n}}}{R} - \frac{\hat{\mathbf{n}}_{\mathrm{phot}}}{R_{\mathrm{phot}}}. \tag{14}$$

If we use Eq. (13) to find $\hat{\mathbf{n}}_{\mathrm{phot}}/R_{\mathrm{phot}}$ (setting $\mathbf{F}_\perp = 0$ and $v = 1$) and substitute the expression for $\hat{\mathbf{n}}/R$ from Eq. (14) into Eq. (13), we obtain

$$\frac{\mathbf{F}_\perp}{m} = -\mathbf{g}_\perp + \gamma^2 v^2\frac{\hat{\mathbf{n}}_{\mathrm{rel}}}{R_{\mathrm{rel}}}. \tag{15}$$

We see that Eqs. (4) and (7), which were previously derived only for effectively two-dimensional scenarios, are also valid for arbitrary three-dimensional scenarios. For the case where the observers at rest in the accelerating reference frame provide the pushing needed to keep the particle on track, we obtain the given force as before by dividing the received force by a factor of γ.

X Parallel accelerations

Now that we know how the spatial curvature depends on the perpendicular force, it would be useful also to know how forces in the forward direction affect the speed v of the particle relative to the accelerated reference frame. We could derive this relation using four-velocities,[16] but for simplicity, we will use only standard results that follow from the Lorentz-transformation.

To determine dv/dt, where v is the local velocity relative to the accelerating reference frame, we must take into account that the derivative implies that we are comparing the velocity at two different times, relative to two different systems (effectively) because the reference system is accelerating. Consider a scenario where the acceleration a of the reference frame is aligned with the direction of motion. Relative to an inertial system S in which the reference frame is at rest at $t = 0$, the reference frame gains a velocity $\delta u = a\delta t$ after a time δt. We denote by δv_s the velocity difference of the particle relative to S from $t = 0$ to $t = \delta t$ (see Fig. 15).

The relation between an arbitrary object's velocity w (along the x-axis) as observed from S and the corresponding velocity w' as observed from an inertial system

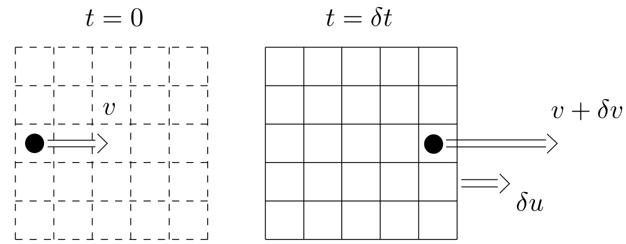

Figure 15: An object moving relative to an accelerated reference frame. At $t = 0$ the velocity of the particle is v. In a time δt the reference frame is accelerated to a velocity δu, and the velocity of the particle relative to the accelerated reference frame is $v + \delta v$.

S' moving with velocity δu relative to S (along the x-axis) follows from the Lorentz transformation (see for example, Ref. 17, p. 31)

$$w' = \frac{w - \delta u}{1 - w\delta u}. \tag{16}$$

If we substitute $w = v + \delta v_s$ and $w' = v + \delta v$ in Eq. (16) and do a Taylor expansion to first order in δu and δv_s, we obtain

$$\delta v = \delta v_s - (1 - v^2)\delta u. \tag{17}$$

We also know (see for example, Ref. 17, p. 33), that the proper acceleration α of the object, that is, the acceleration as observed in a system comoving with the object, is related to the acceleration dv_s/dt relative to S by

$$\alpha = \gamma^3\frac{dv_s}{dt}. \tag{18}$$

If we denote the received forward thrust by $F_\parallel$, we have $F_\parallel = m\alpha$. We use this relation in Eqs. (17) and (18), take the limit where δt is infinitesimal, together with $a = du/dt$ and find

$$\frac{dv}{dt} = \frac{1}{m\gamma^3}F_\parallel - \frac{a}{\gamma^2}. \tag{19}$$

Consider now a more general case where the acceleration of the reference frame need not be aligned with the direction of motion. It is easy to realize (or at least guess) that the acceleration of the reference frame perpendicular to the direction of motion will not affect the local speed derivative.[18] We let $\mathbf{g} = -\mathbf{a}$, where $\mathbf{a}$ is the acceleration of the reference frame relative to an inertial system in which the reference frame is momentarily at rest, and write

$$\frac{dv}{dt} = \frac{1}{m\gamma^3}F_\parallel + \frac{g_\parallel}{\gamma^2}. \tag{20}$$

Here $g_\parallel$ is minus the part of the reference frame acceleration that is parallel to the particle's direction of motion. Thus we now have a general expression for the speed change relative to the accelerating reference system. Note that t is the local time relative to the reference frame (so $dt = \gamma d\tau$).

A Combining the force equations

From the form of Eqs. (20) and (13), we see that we can combine them into a single vector relation. Let $\hat{\mathbf{t}}$ be a normalized vector in the forward direction of motion (to conform with the notation of Ref. 10). By multiplying Eq. (20) by $\gamma^2\hat{\mathbf{t}}$ and adding the resulting equation to Eq. (13), we can form a single term $\mathbf{g}$ (by adding the $\mathbf{g}_\perp$ and $g_\parallel\hat{\mathbf{t}}$ terms) and obtain

$$\frac{1}{m\gamma^2}(\gamma F_\parallel \hat{\mathbf{t}} + F_\perp \hat{\mathbf{m}}) = -\mathbf{g} + \gamma^2 \frac{dv}{dt}\hat{\mathbf{t}} + \frac{v^2}{R}\hat{\mathbf{n}}. \tag{21}$$

Here $\hat{\mathbf{m}}$ is a unit vector perpendicular to $\hat{\mathbf{t}}$. We thus have an expression for the spatial curvature and the speed derivative in terms of the received forces. Note that $\mathbf{g}$ may be interpreted as an inertial (fictitious) force; we will discuss this interpretation in Sec. XI.

We have previously considered a rocket in circular orbit with constant speed around a black hole. Now we consider a rocket in radial motion with constant speed outward from a black hole. From the parallel part of Eq. (21) we find

$$F_\parallel = mg\gamma, \tag{22}$$

where g is the magnitude of the acceleration of the local freely falling frames (g is a function of the radius that can readily be found from the spacetime metric). Here there are no reversal issues. However, we can see that (unlike in Newtonian theory), a greater thrust is needed to keep a constant speed the faster the rocket moves.

B The given parallel force

If we would like an expression of the type Eq. (21) for the parallel given force, we need to know how the given force along the direction of motion is related to the received force along the direction of motion. We can make an argument similar to the one we made in Sec. VI. Let S denote a certain rest system, and let S' be a system in a standard (non-rotated) configuration relative to S, which comoves with the object in question along the x-axis of S. Consider the force parallel to the direction of motion to be mediated by (very light) particles bouncing elastically on the object. For simplicity let us assume that in a system comoving with the object, each bouncing particle is reflected in such a way that the energy of the bouncing particle is unaffected by the bounce (so $\Delta p'^0 = 0$). If we consider motion along the x-axis and use the fact that the change of momentum four-tensor transforms according to the Lorentz transformation, we have

$$\Delta p^x = \gamma(\Delta p'^x + v \underbrace{\Delta p'^0}_{0}). \tag{23}$$

Thus the received impulse $\Delta p'^x$ is smaller than the given impulse Δp^x by a factor of γ. On the other hand, due to time dilation the frequency at which these impulses are received (assuming several bouncing particles) is greater in the comoving system S' than in S by a factor of γ. These two factors of γ cancel each other, and we conclude that the given and the received force in the direction of motion are the same. One can easily give a formal proof of this fact (see for example, Ref. 10).

We can now express Eq. (21) in terms of the given forces. We let $F_{c\parallel}$ denote the given force in the direction of motion and write

$$\frac{1}{m\gamma^2}(\gamma F_{c\parallel} \hat{\mathbf{t}} + \gamma F_{c\perp} \hat{\mathbf{m}}) = -\mathbf{g} + \gamma^2 \frac{dv}{dt}\hat{\mathbf{t}} + \frac{v^2}{R}\hat{\mathbf{n}}. \tag{24}$$

We define $\mathbf{F}_c = F_{c\parallel}\hat{\mathbf{t}} + F_{c\perp}\hat{\mathbf{m}}$ and write Eq. (24) as

$$\frac{\mathbf{F}_c}{m\gamma} = -\mathbf{g} + \gamma^2 \frac{dv}{dt}\hat{\mathbf{t}} + \frac{v^2}{R}\hat{\mathbf{n}}. \tag{25}$$

If we compare (25) with Eq. (21), we see that the formalism is a bit cleaner if we consider the given force rather than the received force.

XI Rotating reference frame

Suppose that we would also like to consider stationary spacetimes, such as the spacetime of a rotating (Kerr) black hole. For this case we have a spatial geometry defined by the stationary (Killing) observers. In this case through frame dragging, the local reference frame connected to the stationary observers is not only accelerating, but also rotating.

Consider (in special relativity) a reference frame that rotates around its origin relative to an inertial system S. For simplicity, we consider motion along a straight line that passes the origin and is fixed to the rotating frame. The particle is assumed to be at the origin at $t = 0$. This scenario is depicted in Fig. 16.

Let $\delta\mathbf{x}$ denote the perpendicular separation from the particle to a line that is fixed in the inertial system S and that at $t = 0$ was aligned with the rotating line. We let $\delta\mathbf{u}$ denote the velocity of the line fixed to the rotating system at the position of the particle after a time δt. To lowest order in $\delta\mathbf{u}$ we have

$$\delta\mathbf{x} = \delta\mathbf{u}\,\delta t. \tag{26}$$

The position of the particle after a time δt is $\mathbf{v}\delta t$ (to lowest order in δt), where $\mathbf{v} = v\hat{\mathbf{t}}$. We thus have $\delta\mathbf{u} = \boldsymbol{\omega}\times(\mathbf{v}\delta t)$. We use this relation in Eq. (26) and obtain $\delta\mathbf{x} = \boldsymbol{\omega}\times\mathbf{v}\delta t^2$. In the limit where the time step is infinitesimal, the perpendicular acceleration coming from the rotation is

$$\mathbf{a}_{\text{rel}\perp} = 2\boldsymbol{\omega}\times\mathbf{v}. \tag{27}$$

For the low reference frame velocities that occur during the short time δt, the effects of length contraction and time dilation will not enter the expressions for the perpendicular deviations (to lowest nonzero order). Therefore we can add the effect of rotation to the effects of curvature and acceleration. The generalization of Eq. (12) is

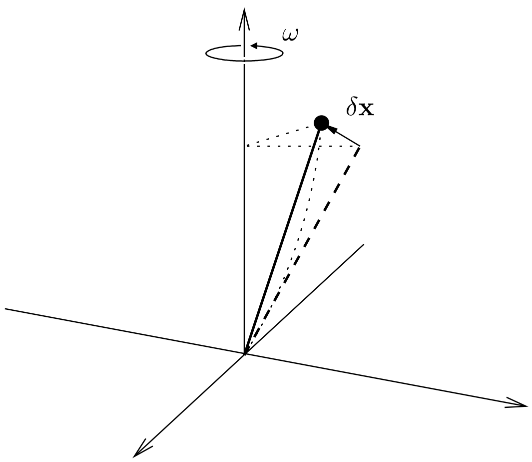

Figure 16: A particle (the black dot) moving along a rotating straight line (depicted at two successive time steps – the dashed and the solid line), as observed relative to an inertial system S. Relative to S the particle trajectory (the dotted line) curves.

thus

$$\mathbf{a}_{\mathrm{rel}\perp} = -\mathbf{g}_{\perp} + 2\boldsymbol{\omega} \times \mathbf{v} + v^2 \frac{\hat{\mathbf{n}}}{R}. \quad (28)$$

Here $\mathbf{a}_{\mathrm{rel}\perp}$ is the perpendicular acceleration of the test particle relative to the inertial system S.

Because the changes in the reference frame velocity (as observed from the inertial system in question) are perpendicular to the direction of motion, the derivative of the speed will not be affected by the rotation. Thus we can write the generalization of Eq. (21) as

$$\frac{1}{m\gamma^2}(\gamma F_{\|}\hat{\mathbf{t}} + F_{\perp}\hat{\mathbf{m}}) = -\mathbf{g} + 2\boldsymbol{\omega} \times \mathbf{v} + \gamma^2 \frac{dv}{dt}\hat{\mathbf{t}} + \frac{v^2}{R}\hat{\mathbf{n}}. \quad (29)$$

Equation (29) relates the real received forces to both the curvature and the speed change per unit time relative to the accelerating and rotating reference frame. Note that although $\mathbf{g}$ is minus the acceleration of the reference frame, $\boldsymbol{\omega}$ is the rotation vector of the reference frame.

As an application we consider a person walking on a straight line through the center of a rotating flat merry-go-round (in special relativity). The perpendicular force experienced as he/she passes the center (where $\mathbf{g}$ is zero) is given by Eq. (29) as

$$F_{\perp} = 2\omega_0 v\gamma^2. \quad (30)$$

Here ω_0 is the angular frequency of the merry-go-round. Apart from the γ^2 factor, Eq. (30) is the same as the corresponding equation in Newtonian mechanics. For points other than the central point we must consider that the proper rotation ω (as measured by an observer riding the merry-go-round at the point in question) is different from the rotation ω_0 as observed from the outside.[19]

XII Discussion

On the left-hand side of Eq. (29) there are real forces as experienced in a system comoving with the object in question. On the right-hand side the first two terms multiplied by $-m$ may be interpreted as inertial forces

$$\text{Acceleration:} \quad m\mathbf{g}, \quad (31a)$$

$$\text{Coriolis:} \quad -2m\boldsymbol{\omega} \times \mathbf{v}. \quad (31b)$$

We might be tempted to denote the first term by "gravity" rather than "acceleration," but if we consider a rotating merry-go-round as a reference frame, this term would correspond to what is commonly called the centrifugal force. To avoid confusion we therefore label this term "acceleration." For the second term the name Coriolis is obvious in analogy with the standard notation for inertial forces in non-relativistic mechanics.

Note that what we call an inertial force is ambiguous. For example, we could multiply the perpendicular part of Eq. (29) by γ. By defining $\mathbf{F} = F_{\|}\hat{\mathbf{t}} + F_{\perp}\hat{\mathbf{m}}$, we could then simplify the left-hand side of Eq. (29) to $\mathbf{F}/m\gamma$. However, because of the γ-multiplication we would need to express the $\mathbf{g}$-term as a sum of a parallel and a perpendicular part (with different factors of γ), thus creating two different acceleration terms. There is is thus more than one way of expressing Eq. (29), and identifying inertial forces, that reduce to the Newtonian analogue by setting $\gamma = 1$.

We do not regard the last two terms on the right-hand side of Eq. (29) as inertial forces, but rather as descriptions of the motion (acceleration) relative to the frame of reference. There are alternative interpretations; see Ref. 10 for further discussion.

Note that dt is the local time (for the local reference frame observers) and is related to the proper time $d\tau$ for the particle in question by $dt = \gamma d\tau$. Equation (29) is identical to the more formally derived corresponding expression in Ref. 10.

We have considered accelerating and rotating reference frames, but not shearing or expanding reference frames. The extension is straightforward for an isotropically expanding reference frame, but for brevity we refer to Ref. 10.

In summary, we have seen how we can derive a formalism of inertial forces that applies to arbitrary rigid reference frames in special and general relativity. Apart from factors of gamma, the formalism is locally equal to its Newtonian counterpart. We have also applied the insights and formalism of this paper to various examples, such as moving trains and submarines.

A A comment on static spacetimes, index notation, and the optical geometry

For the purposes of this article it is not necessary to discuss a formalism known as optical geometry. However, because the latter is the inspiration for this article and the formalisms are very similar, a comment is in order. The index formalism (which distinguishes between covariant and contravariant vectors) is vital for the comparison.

Suppose that we have a static spacetime with the line element

$$ds^2 = -e^{2\Phi} dt_c^2 + g_{ij} dx^i dx^j. \tag{A1}$$

We denote coordinate time by t_c so as not to confuse it with the local time of the reference frame which we denote by t. Also, Latin indices are spatial indices running from 1–3. It is easy to show (see for example, Ref. 10, Appendix E) that the acceleration of the freely falling frames for a line element of this form is given by $\mathbf{g} = -\nabla\Phi$. We can equivalently write this relation as $g^k = -g^{kj}\nabla_j\Phi$. For later convenience we define $F_\perp^k = F_\perp m^k$, where m^k is a normalized spatial vector. If we use these results and definitions, we can rewrite Eq. (15) as

$$\frac{F_\perp}{m} m^k = [g^{kj}\nabla_j\Phi]_\perp + \gamma^2 v^2 \frac{n^k_{\text{rel}}}{R_{\text{rel}}}. \tag{A2}$$

Here $\perp$ means that we should select the part perpendicular to the spatial direction of motion t^k. For a line element such as Eq. (A1), the optical geometry (see for example, Ref. 20 although a different sign convention for Φ is used) is given by a rescaling of the standard spatial geometry

$$\tilde{g}_{ij} = e^{-2\Phi} g_{ij}. \tag{A3}$$

We thus stretch space by a factor $e^{-\Phi}$ to create a new spatial geometry. We may consider both metrics to live on the same (sub)manifold. Relative to the rescaled geometry, the curvature of a given spatial (coordinate) trajectory is in general different from that relative to the standard spatial geometry. In particular, the spatial trajectories of geodesic photons are straight with respect to the optically rescaled space. It follows that the curvature and curvature direction with respect to the rescaled (optical) space gives how fast (with respect to the distance along the trajectory) and in what direction a trajectory deviates from that of a geodesic photon. This curvature and curvature direction thus correspond to the relative curvature and curvature direction introduced in Sec. V and Sec. IX, except that the deviation and the distance along the trajectory are now rescaled. The optical spatial curvature $\tilde{R}$, the optical curvature direction $\tilde{n}^k$, and the optically normalized direction of the perpendicular force $\tilde{m}^k$ for a certain (coordinate) trajectory are related to R_{rel}, n^k_{rel}, and m^k by[21]

$$\tilde{R} = e^{-\Phi} R_{\text{rel}} \tag{A4a}$$

$$\tilde{n}^k = e^{\Phi} n^k_{\text{rel}} \tag{A4b}$$

$$\tilde{m}^k = e^{\Phi} m^k. \tag{A4c}$$

If we use $\tilde{g}^{ij} = e^{2\Phi} g^{ij}$, we may rewrite Eq. (A2) as (multiply the entire expression by $e^{2\Phi}$)

$$\frac{F_\perp}{m} e^{\Phi} \tilde{m}^k = [\tilde{g}^{kj}\tilde{\nabla}_j\Phi]_\perp + \gamma^2 v^2 \frac{\tilde{n}^k}{\tilde{R}}. \tag{A5}$$

Note that because the covariant derivative acts on a scalar (in contrast to a vector for example), we have $\tilde{\nabla}_j = \nabla_j\Phi$ (although $\tilde{\nabla}^j\Phi = e^{2\Phi}\nabla^j\Phi$). By comparing Eq. (A5) with the more general (and more formally derived) corresponding equation in Ref. 10, we have a perfect match.[22] In covariant form (lower indices with $\tilde{g}_{ij}$) Eq. (A5) becomes slightly more compact:

$$\frac{F_\perp}{m} e^{\Phi} \tilde{m}_k = [\tilde{\nabla}_k\Phi]_\perp + \gamma^2 v^2 \frac{\tilde{n}_k}{\tilde{R}}. \tag{A6}$$

Because $\tilde{m}_k = e^{-\Phi} m_k$, the left-hand side of Eq. (A6) can be expressed as $F_{\perp k}/m$. On the other hand, the left-hand side of Eq. (A5) can be written as $F_\perp^k e^{2\Phi}/m$. Expressed in these forms, but using the boldface vector notation, the right-hand sides of Eq. (A5) and Eq. (A6) are identical and the left-hand sides differ by a factor $e^{2\Phi}$. We hence understand the hazard of using the bold face vector notation, at least if we use vectors that naturally "belong" to two different metrics in the same expression. As Eqs. (A5) and (A6) are written, only vectors belonging to the optical geometry are used, and we could use vector notation after all.

The parallel part of Eq. (21) in index notation (for the line element in question and a static reference frame) takes the form

$$\frac{1}{m\gamma} F_\parallel t^k = [g^{kj}\nabla_j\Phi]_\parallel + \gamma^2 \frac{dv}{dt} t^k. \tag{A7}$$

Here we have $dt = e^{\Phi} dt_c$. We use the latter relation, rewrite the tensors in terms of their rescaled analogues, multiply the entire expression by $e^{2\Phi}$, and add it to Eq. (A5). The result is

$$\frac{e^{\Phi}}{m}\Big(\frac{F_\parallel}{\gamma}\tilde{t}^k + F_\perp \tilde{m}^k\Big) = \tilde{g}^{kj}\tilde{\nabla}_j\Phi + \gamma^2 \frac{dv}{dt_c}\tilde{t}^k + \gamma^2 v^2 \frac{\tilde{n}^k}{\tilde{R}}. \tag{A8}$$

Equation (A8) is the inertial force formalism in terms of the optical geometry. Again it agrees with the corresponding equation of Ref. 10.[23]

B Some basic concepts

This appendix is included for readers with little or no background in differential geometry or Einstein's theory of gravity.

Curvature. Consider a curved path on a plane. At any point along the path we can find a circle that is precisely tangent to the path and whose curvature matches that of the path (see Fig. 17). At any point along the curve we can thus introduce a curvature direction $\hat{\mathbf{n}}$, a unit vector, and a curvature radius R as shown. The greater the curvature, the smaller the curvature radius. For paths that are not in a plane we can locally match a circle to every point along the path and define the curvature direction and curvature radius analogously. Note that the curvature direction $\hat{\mathbf{n}}$ is always perpendicular to the path.

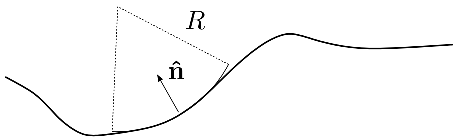

Figure 17: A path on a plane always corresponds locally to a circle as far as direction and curvature are concerned.

Spatial geometry. Consider a symmetry plane through a black hole. For the purposes of this article we may illustrate the black hole as a black sphere (see Fig. 18).

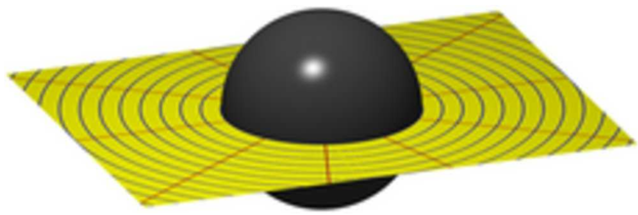

Figure 18: A symmetry plane through a black hole.

If we could walk around on the plane and measure distances, we would notice that the distances would not match those we would expect from a flat plane. Rather, the apparent geometry would be as that depicted in Fig. 19. In particular, we would note that as one walks outward from the surface of the black hole, the circumference would initially hardly change. Although the geometry of the curved surface corresponds to the geometry of the symmetry plane, the symmetry plane neither curves upward nor downward in reality. Distances on the plane are *as if* the plane curves as depicted in Fig. 19.

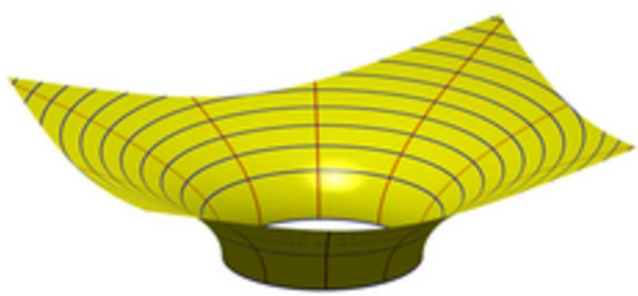

Figure 19: Sketch of the apparent geometry of a symmetry plane through a black hole. The innermost circle is at the surface of the black hole.

Straight lines as geodesics. On a curved surface we can determine if a line is straight or curved at a certain point by looking at the line. We position our eye somewhere on an imagined line extending from the point in the direction of the normal to the surface, and look down along this imagined line at the surface. If the line on the surface looks straight, it is straight. If the line looks curved, it is curved. A line that everywhere, as seen from the local normal, looks straight, is known as a *geodesic*. For a spherical surface like the surface of the Earth, the equator is a geodesic.

For a line that is not straight, we can introduce a curvature direction and a curvature radius by considering how fast and in what direction the line deviates from a corresponding straight line on the surface, analogous to the definition for flat surfaces.

In Einstein's theory of relativity, the motion of particles whose motion is determined by gravity alone corresponds to geodesics in curved spacetime. For the purposes of this article it is sufficient to know that a geodesic particle is a particle that is free to move as gravity alone dictates. Examples are a dropped apple or a flying cannonball (assuming that we neglect air resistance). In general relativity there is no gravitational force, but there are forces such as air resistance. These forces cause objects to deviate from the motion determined by gravity.

The *equivalence principle* can be formulated as follows: At any point in space and time we can introduce freely falling coordinates relative to which special relativity holds. As an example we consider an elevator whose support cables have just snapped at the topmost level of a high building. An observer dropping a coin inside the elevator will note that the coin will float in front of him. If he tosses the coin, he will note that the coin moves away from him on a straight line with constant speed just as it would if the elevator was in outer space where there is no gravity and special relativity holds. We can alternatively say that being in an elevator at rest on Earth is equivalent to being in an accelerated elevator in outer space (see Fig. 20).

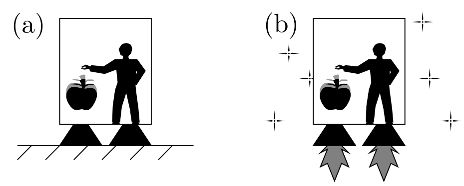

Figure 20: Dropping an apple inside an elevator on Earth gives the same motion relative to the elevator as dropping it inside a (properly) accelerated elevator in outer space. In both cases we can introduce an inertial (fictitious) gravitational force – but there is (in either case) no real gravitational force (in Einstein's theory).

It is a standard technique of Einstein's general theory of relativity to first understand how a scenario will work relative to a freely falling frame where everything is simple, and then express the result with respect to the coordinates that really interest us. These freely falling frames are however not falling relative to a flat spatial geometry. For the particular case of a symmetry plane of a static black hole (see Fig. 18), we can imagine the freely falling frames (a coordinate grid in this case) to be falling relative to the curved geometry depicted in Fig. 19. How fast the freely falling frames accelerate depends on the position (the radius). At spatial infinity the acceleration is zero and at the horizon it is infinite. The idea is illustrated in Fig. 21.

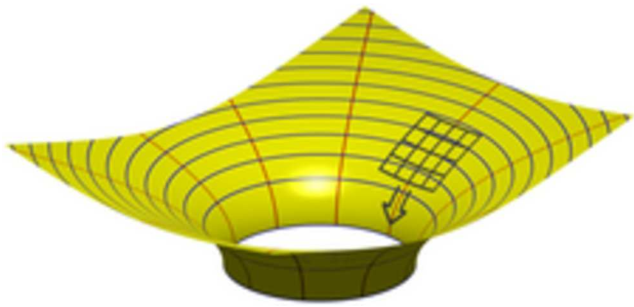

Figure 21: A coordinate system accelerating (falling) relative to the curved spatial geometry of a black hole.

The depicted freely falling frame coordinate lines are geodesics[24] on the curved surface. With respect to the falling coordinate grid a free particle, that is, a particle whose motion is determined by nothing but gravity, will move in a straight line. This law of motion applies to all free particles, including free photons. Because the freely falling system is accelerating relative to the spatial geometry, the paths of free particles will curve relative to the spatial geometry. The fact that the spatial geometry is curved does not complicate the analysis as far as this paper is concerned. The point is that locally we can always consider the geometry to be flat. Living on a small patch of the curved surface is like living in an accelerated reference system in special relativity. It is only when we consider circles around the black hole that we need to think about the spatial geometry to determine the correct curvature of the circular path. For instance, due to the curved spatial geometry, the innermost circle (at the surface of the black hole) is not curved at all.

References

[1] For the purposes of this article we may consider the event horizon to be an (invisible) sphere. If one ventures inside of this sphere, one cannot come back out again.

[2] There is a radius where free photons, that is, photons whose motion is determined only by gravity, can move on circular orbits around a black hole. The circumference of this circle is 1.5 times the circumference of the surface of the black hole (the event horizon).

[3] Marek A. Abramowicz, "Relativity of inwards and outwards: An example," Month. Not. Roy. astr. Soc. **256**, 710-718 (1992).

[4] Marek A. Abramowicz and Jean-Pierre Lasota, "A note of a paradoxical property of the Schwarzschild solution," Acta Phys. Pol. **B5**, 327-329 (1974).

[5] Marek A. Abramowicz, Brandon Carter, and Jean-Pierre Lasota, "Optical reference geometry for stationary and static dynamics," Gen. Relativ. Gravit. **20**, 1173–1183 (1988).

[6] Marek A. Abramowicz and A. R. Prasanna, "Centrifugal force reversal near a Schwarzschild black-hole," Mon. Not. R. Astr. Soc. **245**, 720–728 (1990).

[7] Marek A. Abramowicz, "Centrifugal force: A few surprises," Mon. Not. R. Astr. Soc. **245**, 733–746 (1990).

[8] Bruce Allen, "Reversing centrifugal forces," Nature **347**, 615–616 (1990).

[9] George E. Matsas, "Relativistic Archimedes law for fast moving bodies and the general-relativistic resolution of the 'submarine paradox'," Phys. Rev. D **68**, 027701-1–4 (2003).

[10] Rickard Jonsson, "Inertial forces and the foundations of optical geometry," Class. Quantum Grav. **23**, 1–36 (2006).

[11] The proper acceleration of an object is the acceleration measured relative to an inertial system (a freely falling system) momentarily comoving with the object.

[12] Strictly speaking we are using geometrized units in which $c = 1$. In these units time has the same dimensions as distance. If we want to express distances and times in terms of standard units, we should replace any instance of v by v/c, where c is the velocity of light in standard units.

[13] The embedded geometry ($t = 0$ and $\theta = \pi/2$ in Schwarzschild coordinates) corresponds to a section of a parabola (see Ref. 25), $z = 2\sqrt{R_{\mathrm{G}}}\sqrt{r - R_{\mathrm{G}}}$, revolved around the vertical (z) axis (R_{G} is the radius at the event horizon).

[14] In the derivation we divided by v, thus assuming $v \neq 0$. For $v = 0$ any $\hat{\mathbf{n}}/R$ will do.

[15] Strictly speaking the argument only holds exactly as long as the velocity is purely horizontal.

[16] Consider a $1 + 1$ dimensional scenario. Let u be the velocity of the reference frame relative to an inertial system in which the reference frame is momentarily at rest, and let $\eta^\mu = (\gamma(u), \gamma(u)u)$ be the corresponding four-velocity. Let v_s be the velocity of the test particle relative to the inertial system in question, and let $u^\mu = (\gamma(v_s), \gamma(v_s)v_s)$ be the corresponding four-velocity. Let v be the velocity of the test particle relative to the reference frame. We

have $\gamma(v) = -\eta^\mu u_\mu$ (using the $(-,+,+,+)$ convention). If we differentiate both sides of this expression by d/dt and use the fact that $u = 0$ and $v = v_s$ momentarily, we find $v\frac{dv}{dt}\gamma^4 = -\gamma^2 v\frac{du}{dt} + v\frac{dv_s}{dt}\gamma^4$. This result corresponds to Eq. (17) (multiply Eq. (17) by γ^4, divide by δt, and take the limit where δt is infinitesimal). Also we know that the proper acceleration α is given by $\alpha^2 = -\frac{du^\mu}{d\tau}\frac{du_\mu}{d\tau}$. If we differentiate $u^\mu = (\gamma(v_s), \gamma(v_s)v_s)$ with respect to τ, we find $\frac{du^\mu}{d\tau} = \gamma^3\frac{dv_s}{d\tau}(v,1)$. It follows that $\alpha = \gamma^3\frac{dv_s}{dt}$, which is Eq. (18).

[17] Wolfgang Rindler, *Introduction to Special Relativity* (Clarendon Press, Oxford, 1982), 2nd ed.

[18] It is true in the Newtonian limit that accelerations of the reference frame perpendicular to the direction of motion do not affect the local speed derivative. We can also reason this strictly relativistically knowing a little about time dilation and simultaneity. We could also understand it using four-tensors knowing that $\gamma = \eta^\mu u_\mu$, where η^μ is the reference frame four-velocity and u^μ is the particle four-velocity. Consider a particle moving in the x-direction and consider the reference frame to reach a velocity δv_y in the y-direction after a time δt. To first order in δt we have $\eta^\mu : (1, 0, \delta v_y, 0)$ and u^μ: $(\gamma_0, v_x, 0, 0)$. Here γ_0 is the value of γ at $t = 0$. We see that to first order $\gamma = \eta^\mu u_\mu = \gamma_0$, and thus a perpendicular acceleration of the reference grid does not affect the local speed derivative. Similarly an infinitesimal perpendicular velocity of the *particle* will have no first order effect on the speed. Any one-dimensional reasoning of how the speed derivative is related to force parallel to the direction of motion thus holds also when there are perpendicular effects.

[19] The difference between the proper rotation and the rotation as observed from outside of the merry-go-round, for the non-central points, is not only due to time dilation, but also to relativistic precession (rotation) effects. We can also use the formalism of this paper for these points assuming that we correctly express the proper local reference frame rotation $\boldsymbol{\omega}$, that is, the rotation as experienced by an observer at rest relative to the reference frame at the points in question.

[20] Marek A. Abramowicz and Jean-Pierre Lasota, "A brief story of a straight circle," Class. Quantum Grav. **14**, A23–A30 (1997).

[21] It is easy to make a formal proof of how the factors should enter. For contravariant vectors one can reason it out instead. Consider two coordinate points separated by an infinitesimal vector dx^k. Assume that we stretch space by a factor $e^{-\Phi}$. A coordinate vector that has the same norm (length) relative to the stretched space that dx^k had relative to the standard space would have to be componentwise smaller by a factor $e^{-\Phi}$. Hence given a normalized vector relative to the standard space, we obtain a corresponding normalized vector relative to the rescaled space by dividing by the stretching factor $e^{-\Phi}$.

[22] Here we consider the perpendicular part of the spatial part of Eq. (44) of Ref. 10 and set $\tilde{\theta}^{\alpha\beta} = 0$ as is appropriate for this case.

[23] Set $\tilde{\theta}_{\alpha\beta} = 0$, $\tilde{\eta}^\alpha\tilde{\nabla}_\alpha\Phi = 0$ and identify $\tilde{\tau}_0 = t_\mathrm{c}$ in Eq. (44) of Ref. 10.

[24] Actually they are not all exact geodesics. They cannot be in general (while at the same time being orthogonal) when the surface is curved. But at the center of coordinates they are orthogonal, and the curvature of the coordinate lines vanishes, which is sufficient for the type of arguments made in this article.

[25] Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (W. H. Freeman, New York, 1973), p. 615, Eq. (23.34b).