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On the Derivation of Shell Theories by Direct Approach

A constrained theory of shells by a direct approach, based on a general theory of a Cosserat surface, is derived and its relation to other recent developments obtained by direct procedures is indicated.

Introduction

IN recent years there has been much interest in the derivation of theories of shells and plates by a direct procedure, rather than from the three-dimensional equations of the classical (nonpolar) continuum mechanics. However, with reference to the former approach, the basic concepts and definitions differ from one paper to another and this has created a somewhat confusing situation. A number of recent works on the subject claim to present new shell theories when, in fact, they are special cases of others which have appeared previously or are closely related to previous papers. There is some value in an *ab initio* development of a special theory (in contrast to one resulting from specialization of a more general theory) but it is also important to show what relation, if any, one theory has to another.

It is not the purpose of this paper to decide either on the desirability of different forms or even the most appropriate form of derivation of shell theory by a direct approach. We restrict our attention mainly to an examination of the relation between a theory in which the shell is defined as a surface with a finite rotation vector associated with every point of the surface, and the theory of a two-dimensional directed continuum called a *Cosserat surface,* i.e., a surface with a single deformable director assigned to every point of the surface. We show that the field equations of the former, including the energy equation or the expression for rate of work in the purely mechanical theory, can be obtained as a special (constrained) case of the general theory of a Cosserat surface. The relationship between the general constitutive equations for elastic shells in the two developments is, however, clear only in special circumstances.

While each of the foregoing developments has its own interest, it is essential that a proper interpretation should be given to the results of each if they are to reflect adequately the three-dimensional effects in thin shells. For interpretations of the various results in the theory of a Cosserat surface and other details and definitions we refer the reader to [3],[1] which contains extensive references on the subject. Authors employing a theory in which the rotation vector is a kinematic ingredient appear to interpret the vector $\hat{\mathbf{M}}$ [see equation (10)] as a couple vector with three components, rather than a couple having only tangential components at each point of the surface. The third component would appear to be of the type ordinarily associated with a "couple-stress" or "multipolar stress," i.e., one arising in a theory which is deduced from the three-dimensional equations of polar media. To this extent, it is not a suitable property of a conventional shell theory and is, therefore, unsatisfactory. In this connection, it should be noted that a *restricted* theory derived by a direct approach and involving only a tangential couple vector is discussed in [3]. We also remark that the vectors $\mathbf{N}^\alpha$, $\mathbf{m}$, $\mathbf{M}^\alpha$ [see equations (3)–(5)] in the theory of a Cosserat surface have simple interpretations as force and couple resultants, defined in terms of three-dimensional stresses [3, Section 22], and that only two components of the vector $\mathbf{M}$ in (2) represent couples.

Summary of Theory of a Cosserat Surface

A Cosserat surface is a body $\mathcal{C}$ comprising a surface (embedded in Euclidean 3-space) and a single deformable director attached to every point of the surface. Let the particles of the material surface of $\mathcal{C}$ be identified with a system of convected coordinates θ^α ($\alpha = 1, 2$) and let the surface occupied by the material surface of $\mathcal{C}$ in the present configuration at time t be referred to by $\mathcal{S}$.[2] Let $\mathbf{r}$ and $\mathbf{d}$, each a function of θ^α and t, denote the position vector of a typical point of $\mathcal{S}$ and the director at the same point, respectively. We define the base vectors along the θ^α-curves on $\mathcal{S}$ by

$$\mathbf{a}_\alpha = \mathbf{a}_\alpha(\theta^\alpha, t) = \frac{\partial \mathbf{r}}{\partial \theta^\alpha}$$

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[1] Numbers in brackets designate References at end of paper.

[2] The use of the symbol cap script "$\mathcal{S}$" used here differs from the corresponding usage for the surface in the present configuration in reference [3].

and denote by $\mathbf{a}_3(\theta^\alpha, t)$ the unit normal to $\mathcal{S}$. The motion of a Cosserat surface is defined by

$$\mathbf{r} = \mathbf{r}(\theta^\alpha, t), \qquad \mathbf{d} = \mathbf{d}(\theta^\alpha, t)$$

and we assume that the director $\mathbf{d}$ is nowhere tangent to $\mathcal{S}$. The velocity and the director velocity vectors are given by

$$\mathbf{v} = \dot{\mathbf{r}}, \qquad \mathbf{w} = \dot{\mathbf{d}},$$

where a superposed dot denotes differentiation with respect to t holding θ^α fixed. Throughout this paper, we use a standard vector and tensor notation. In particular, all Greek indices take the values 1, 2, and the usual summation over a subscript and a superscript is employed.

A general theory of a Cosserat surface given by Green, Naghdi, and Wainwright [1] is developed within the framework of thermodynamics; the derivation in [1] is carried out mainly from an appropriate (two-dimensional) energy equation, together with invariance requirements under superposed rigid-body motions. A related development utilizing three directors at each point of the surface, carried out in the context of a purely mechanical theory and with the use of a (two-dimensional) virtual work principle, is given by Cohen and DeSilva [2]. Here we adopt the mode of derivation of the basic theory employed by Naghdi [3, Section 8]. Let $\mathcal{P}$, bounded by a closed curve $\partial\mathcal{P}$, be a part of $\mathcal{S}$ occupied by an arbitrary material region of the surface of $\mathcal{C}$ in the present configuration at time t and let

$$\boldsymbol{\nu} = \nu_\alpha \mathbf{a}^\alpha$$

be the outward unit normal to the closed curve $\partial\mathcal{P}$. It is convenient at this point to define certain additional quantities as follows: The mass density $\rho = \rho(\theta^\gamma, t)$ of the surface $\mathcal{S}$ in the present configuration; the contact force $\mathbf{N} = \mathbf{N}(\theta^\gamma, t; \boldsymbol{\nu})$ and the contact director couple $\mathbf{M} = \mathbf{M}(\theta^\gamma, t; \boldsymbol{\nu})$, each per unit length of a curve in the present configuration; the assigned force $\mathbf{f} = \mathbf{f}(\theta^\gamma, t)$ and the assigned director couple $\mathbf{l} = \mathbf{l}(\theta^\gamma, t)$, each per unit mass of the surface $\mathcal{S}$; the intrinsic (surface) director couple $\mathbf{m}$ per unit area of $\mathcal{S}$ which makes no contribution to the supply of momentum; the inertia coefficient $\alpha = \alpha(\theta^\gamma)$ which is independent of time and is associated with the director velocity; the specific internal energy $\epsilon = \epsilon(\theta^\gamma, t)$; the heat flux $h = h(\theta^\gamma, t; \boldsymbol{\nu})$ per unit time and per unit length of a curve in the present configuration; the specific heat supply $r = r(\theta^\gamma, t)$ per unit time; and the element of area $d\sigma$, and the line element ds of the surface $\mathcal{S}$.

With the foregoing definitions of the various field quantities and with reference to the present configuration, the conservation laws for a Cosserat surface are:[3]

$$\frac{d}{dt}\int_{\mathcal{P}} \rho d\sigma = 0,$$

$$\frac{d}{dt}\int_{\mathcal{P}} \rho \mathbf{v} d\sigma = \int_{\mathcal{P}} \rho \mathbf{f} d\sigma + \int_{\partial\mathcal{P}} \mathbf{N} ds,$$

$$\frac{d}{dt}\int_{\mathcal{P}} \rho\alpha \mathbf{w} d\sigma = \int_{\mathcal{P}} (\rho \mathbf{l} - \mathbf{m}) d\sigma + \int_{\partial\mathcal{P}} \mathbf{M} ds,$$

$$\frac{d}{dt}\int_{\mathcal{P}} \rho(\mathbf{r}\times\mathbf{v} + \mathbf{d}\times\alpha\mathbf{w}) d\sigma = \int_{\mathcal{P}} \rho(\mathbf{r}\times\mathbf{f} + \mathbf{d}\times\mathbf{l}) d\sigma + \int_{\partial\mathcal{P}} (\mathbf{r}\times\mathbf{N} + \mathbf{d}\times\mathbf{M}) ds, \tag{1}$$

$$\frac{d}{dt}\int_{\mathcal{P}} \rho\left[\epsilon + \frac{1}{2}(\mathbf{v}\cdot\mathbf{v} + \alpha\mathbf{w}\cdot\mathbf{w})\right] d\sigma = \int_{\mathcal{P}} \rho(r + \mathbf{f}\cdot\mathbf{v} + \mathbf{l}\cdot\mathbf{w}) d\sigma + \int_{\partial\mathcal{P}} (\mathbf{N}\cdot\mathbf{v} + \mathbf{M}\cdot\mathbf{w} - h) ds.$$

[3] The conservation laws in (1) correspond to those given by equations (8.17) in [3].

The first of (1) is a statement of the conservation of mass, the second the conservation of linear momentum, the third that of the director momentum, the fourth the conservation of moment of momentum, and the fifth represents the conservation of energy. The left-hand sides of the last four in (1) represent, respectively, the rate of increase of the linear momentum, the director momentum, the moment of momentum (including contributions from both ordinary and director momentum) and the total energy, i.e., the sum of internal energy and the kinetic energy due to both velocity $\mathbf{v}$ and director velocity $\mathbf{w}$.

Under suitable continuity assumptions, the curve force vector $\mathbf{N}$, the director couple vector $\mathbf{M}$, and the heat flux h can be expressed as[4]

$$\mathbf{N} = \mathbf{N}^\alpha \nu_\alpha, \quad \mathbf{M} = \mathbf{M}^\alpha \nu_\alpha, \quad h = q^\alpha \nu_\alpha. \tag{2}$$

The first four equations in (1) are then equivalent to

$$\rho a^{1/2} = k, \tag{3}$$

$$\mathbf{N}^\alpha{}_{|\alpha} + \rho\mathbf{f} = \rho\dot{\mathbf{v}}, \tag{4}$$

$$\mathbf{M}^\alpha{}_{|\alpha} + \rho\mathbf{l} = \mathbf{m} + \rho\alpha\dot{\mathbf{w}}, \tag{5}$$

$$\mathbf{a}_\alpha \times \mathbf{N}^\alpha + (\mathbf{d}\times\mathbf{M}^\alpha)_{|\alpha} + \rho\mathbf{d}\times(\mathbf{l} - \alpha\dot{\mathbf{w}}) = 0, \tag{6}$$

where k is a function of θ^α only, $[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3] = a^{1/2}$, a vertical line denotes covariant differentiation with respect to the first fundamental form of $\mathcal{S}$ and the inertia coefficient α is a function of θ^α and independent of time. Also, with the help of $(2)_3$, (3), and (5), the energy equation $(1)_5$ can be reduced to

$$\rho r - q^\alpha{}_{|\alpha} - \rho\dot{\epsilon} + \mathbf{N}^\alpha\cdot\mathbf{v}_{,\alpha} + \mathbf{m}\cdot\mathbf{w} + \mathbf{M}^\alpha\cdot\mathbf{w}_{,\alpha} = 0, \tag{7}$$

where a comma denotes partial differentiation with respect to the surface coordinates θ^α.

Constrained Theory of a Cosserat Surface

We consider now a special case of the foregoing results in which the director is constrained to be of constant length at each point of $\mathcal{S}$, i.e.,

$$\mathbf{d}\cdot\mathbf{d} = d^2, \tag{8}$$

where d is a function of θ^α, independent of t. It follows from (8) that

$$\mathbf{w}\cdot\mathbf{d} = 0.$$

Hence there exists an axial vector $\boldsymbol{\omega} = \boldsymbol{\omega}(\theta^\alpha, t)$, called angular velocity, such that

$$\mathbf{w} = \boldsymbol{\omega}\times\mathbf{d}. \tag{9}$$

Next, we introduce the notations

$$\hat{\mathbf{M}} = \mathbf{d}\times\mathbf{M}, \quad \hat{\mathbf{M}}^\alpha = \mathbf{d}\times\mathbf{M}^\alpha, \quad \hat{\mathbf{M}} = \hat{\mathbf{M}}^\alpha\nu_\alpha, \quad \hat{\mathbf{l}} = \mathbf{d}\times\mathbf{l} \tag{10}$$

and note that

$$\mathbf{M}\cdot\mathbf{w} = \hat{\mathbf{M}}\cdot\boldsymbol{\omega}, \qquad \mathbf{l}\cdot\mathbf{w} = \hat{\mathbf{l}}\cdot\boldsymbol{\omega}. \tag{11}$$

Also, let

$$\boldsymbol{\sigma} = \mathbf{d}\times\alpha\mathbf{w}, \qquad \dot{\boldsymbol{\sigma}} = \mathbf{d}\times\alpha\dot{\mathbf{w}}, \tag{12}$$

so that

$$\boldsymbol{\sigma}\cdot\boldsymbol{\omega} = \alpha\mathbf{w}\cdot\mathbf{w}, \qquad \dot{\boldsymbol{\sigma}}\cdot\boldsymbol{\omega} = \boldsymbol{\sigma}\cdot\dot{\boldsymbol{\omega}} = \alpha\mathbf{w}\cdot\dot{\mathbf{w}}. \tag{13}$$

Introducing (9) into $(12)_1$ and using the expansion formula for the vector triple product, $\boldsymbol{\sigma}$ can be expressed in terms of $\boldsymbol{\omega}$ and an inertia tensor $\mathbf{J}$ in the form

[4] The director couple $\mathbf{M}$ is an ordinary vector and not an axial vector, whereas the vector $\hat{\mathbf{M}}$ in (10) is an axial vector.

$$\dot{\boldsymbol{\sigma}} = \mathbf{J}\boldsymbol{\omega}, \tag{14}$$

where

$$\mathbf{J} = \alpha(d^2\mathbf{1} - \mathbf{d}\mathbf{d}) \tag{15}$$

and $\mathbf{1}$ is the unit tensor. Moreover, suppose that $\boldsymbol{\Omega}$ is the skew-symmetric tensor associated with the axial vector $\boldsymbol{\omega}$ such that for any vector $\mathbf{u}$:

$$\boldsymbol{\Omega}\mathbf{u} = \boldsymbol{\omega} \times \mathbf{u}. \tag{16}$$

Then, using (9) and (15), it can be verified that $\mathbf{J}$ satisfies the differential equation

$$\dot{\mathbf{J}} = \boldsymbol{\Omega}\mathbf{J} - \mathbf{J}\boldsymbol{\Omega}. \tag{17}$$

Some writers use an inertia term in the form (14) with $\mathbf{J}$ satisfying (17) instead of that given by (15). Although the inertia tensor in the form (15) is to some extent less general, the inertia coefficient α associated with the director velocity $\mathbf{w}$ in the unconstrained theory is well established for thin shells [3].

Returning to the energy equation, we now substitute (11) and (13) into the last equation in (1) to obtain

$$\frac{d}{dt}\int_{\mathcal{P}} \rho\left[\epsilon + \frac{1}{2}(\mathbf{v}\cdot\mathbf{v} + \boldsymbol{\sigma}\cdot\boldsymbol{\omega})\right]d\sigma$$

$$= \int_{\mathcal{P}} \rho(r + \mathbf{f}\cdot\mathbf{v} + \hat{\mathbf{l}}\cdot\boldsymbol{\omega})d\sigma + \int_{\partial\mathcal{P}}(\mathbf{N}\cdot\mathbf{v} + \hat{\mathbf{M}}\cdot\boldsymbol{\omega} - h)ds. \tag{18}$$

Conservation equations for mass and linear and angular momentum can be deduced from (18), with the help of invariance conditions under superposed rigid-body motions. Alternatively, the local forms of these equations can be obtained directly as a special case of (3)–(6). Thus, under the constraint condition (8), equations (3) and (4) remain unaltered while (6) becomes

$$\mathbf{a}_\alpha \times \mathbf{N}^\alpha + \hat{\mathbf{M}}^\alpha{}_{|\alpha} + \rho\hat{\mathbf{l}} = \rho\dot{\boldsymbol{\sigma}}. \tag{19}$$

Also, under the constraint condition (8), the energy equation (7) can be reduced to the form

$$\rho r - q^\alpha{}_{|\alpha} - \rho\dot{\epsilon} + \hat{\mathbf{N}}^\alpha\cdot(\mathbf{v}_{,\alpha} - \boldsymbol{\omega}\times\mathbf{a}_\alpha) + \hat{\mathbf{M}}^\alpha\cdot\boldsymbol{\omega}_{,\alpha} = 0. \tag{20}$$

Within the scope of the previous constrained thermomechanical theory, the field equations (3), (4), (19), and (20) are sufficient if we are not interested in the detailed behavior of the director. From this point of view, the director momentum equation (5) can be discarded. Of course, equation (5) does not arise if we proceed directly from the energy equation (18) and invariance requirements under superposed rigid-body motions. Equations such as (3), (4), and (19), or their corresponding integral forms, are the starting point of some authors for a special shell theory by a direct approach.[5]

Before proceeding further, we observe that the foregoing constrained theory can also be developed in quite a different manner and without the use of the angular velocity vector as a kinematic ingredient. For example, as in the three-dimensional theory of liquid crystals which is based on a constrained directed media, the original structure of a Cosserat surface is retained and some indeterminancies are introduced into the constitutive equations for $\mathbf{m}$ and $\mathbf{M}^\alpha$ (see, e.g., [5]). This procedure has the advantage of providing a simple interpretation of the theory in relation to available results for shells derived from the nonpolar three-dimensional equations and does not introduce ingredients associated with "couple stresses."

The foregoing clearly shows the relationship between the (dynamical and thermodynamical) field equations of a theory such as that in [4], where the basic kinematic ingredients are taken as a velocity vector and an angular velocity vector at each point of the surface, and the constrained theory of a Cosserat surface obtained under the condition (8). For the purpose of a further comparison, we consider below constitutive equations for elastic shells. In this connection, we observe that from the point of view of thermodynamics, an entropy production inequality must be added to the list of conservation equations (1) and this inequality is the same for both the unconstrained or the constrained Cosserat surface. Also, within the framework of thermomechanical theory, it is more convenient to employ the specific Helmholtz free energy ψ in place of the specific internal energy ϵ.

Elastic Shells

Simmonds and Danielson [4] do not consider constitutive equations for their general theory but limit their attention to a special case, which is in line with the conventional shell theory. We deal with this special case later but examine first the nature of the constitutive equations for an elastic shell associated with the field equations (3), (4), (19), and (20). Simmonds and Danielson [4] employ a rigid frame of reference $\{\bar{\mathbf{A}}_\alpha, \mathbf{N}\}$ rotating with angular velocity $\boldsymbol{\omega}$, and which coincides with $\{\mathbf{A}_\alpha, \mathbf{A}_3\}$ in the reference configuration so that[6]

$$A_{\alpha\beta} = \mathbf{A}_\alpha\cdot\mathbf{A}_\beta = \bar{\mathbf{A}}_\alpha\cdot\bar{\mathbf{A}}_\beta = \bar{A}_{\alpha\beta}, \tag{21}$$

We set

$$\hat{\mathbf{N}}^\alpha = \hat{N}^{\alpha\beta}\bar{\mathbf{A}}_\beta + \hat{Q}^\alpha\mathbf{N},$$
$$\hat{\mathbf{M}}^\alpha = \hat{M}^{\alpha\beta}\bar{\mathbf{A}}_\beta \times \mathbf{N} + \hat{M}^\alpha\mathbf{N} \tag{22}$$

and quote the following three formulas (in a slightly different notation) from [4]:[7]

$$\mathbf{r}_{,\alpha} = (A_{\alpha\beta} + \Gamma_{\alpha\beta})\bar{\mathbf{A}}^\beta + \gamma_\alpha\mathbf{N},$$
$$\mathbf{v}_{,\alpha} - \boldsymbol{\omega}\times\mathbf{a}_\alpha = \dot{\Gamma}_{\alpha\beta}\bar{\mathbf{A}}^\beta + \dot{\gamma}_\alpha\mathbf{N}, \tag{23}$$
$$\boldsymbol{\omega}_{,\alpha} = \dot{\bar{B}}_{\lambda\alpha}\bar{\mathbf{A}}^\lambda \times \mathbf{N} + \dot{\Gamma}_\alpha\mathbf{N}.$$

Then, the energy equation (20) reduces to

$$\rho r - \rho q^\alpha{}_{|\alpha} - \rho\dot{\epsilon} + \hat{N}^{\alpha\beta}\dot{\Gamma}_{\alpha\beta} + \hat{Q}^\alpha\dot{\gamma}_\alpha + \hat{M}^{\alpha\beta}\dot{\bar{B}}_{\beta\alpha} + \hat{M}^\alpha\dot{\Gamma}_\alpha = 0. \tag{24}$$

We now restrict attention to the isothermal theory and consider the constitutive equations for an elastic shell, which is homogeneous in its reference state. Guided by (24), we assume that

$$\psi = \hat{\psi}(\Gamma_{\alpha\beta}, \gamma_\alpha, \bar{B}_{\alpha\beta}, \Gamma_\alpha; A_{\alpha\beta}, B_{\alpha\beta}), \tag{25}$$

where $B_{\alpha\beta}$ in the argument of $\hat{\psi}$ are the components of the second fundamental form of the surface in the reference configuration. Then, with similar constitutive assumptions for $\hat{N}^{\alpha\beta}$, $\hat{Q}^\alpha$ $\hat{M}^{\alpha\beta}$, and $\hat{M}^\alpha$, from (24) and the appropriate entropy inequality, we obtain[8]

$$\hat{N}^{\alpha\beta} = \rho\frac{\partial\hat{\psi}}{\partial\Gamma_{\alpha\beta}}, \quad \hat{Q}^\alpha = \rho\frac{\partial\hat{\psi}}{\partial\gamma_\alpha},$$
$$\hat{M}^{\alpha\beta} = \rho\frac{\partial\hat{\psi}}{\partial\bar{B}_{\beta\alpha}}, \quad \hat{M}^\alpha = \rho\frac{\partial\hat{\psi}}{\partial\Gamma_\alpha}. \tag{26}$$

[5] For example see Simmonds and Danielson [4]. These authors consider only a mechanical theory and, apart from the thermodynamic terms, (18) and (20) correspond to equations (23) and (24) in [4].

[6] Most of the notations used here differ from those in [4]. The triad $\{\mathbf{A}_\alpha, \mathbf{A}_3\}$ consist of base vectors $\mathbf{A}_\alpha$ in the reference surface and the unit normal $\mathbf{A}_3$ to the reference surface.

[7] Equations (23) correspond to equations (11), (26), and (27) of [4].

[8] For the present purpose, this is the local Clausius-Duhem inequality. Instead of using the entropy production inequality, the results (26) may also be obtained in the context of the purely mechanical theory and from a consideration of rate of work in which ψ is regarded as the strain-energy density. The latter development would be similar to that discussed in [3, Section 14].

The specific Helmholtz free energy and the various resultants in (25)-(26) are measured per unit mass and per unit length in the present configuration but corresponding results in terms of quantities measured with respect to the reference configuration can easily be obtained from (26).

We now turn to the theory of a Cosserat surface, again restrict attention to the isothermal theory, and for simplicity limit the discussion of constitutive equations to that for an elastic shell which is homogeneous in its reference configuration. Thus we begin by assuming that [3, Section 13]

$$\psi = \bar{\psi}(\mathbf{r}_{,\alpha}, \mathbf{d}, \mathbf{d}_{,\alpha}; \mathbf{A}_\alpha, \mathbf{D}, \mathbf{D}_{,\alpha}), \tag{27}$$

where $\mathbf{A}_\alpha$, $\mathbf{D}$, $\mathbf{D}_{,\alpha}$ are the values of the kinematic variables $\mathbf{a}_\alpha$ $(=\mathbf{r}_{,\alpha})$, $\mathbf{d}$, $\mathbf{d}_{,\alpha}$, respectively, in the reference configuration. Then, either in the context of the isothermal theory or within the scope of the purely mechanical theory, using (27) and similar assumptions for $\mathbf{N}^\alpha$, $\mathbf{m}$, $\mathbf{M}^\alpha$, it can be shown that

$$\mathbf{N}^\alpha = \rho \frac{\partial \bar{\psi}}{\partial \mathbf{r}_{,\alpha}}, \qquad \mathbf{m} = \rho \frac{\partial \bar{\psi}}{\partial \mathbf{d}}, \qquad \mathbf{M}^\alpha = \rho \frac{\partial \bar{\psi}}{\partial \mathbf{d}_{,\alpha}}, \tag{28}$$

where $\bar{\psi}$ satisfies the invariance condition

$$\mathbf{r}_{,\alpha} \times \frac{\partial \bar{\psi}}{\partial \mathbf{r}_{,\alpha}} + \mathbf{d} \times \frac{\partial \bar{\psi}}{\partial \mathbf{d}} + \mathbf{d}_{,\alpha} \times \frac{\partial \bar{\psi}}{\partial \mathbf{d}_{,\alpha}} = 0. \tag{29}$$

Further, by invariance arguments (under superposed rigid-body motions), equation (27) can be reduced to the form used in [1] or to the various forms obtained in [3]. The constitutive equations (28) may be expressed in terms of tensor components with respect to any desired set of base vectors. For example, we may consider the components of (28) with respect to base vectors $\{\mathbf{a}_\alpha, \mathbf{a}_3\}$ or with respect to their duals $\{\mathbf{A}_\alpha, \mathbf{A}_3\}$ in the reference configuration.

The results (28) hold in the unconstrained theory of a Cosserat surface. However, with the introduction of the constraint condition (8) and the relations (10), it does not appear to be possible to deduce the results (25)-(26) from (27)-(28) except in certain restricted circumstances. For example, since $\hat{\mathbf{M}} \cdot \mathbf{d} = 0$ by $(10)_1$, we could deduce (25) from (27), provided $\bar{B}_{\alpha\beta}$ and Γ_α occur in the argument of $\hat{\psi}$ in (25) in such a way that this condition is satisfied. It should be observed that the results (26) include the couple components $\hat{M}^\alpha$, which is of the type ordinarily associated with a "couple-stress" theory and to this extent is an unsatisfactory property for a conventional shell theory.

Simmonds and Danielson early in their work concentrate their attention to the case in which (see the statements between equations (39)-(40) in [4])

$$\hat{M}^\alpha = 0, \tag{30}$$

and in this situation we can pass from (27) to the results (26).[9] With reference to the constitutive equation (27) for a Cosserat surface, we now choose the special values

$$\mathbf{d} = \mathbf{N}, \qquad \mathbf{D} = \mathbf{A}_3 \tag{31}$$

and recall from equation (7) of [4] (in a slightly different notation) the expression

$$\mathbf{N}_{,\alpha} = -\bar{B}_{\lambda\alpha}\bar{\mathbf{A}}^\lambda.$$

We observe that the constraint condition (8) is satisfied with d a constant. Moreover, since $\mathbf{d} \cdot \hat{\mathbf{M}}^\alpha = 0$, we recover (30) using $(10)_1$ and (22). Then, with the help of invariance conditions under superposed rigid-body motions, ψ in (27) reduces to the form

$$\psi = \tilde{\psi}(\Gamma_{\alpha\beta}, \gamma_\alpha, \bar{B}_{\alpha\beta}; A_{\alpha\beta}, B_{\alpha\beta}), \tag{32}$$

and the results $(26)_{1,2,3}$ follow.

In addition to (30), Simmonds and Danielson [4] also add the restrictions

$$\Gamma_{\alpha\beta} = \Gamma_{\beta\alpha}, \qquad \gamma_\alpha = 0, \tag{33}$$

which imply that $\mathbf{N}$ coincides with the unit normal $\mathbf{a}_3$ and that $\boldsymbol{\omega}$ is no longer independent of $\mathbf{v}$ in view of $(23)_2$. In this way, they suppress the effect of "transverse shear" deformation, but also avoid retaining the effect of the third component of the couple $\hat{\mathbf{M}}^\alpha$ around the normal to the surface. This special form of the theory also follows from the restricted theory discussed by Naghdi [3, Sections 10 and 15] employing methods similar to those used above in deriving the special constrained theory. To see this, we recall that the rate of work by tangential couple vector $\dot{\mathbf{M}}$ and the assigned tangential couple vector $\dot{\mathbf{l}}$ which occur in the energy equation of the restricted theory are

$$\dot{\mathbf{M}} \cdot \dot{\mathbf{w}}, \qquad \dot{\mathbf{l}} \cdot \dot{\mathbf{w}}, \tag{34}$$

where $\dot{\mathbf{w}} = \dot{\mathbf{a}}_3$ and all symbols are those defined in [3]. Since $\dot{\mathbf{a}}_3 \cdot \mathbf{a}_3 = 0$, it follows that an angular velocity vector $\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}(\theta^\alpha, t)$ exists such that

$$\dot{\mathbf{w}} = \dot{\boldsymbol{\omega}} \times \mathbf{a}_3. \tag{35}$$

Now, if we identify $\mathbf{a}_3 \times \dot{\mathbf{M}}^\alpha$ with $\hat{\mathbf{M}}^\alpha$, $\mathbf{a}_3 \times \dot{\mathbf{l}}$ with $\hat{\mathbf{l}}$ and the angular velocity $\dot{\boldsymbol{\omega}}$ with $\boldsymbol{\omega}$, a 1-1-correspondence holds between all field equations of the mechanical theory in [4] under the conditions (30), (33) and the corresponding equations of the restricted theory in [3].[10] Moreover, the rate of work terms in (34) become

$$\dot{\mathbf{M}} \cdot \dot{\mathbf{w}} = \dot{\mathbf{M}} \cdot \dot{\boldsymbol{\omega}} \times \mathbf{a}_3 = \mathbf{a}_3 \times \dot{\mathbf{M}} \cdot \dot{\boldsymbol{\omega}} = \hat{\mathbf{M}} \cdot \boldsymbol{\omega},$$

with a similar reduction for $\dot{\mathbf{l}} \cdot \dot{\mathbf{w}}$.

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[9] Simmonds and Danielson [4], in addition to (30), also impose the restrictions $\gamma_\alpha = 0$ and $\Gamma_{\alpha\beta} = \Gamma_{\beta\alpha}$ which are not necessary for our present purpose.

[10] A contribution to the kinetic energy of the form $\beta \mathbf{w}.\mathbf{w}$, where β is a function of θ^α only, is not explicitly indicated in the restricted theory given in [3] but this can be accounted for as in the case of the more general theory. See, in this connection, the remark made following equation (10.5) in [3].