View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by CiteSeerX

Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**
Website: http://AIMsciences.org
pp. **X–XX**

ON A CLASS OF INTEGRO-DIFFERENTIAL PROBLEMS

Michel Chipot and Senoussi Guesmia

University of Zürich
Institute of Mathematics
Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

Abstract. The paper is concerned with the existence of solutions to an integro-differential problem arising in the neutron transport theory. By an anisotropic singular perturbations method we show that solutions of such a problem exist and are close to those of some nonlocal elliptic problem. The existence of the solutions of the nonlocal elliptic problem with bounded data is ensured by the Schauder fixed point theorem. Then an asymptotic method is applied in the general case. The limits of the solutions of the nonlocal elliptic problems are solutions of our integro-differential problem.

1. **Introduction and motivation.** The Boltzmann transport equation, governing the neutron distribution in a nuclear reactor leads, by using the Vladimirov method given in [12], to the even-parity second order transport equation. So, let us consider the within-group transport equation for the neutron angular flux $\psi\left(r,\hat{\Omega}\right)$ with the simplifying restriction that scattering be isotropic

$$\hat{\Omega}\cdot\nabla\psi\left(r,\hat{\Omega}\right)+\sigma\left(r\right)\psi\left(r,\hat{\Omega}\right)-\sigma_s\left(r\right)\phi\left(r\right)=s\left(r,\hat{\Omega}\right) \tag{1}$$

coupled with some boundary conditions. The unit vector $\hat{\Omega}$ represents the traveling direction of a neutron, the gradient operator ∇ acts on the spatial variable r only, $\sigma\left(r\right)$ is the total macroscopic cross section, $\sigma_s\left(r\right)$ is the macroscopic scattering cross section and $s\left(r,\hat{\Omega}\right)$ is the source term. The scalar flux $\phi\left(r\right)$ is given by

$$\phi\left(r\right)=\int_{\hat{\Omega}}\psi\left(r,\hat{\Omega}\right)d\hat{\Omega}.$$

We used the physical notation where $\int_{\hat{\Omega}}$ denotes the integration on the set representing the traveling directions. To derive the even-parity transport equation let us decompose $\psi\left(r,\hat{\Omega}\right)$ into the sum of even and odd angular-parity components

$$\begin{aligned}\psi^{+}\left(r,\hat{\Omega}\right)&=\frac{1}{2}\left(\psi\left(r,\hat{\Omega}\right)+\psi\left(r,-\hat{\Omega}\right)\right)\\ \psi^{-}\left(r,\hat{\Omega}\right)&=\frac{1}{2}\left(\psi\left(r,\hat{\Omega}\right)-\psi\left(r,-\hat{\Omega}\right)\right)\end{aligned}$$

2000 *Mathematics Subject Classification.* Primary: 35B25, 35B40, 45K05 ; Secondary: 35J25, 82D75.

Key words and phrases. Integro-partial differential equations, anisotropic singular perturbations, asymptotic behaviour, neutron transport, Schauder fixed point theorem, elliptic problems.

The authors are supported by the Swiss National Science Foundation.

i.e.

$$\psi\left(r,\hat{\Omega}\right)=\psi^{+}\left(r,\hat{\Omega}\right)+\psi^{-}\left(r,\hat{\Omega}\right).$$

Then the scalar flux is only written in terms of ψ^{+} by

$$\phi\left(r\right)=\int_{\hat{\Omega}}\psi\left(r,\hat{\Omega}\right)d\hat{\Omega}=\int_{\hat{\Omega}}\psi^{+}\left(r,\hat{\Omega}\right)d\hat{\Omega}.$$

On the other hand, rewriting the transport equation (1) for $-\hat{\Omega}$

$$-\hat{\Omega}\cdot\nabla\psi\left(r,-\hat{\Omega}\right)+\sigma\left(r\right)\psi\left(r,-\hat{\Omega}\right)-\sigma_{s}\left(r\right)\phi\left(r\right)=s\left(r,-\hat{\Omega}\right) \tag{2}$$

and summing (1) and (2) term by term, we get

$$\hat{\Omega}\cdot\nabla\psi^{-}\left(r,\hat{\Omega}\right)+\sigma\left(r\right)\psi^{+}\left(r,\hat{\Omega}\right)-\sigma_{s}\left(r\right)\phi\left(r\right)=s^{+}\left(r,\hat{\Omega}\right), \tag{3}$$

then subtracting them leads to

$$\hat{\Omega}\cdot\nabla\psi^{+}\left(r,\hat{\Omega}\right)+\sigma\left(r\right)\psi^{-}\left(r,\hat{\Omega}\right)=s^{-}\left(r,\hat{\Omega}\right), \tag{4}$$

whence

$$\psi^{-}\left(r,\hat{\Omega}\right)=\frac{1}{\sigma\left(r\right)}\left[s^{-}\left(r,\hat{\Omega}\right)-\hat{\Omega}\cdot\nabla\psi^{+}\left(r,\hat{\Omega}\right)\right],$$

where $\sigma\left(r\right)$ is assumed different from 0. Replacing the term $\psi^{-}\left(r,\hat{\Omega}\right)$ in (3) using the above identity yields the second order form of the transport equation

$$\begin{aligned}-\hat{\Omega}\cdot\nabla\left(\frac{1}{\sigma\left(r\right)}\hat{\Omega}\cdot\nabla\psi^{+}\left(r,\hat{\Omega}\right)\right)&+\sigma\left(r\right)\psi^{+}\left(r,\hat{\Omega}\right)\\ &=\sigma_{s}\left(r\right)\int_{\hat{\Omega}}\psi^{+}\left(r,\hat{\Omega}\right)d\hat{\Omega}+s^{+}\left(r,\hat{\Omega}\right)-\hat{\Omega}\cdot\nabla\frac{s^{-}\left(r,\hat{\Omega}\right)}{\sigma\left(r\right)}.\end{aligned}$$

This is a second order partial differential equation in r with a nonlocal term given by a partial integral on the angular domain. For more details we refer the reader to [10, 12].

Motivated by the above model we consider in the following some integro-differential problems.

First let ω_1 (resp. ω_2) be a bounded open subset of $\mathbb{R}^m$ (resp. $\mathbb{R}^n$) where m and n are positive integers. We split the components of a point in $x\in\mathbb{R}^{m+n}$ into the m first components and the n last ones i.e.

$$X_1=(x_1,\ldots,x_m)\quad\text{and}\quad X_2=(x_1',\ldots,x_n').$$

With this notation we set

$$\nabla u=\begin{pmatrix}\nabla_{X_1}u\\ \nabla_{X_2}u\end{pmatrix}=\begin{pmatrix}(\partial_{x_1}u,\ldots,\partial_{x_m}u)^T\\ (\partial_{x_1'}u,\ldots,\partial_{x_n'}u)^T\end{pmatrix}.$$

We set

$$\Omega=\omega_1\times\omega_2.$$

Let us denote by $A=(a_{ij}(x))$ a $n\times n$ matrix such that

$$a_{ij}\in L^\infty(\Omega)\quad\forall\, i,j=1,\ldots,n, \tag{5}$$

and for some $\lambda>0$ we have the ellipticity hypothesis

$$A\xi\cdot\xi\geq\lambda|\xi|^2\quad\forall\,\xi\in\mathbb{R}^n,\quad\text{a.e. } x\in\Omega. \tag{6}$$

Let $a \in C(\mathbb{R})$ be a continuous function satisfying

$$a(r) = O(r) \quad \text{when} \quad |r| \to \infty \tag{7}$$

and l be the nonlocal term defined as

$$l(u) = \int_{\omega_1} h(X_1, X_1', X_2)\, u(X_1', X_2)\, dX_1', \qquad \forall u \in L^2(\Omega), \tag{8}$$

where h is a measurable function satisfying

$$h \in L^\infty(\omega_1 \times \Omega). \tag{9}$$

Let us consider the integro-differential problem to find u such that

$$\begin{cases} -\nabla_{X_2}(A\nabla_{X_2} u) + \chi u = a(l(u)) & \text{in} \quad \Omega, \\ u(X_1, \cdot) = 0 \quad \text{on} \quad \partial\omega_2 \quad a.e.\ X_1 \in \omega_1. \end{cases} \tag{10}$$

where χ is nonnegative constant. We say that a function $u_0 \in L^2(\Omega)$ such that $u_0(X_1, \cdot) \in H_0^1(\omega_2)$ for a.e. $X_1 \in \omega_1$, is a weak solution of the problem (10) if the integral identity

$$\int_{\omega_2} A\nabla_{X_2} u_0 \cdot \nabla_{X_2} v\, dx = \int_{\omega_2} a(l(u_0)) v dx, \tag{11}$$

holds for a.e. $X_1 \in \omega_1$ and $\forall v \in H_0^1(\omega_2)$. In order to show the existence of such a solution, we perturb the first equation in (10). This is done by introducing the following anisotropic singular perturbations problem

$$\begin{cases} -\varepsilon^2 \Delta_{X_1} u_\varepsilon - \nabla_{X_2}(A\nabla_{X_2} u_\varepsilon) + \chi u_\varepsilon = a(l(u_\varepsilon)) & \text{in} \quad \Omega, \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{12}$$

where $\varepsilon > 0$ will go to 0, $\Delta_{X_1} = \sum_{i=1}^m \partial_{x_i}^2$ is the usual Laplace operator in X_1. It is clear that the problem (12) is a nonlocal semilinear elliptic problem, subject to homogeneous Dirichlet boundary conditions. The theory of the anisotropic singular perturbations is developed in [1]-[5], [8, 9, 11] where different convergence results are shown for linear boundary value problems. In these references we can see that the limit problem (when $\varepsilon = 0$) is a partial differential equation only in X_2 possibly parameterized by X_1. This is what inspired us to use this asymptotic technique, to show the existence of a solution of the problem (10) as a limit of u_ε when $\varepsilon \to 0$.

The existence of solutions of the nonlocal elliptic problem (12) is shown in two steps in the following section. In the first one we assume that a is bounded which allows the application of the Schauder fixed point theorem, then when a satisfies the initial hypothesis (7) we establish the existence of a solution u_ε by using an asymptotic method. In the last section we deal with the asymptotic behaviour of u_ε solution to (12) when $\varepsilon \to 0$ in order to show that the only possible limits are the solutions of problem (10).

2. **Nonlocal elliptic problems.** Through all this section we assume that ε is fixed.

2.1. **Bounded data.** In this subsection we suppose that a is bounded i.e.

$$|a(s)| \leq \alpha \qquad \forall s \in \mathbb{R}, \tag{13}$$

where α is a positive constant. Under this assumption we can take $\chi = 0$ (instead of $\chi \geq 0$) since it does not play any role. Our aim here is to study the existence of the solution of the nonlocal elliptic problem (12), using a fixed point argument. This is the subject of the following theorem.

Theorem 2.1. *Under the assumptions (5), (6), (9) and (13), there exists at least one weak solution to problem (12).*

Proof. We use the Schauder fixed point theorem. For $w \in L^2(\Omega)$, let $u \in H_0^1(\Omega)$ be the solution to the following linear elliptic problem

$$\begin{cases} -\varepsilon^2 \Delta_{X_1} u - \nabla_{X_2}(A \nabla_{X_2} u) = a(l(w)) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{14}$$

Define the mapping T from $L^2(\Omega)$ into itself by

$$w \mapsto u = T(w). \tag{15}$$

Thus, the pair $(w, u) \in L^2(\Omega) \times H_0^1(\Omega)$ satisfies

$$\int_\Omega \varepsilon^2 \nabla_{X_1} u \cdot \nabla_{X_1} v + A \nabla_{X_2} u \cdot \nabla_{X_2} v \, dx = \int_\Omega a(l(w)) v dx, \qquad \forall v \in H_0^1(\Omega). \tag{16}$$

Taking $v = u$ we get

$$\int_\Omega \varepsilon^2 |\nabla_{X_1} u|^2 + A \nabla_{X_2} u \cdot \nabla_{X_2} u \, dx = \int_\Omega a(l(w)) u dx.$$

The coerciveness assumption (6), (13) and the Cauchy-Schwarz inequality lead to

$$\begin{aligned} \int_\Omega \varepsilon^2 |\nabla_{X_1} u|^2 + \lambda |\nabla_{X_2} u|^2 \, dx &\leq \left(\int_\Omega [a(l(w))]^2 \, dx \right)^{1/2} \left(\int_\Omega u^2 dx \right)^{1/2} \\ &\leq \alpha |\Omega|^{1/2} \left(\int_\Omega u^2 dx \right)^{1/2}. \end{aligned}$$

where $|\Omega|$ denotes the measure of Ω. Applying the Poincaré inequality in the X_2 directions and the Young inequality we derive

$$\begin{aligned} \int_\Omega \varepsilon^2 |\nabla_{X_1} u|^2 + \lambda |\nabla_{X_2} u|^2 \, dx &\leq \alpha |\Omega|^{1/2} C_{\omega_2} \left(\int_\Omega |\nabla_{X_2} u|^2 \, dx \right)^{1/2} \\ &\leq \frac{1}{2\lambda} \alpha^2 C_{\omega_2}^2 |\Omega| + \frac{\lambda}{2} \int_\Omega |\nabla_{X_2} u|^2 \, dx, \end{aligned}$$

whence

$$\int_\Omega \varepsilon^2 |\nabla_{X_1} u|^2 + \frac{\lambda}{2} |\nabla_{X_2} u|^2 \, dx \leq \frac{1}{2\lambda} \alpha^2 C_{\omega_2}^2 |\Omega|,$$

where C_{ω_2} is the Poincaré constant in ω_2. This means that

$$u \text{ is bounded in } H^1(\Omega), \tag{17}$$

this of course independently of w. In particular one has

$$|u|_{L^2(\Omega)} \leq C,$$

where C is a constant independent of w. Let B be the closed ball in $L^2(\Omega)$ centered at the origin with radius C. To apply the Schauder fixed point theorem it only remains to show that T is continuous on B for the $L^2(\Omega)$ topology. Let $w_n \in B$ be a converging sequence such that

$$w_n \to w \text{ in } L^2(\Omega). \tag{18}$$

We set $u_n = T(w_n) \in B$. Then there exist a subsequence n' and $u \in H_0^1(\Omega)$ such that

$$u_{n'} \rightharpoonup u \quad \text{in} \quad H^1(\Omega), \tag{19}$$

$$u_{n'} \to u \quad \text{in} \quad L^2(\Omega), \tag{20}$$

$$w_{n'} \to w \quad \text{a.e. in} \quad \Omega. \tag{21}$$

The pair $(w_{n'}, u_{n'})$ satisfies (16) i.e.

$$\int_\Omega \varepsilon^2 \nabla_{X_1} u_{n'} \cdot \nabla_{X_1} v + A\nabla_{X_2} u_{n'} \cdot \nabla_{X_2} v \, dx = \int_\Omega a(l(w_{n'}))v dx.$$

Passing to the limit in n' we derive

$$\int_\Omega \varepsilon^2 \nabla_{X_1} u \cdot \nabla_{X_1} v + A\nabla_{X_2} u \cdot \nabla_{X_2} v \, dxdt = \lim_{n' \to \infty} \int_\Omega a(l(w_{n'}))v dx.$$

Let us compute the last limit. Thanks to the Lebesgue theorem with (18), (21) we deduce

$$l(w_{n'}) \to l(w) \quad \text{a.e. in } \Omega.$$

Applying again Lebesgue's theorem, taking into account the above limit with the continuity of a, leads to

$$\lim_{n' \to \infty} \int_\Omega a(l(w_{n'}))v dx = \int_\Omega a(l(w))v dx, \quad \forall v \in H_0^1(\Omega).$$

Thus

$$-\varepsilon^2 \Delta_{X_1} u - \nabla_{X_2}(A\nabla_{X_2} u) = a(l(w)) \text{ in } \mathcal{D}'(\Omega).$$

Since $u \in H_0^1(\Omega)$ we deduce that

$$u = T(w).$$

We can easily infer that the convergences (19) and (20) hold for the whole sequence n since problem (16) has a unique solution. Then T is continuous and by the Schauder fixed point theorem we get the existence of a solution to (12). $\square$

In the following we drop the hypothesis (13) and we show the existence theorem in more general case.

2.2. **More general assumptions.** In this section we keep the assumption (7) of the introduction for a and we assume that χ is large enough. For instance we suppose

$$\chi > |\omega_1| \, |h|_{L^\infty(\omega_1 \times \Omega)} \lim_{|r| \to \infty} \sup \left| \frac{a(r)}{r} \right|. \tag{22}$$

Let us introduce a sequence of piecewise linear functions $\theta_n : \mathbb{R} \to \mathbb{R}$ defined as

$$\theta_n(r) = \begin{cases} r \text{ if } |r| \le n \\ sign(r)n \text{ if } |r| \ge n, \end{cases}$$

and construct continuous functions a_n defined as

$$a_n = a \circ \theta_n.$$

Then we have

Theorem 2.2. *Under the assumptions (5)-(7), (9) and (22), there exists at least one weak solution to problem (12).*

Proof. According to the previous subsection and since a_n is a bounded function there exists $u^n \in H_0^1(\Omega)$ solution to

$$\begin{cases} -\varepsilon^2 \Delta_{X_1} u^n - \nabla_{X_2}\left(A \nabla_{X_2} u^n\right) + \chi u^n = a_n(l\left(u^n\right)) \text{ in } \Omega, \\ u^n = 0 \text{ on } \partial\Omega, \end{cases}$$

in the weak sense i.e.

$$\int_\Omega \varepsilon^2 \nabla_{X_1} u^n \cdot \nabla_{X_1} v + A \nabla_{X_2} u^n \cdot \nabla_{X_2} v + \chi u^n v \, dx = \int_\Omega a_n(l\left(u^n\right)) v dx \qquad \forall v \in H_0^1(\Omega). \tag{23}$$

By testing with u^n and using the ellipticity assumption we derive

$$\int_\Omega \varepsilon^2 \left|\nabla_{X_1} u^n\right|^2 + \lambda \left|\nabla_{X_2} u^n\right|^2 + \chi \left(u^n\right)^2 dx \le \int_\Omega \left|a_n(l\left(u^n\right)) u^n\right| dx. \tag{24}$$

Let us first estimate the last term in the above identity. Since a satisfies (7) and χ is large enough (it satisfies (22)) there exists r_0 such that $\forall r, |r| \ge r_0$ we have

$$|a(r)| \le \frac{\chi}{|\omega_1| \, |h|_{L^\infty(\omega_1 \times \Omega)}} |r|. \tag{25}$$

Thus we set

$$\Omega_{r_0} = \{x \in \Omega | \quad |\theta_n(l\left(u^n\right))| \ge r_0\}$$

and rewrite the last term in (24) as

$$\int_\Omega \left|a_n(l\left(u^n\right)) u^n\right| dx = \int_{\Omega/\Omega_{r_0}} \left|a_n(l\left(u^n\right)) u^n\right| dx + \int_{\Omega_{r_0}} \left|a_n(l\left(u^n\right)) u^n\right| dx. \tag{26}$$

Then by (25) we get

$$\int_\Omega \left|a_n(l\left(u^n\right)) u^n\right| dx \le |\Omega|^{1/2} \max_{|r| \le r_0} |a(r)| \left|u^n\right|_{L^2\left(\Omega/\Omega_{r_0}\right)} + \frac{\chi}{|\omega_1| \, |h|_{L^\infty(\omega_1 \times \Omega)}} \int_{\Omega_{r_0}} \left|u^n\right| \left|l\left(u^n\right)\right| dx, \tag{27}$$

since $\left|\theta_n(r)\right| \le |r|, \forall r \in \mathbb{R}$ and $\forall n \in \mathbb{N}$. The last integral in the above inequality can be estimated as follows

$$\begin{aligned} &\int_{\Omega_{r_0}} \left|u^n\right| \left|l\left(u^n\right)\right| dx \\ &\le \int_{\Omega_{r_0}} \left|u^n\right| \int_{\omega_1} \left|h(X_1, X_1', X_2)\right| \left|u^n(X_1', X_2)\right| dX_1' dx \\ &\le |h|_{L^\infty(\omega_1 \times \Omega)} \left(\int_{\Omega_{r_0}} \left|u^n\right|^2 dx\right)^{1/2} \left(\int_{\Omega_{r_0}} \left(\int_{\omega_1} \left|u^n(X_1', X_2)\right| dX_1'\right)^2 dx\right)^{1/2}. \end{aligned}$$

Applying the Young inequality we derive

$$\begin{aligned}
&\int_{\Omega_{r_0}} |u^n| \, |l(u^n)| \, dx \\
&\qquad \leq |h|_{L^\infty(\omega_1\times\Omega)} \left(\int_{\Omega_{r_0}} |u^n|^2 \, dx \right)^{1/2} \left(\int_\Omega |\omega_1| \int_{\omega_1} |u^n(X_1', X_2)|^2 \, dX_1' dx \right)^{1/2} \\
&\qquad \leq |\omega_1| \, |h|_{L^\infty(\omega_1\times\Omega)} \left(\int_{\Omega_{r_0}} |u^n|^2 \, dx \right)^{1/2} \left(\int_\Omega |u^n|^2 \, dx \right)^{1/2} .
\end{aligned}$$

Then using this and Young's inequality in (27) we deduce that

$$\begin{aligned}
&\int_\Omega |a_n(l(u^n))u^n| \, dx \\
&\quad \leq |\Omega|^{1/2} \max_{|r|\leq r_0} |a(r)| \, |u^n|_{L^2(\Omega/\Omega_{r_0})} + \chi \, |u^n|_{L^2(\Omega)} \, |u^n|_{L^2(\Omega_{r_0})} \\
&\quad \leq \frac{1}{2\chi} |\Omega| \left(\max_{|r|\leq r_0} |a(r)| \right)^2 + \frac{\chi}{2} |u^n|^2_{L^2(\Omega/\Omega_{r_0})} + \frac{\chi}{2} |u^n|^2_{L^2(\Omega_{r_0})} + \frac{\chi}{2} |u^n|^2_{L^2(\Omega)} \\
&\quad = C + \chi \, |u^n|^2_{L^2(\Omega)} ,
\end{aligned}$$

where $C = \frac{1}{2\chi} |\Omega| \left(\max_{|r|\leq r_0} |a(r)| \right)^2$. Going back to (24) and using the last estimate we obtain

$$\varepsilon^2 \, |\nabla_{X_1} u^n|^2_{L^2(\Omega)} + \lambda \, |\nabla_{X_2} u^n|^2_{L^2(\Omega)} \leq C, \quad \forall n \in \mathbb{N},$$

which means that

$$|\nabla u^n|^2_{L^2(\Omega)} \text{ is bounded,}$$

this of course independently of n. Then there exist a subsequence n', $g \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ such that

$$u^{n'} \rightharpoonup u \quad \text{in} \quad H^1(\Omega), \tag{28}$$

$$u^{n'} \to u \quad \text{in} \quad L^2(\Omega), \tag{29}$$

$$u^{n'} \to u \quad \text{a.e.} \quad \text{in} \quad \Omega, \tag{30}$$

$$\left| u^{n'} \right| \leq g \quad \text{a.e.} \quad \text{in} \quad \Omega. \tag{31}$$

Passing to the limit in (23) we get

$$\int_\Omega \varepsilon^2 \nabla_{X_1} u \cdot \nabla_{X_1} v + A \nabla_{X_2} u \cdot \nabla_{X_2} v + \chi u v \, dx = \lim_{n'\to\infty} \int_\Omega a_{n'} \left(l \left(u^{n'} \right) \right) v dx. \tag{32}$$

We can compute the last limit using Lebesgue's theorem. Indeed, by (29), (30), (31) we get

$$\left| l(u^{n'})(x) \right| \leq \int_{\omega_1} |h(X_1, X_1', X_2)| \, g(X_1', X_2) dX_1' \quad \text{a.e.} \quad \text{in} \quad \Omega, \tag{33}$$

$$l \left(u^{n'} \right) \to l(u) \quad \text{a.e.} \quad \text{in } \Omega. \tag{34}$$

Then

$$\theta_{n'} \left(l \left(u^{n'} \right) \right) \to l(u) \quad \text{a.e.} \quad \text{in } \Omega.$$

Moreover the right hand side of (33) is a function in $L^2(\Omega)$ which we will denote by H. Then it is easy to remark (cf. (25)) that for some constant C one has

$$\left| a_{n'} l(u^{n'}) \right| \leq \max_{|r| \leq r_0} |a\,(r)| + CH.$$

Thanks to the continuity of a we deduce also

$$a_{n'}\left(l\left(u^{n'}\right)\right) \to a\,(l\,(u)) \quad \text{a.e.} \quad \text{in} \quad \Omega.$$

Thus by the Lebesgue theorem it follows that

$$\int_{\Omega} \varepsilon^2 \nabla_{X_1} u \cdot \nabla_{X_1} v + A \nabla_{X_2} u \cdot \nabla_{X_2} v + \chi uv\, dx = \int_{\Omega} a\,(l\,(u))\, v dx. \tag{35}$$

This completes the proof since we already have $u \in H_0^1\,(\Omega)$. □

Note that this technique of bounded and a.e. converging subsequence as in (30), (31) will be used subsequently in this paper.

3. **Anisotropic singular perturbations method.** Before stating our asymptotic behaviour result, let us introduce the functional space V defined as

$$V := \left\{ v \in L^2\,(\Omega)\, | \partial_{x_i'} v \in L^2\,(\Omega)\,,\ i = 1, \cdots, n \text{ and } v(X_1, \cdot) \in H_0^1(\omega_2) \text{ a.e. } X_1 \in \omega_1 \right\},$$

equipped with the norm

$$|u|_V^2 = |\nabla_{X_2} u|_{L^2(\Omega)}^2 + |u|_{L^2(\Omega)}^2\,. \tag{36}$$

It is clear that V is a Hilbert space since if u_n is a Cauchy sequence in V, there exist $u \in L^2\,(\Omega)$ with $\partial_{x_i'} u \in L^2\,(\Omega)\,,\ i = 1, \cdots, n$ such that $u_n \to u$ with respect to the norm (36), and for a.e. $X_1 \in \omega_1$ -up to a subsequence-

$$u_n(X_1, \cdot) \to u(X_1, \cdot) \quad \text{in} \quad H^1(\omega_2).$$

In this section we also assume that

$$\nabla h \in L^\infty(\omega_2; L^2(\omega_1 \times \omega_1)). \tag{37}$$

Then let us show the following lemma that plays a principal role in this study.

Lemma 3.1. *Let $w_n \in V$ be a sequence converging weakly toward w in V. Then we have*

$$l\,(w_n) \rightharpoonup l\,(w) \quad \textit{in } H^1\,(\Omega)$$

and

$$l\,(w_n) \to l\,(w) \quad \textit{in } L^2\,(\Omega)\,.$$

Proof. Using (37) we have

$$\begin{aligned}
&\int_\Omega \left|\nabla_{X_1} l\left(w_n\right)\right|^2 dx \\
&= \int_\Omega \left|\int_{\omega_1} \nabla_{X_1} h\left(X_1, X_1', X_2\right) w_n\left(X_1', X_2\right) dX_1'\right|^2 dx \\
&\leq \int_{\omega_1\times\omega_2} \left(\int_{\omega_1} \left|\nabla_{X_1} h\left(X_1, X_1', X_2\right)\right| \left|w_n\left(X_1', X_2\right)\right| dX_1'\right)^2 dx \\
&\leq \int_{\omega_2}\int_{\omega_1}\int_{\omega_1} \left(\left|\nabla_{X_1} h\left(X_1, X_1', X_2\right)\right|^2 dX_1'\right)\left(\int_{\omega_1} \left(w_n\left(X_1', X_2\right)\right)^2 dX_1'\right) dx \\
&= \int_{\omega_2}\left(\int_{\omega_1} \left(w_n\left(X_1', X_2\right)\right)^2 dX_1'\right)\left(\int_{\omega_1\times\omega_1} \left|\nabla_{X_1} h\left(X_1, X_1', X_2\right)\right|^2 dX_1' dX_1\right) dX_2 \\
&\leq \left(ess\sup_{\omega_2} \int_{\omega_1\times\omega_1} \left|\nabla_{X_1} h\left(X_1, X_1', X_2\right)\right|^2 dX_1' dX_1\right)\int_\Omega w_n^2(x) dx.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\int_\Omega \left|\nabla_{X_2} l\left(w_n\right)\right|^2 dx \\
=\ & \int_\Omega \left|\int_{\omega_1} \nabla_{X_2} h\left(X_1, X_1', X_2\right) w_n\left(X_1', X_2\right) + h\nabla_{X_2} w_n\left(X_1', X_2\right) dX_1'\right|^2 dx \\
\leq\ & \int_{\omega_1\times\omega_2} \left(\int_{\omega_1} \left|\nabla_{X_2} h\right| \left|w_n\left(X_1', X_2\right)\right| + |h| \left|\nabla_{X_2} w_n\left(X_1', X_2\right)\right| dX_1'\right)^2 dx \\
\leq\ & 2\int_{\omega_1\times\omega_2} \left(\int_{\omega_1} \left|\nabla_{X_2} h\left(X_1, X_1', X_2\right)\right| \left|w_n\left(X_1', X_2\right)\right| dX_1'\right)^2 dx \\
& +2\int_{\omega_1\times\omega_2} \left(\int_{\omega_1} \left|h\left(X_1, X_1', X_2\right)\right| \left|\nabla_{X_2} w_n\left(X_1', X_2\right)\right| dX_1'\right)^2 dx \\
\leq\ & 2\left(ess\sup_{\omega_2} \int_{\omega_1\times\omega_1} \left|\nabla_{X_2} h\left(X_1, X_1', X_2\right)\right|^2 dX_1' dX_1\right)\int_\Omega \left(w_n\right)^2 dx \\
& +2\left(ess\sup_{\omega_2} \int_{\omega_1\times\omega_1} \left|h\left(X_1, X_1', X_2\right)\right|^2 dX_1' dX_1\right)\int_\Omega \left|\nabla_{X_2} w_n\right|^2 dx.
\end{aligned}$$

Then since

$$w_n \rightharpoonup w \ \text{ in } V,$$

it follows that w_n is bounded in V and by the above estimates we deduce that the sequence $l\left(w_n\right)$ is bounded in $H^1\left(\Omega\right)$. Thus, there exist a subsequence n' and $W \in H^1\left(\Omega\right)$, such that

$$l\left(w_{n'}\right) \rightharpoonup W \ \text{ in } H^1\left(\Omega\right), \tag{38}$$

$$l\left(w_{n'}\right) \to W \ \text{ in } L^2\left(\Omega\right). \tag{39}$$

On the other hand we have for every $v \in \mathcal{D}\left(\Omega\right)$

$$\begin{aligned}
\int_\Omega l\left(w_{n'}\right) v dx &= \int_\Omega v\left(X_1, X_2\right)\left(\int_{\omega_1} h\left(X_1, X_1', X_2\right) w_{n'}\left(X_1', X_2\right) dX_1'\right) dx \\
&= \int_{\omega_1}\left(\int_\Omega h\left(X_1, X_1', X_2\right) v\left(X_1, X_2\right) w_{n'}\left(X_1', X_2\right) dX_1' dX_2\right) dX_1.
\end{aligned}$$

Since $w_n \rightharpoonup w$ in V it follows that for a.e. $X_1 \in \omega_1$

$$\int_\Omega h(X_1, X_1', X_2)\, v(X_1, X_2)\, w_{n'}(X_1', X_2)\, dX_1' dX_2 \to \int_\Omega h(X_1, X_1', X_2)\, v(X_1, X_2)\, w(X_1', X_2)\, dX_1' dX_2.$$

Using Lebesgue's theorem we derive

$$\int_{\omega_1} \left(\int_\Omega h(X_1, X_1', X_2)\, v(X_1, X_2)\, w_{n'}(X_1', X_2)\, dX_1' dX_2 \right) dX_1 \to \int_{\omega_1} \left(\int_\Omega h(X_1, X_1', X_2)\, v(X_1, X_2)\, w(X_1', X_2)\, dX_1' dX_2 \right) dX_1,$$

i.e.

$$\int_\Omega l(w_{n'})\, v dx \to \int_\Omega l(w)\, v dx \quad \forall v \in \mathcal{D}(\Omega).$$

Thus we have

$$l(w_{n'}) \to l(w) \quad \text{in } \mathcal{D}'(\Omega).$$

Combining this with (38) and (39) we deduce

$$W = l(w).$$

Since w is the unique limit of w_n, the whole sequence $l(w_n)$ converges to $l(w)$ i.e.

$$\begin{array}{rcll} l(w_n) & \rightharpoonup & l(w) & \text{in } H^1(\Omega), \\ l(w_n) & \to & l(w) & \text{in } L^2(\Omega). \end{array}$$

This completes the proof of the lemma. □

Next, let us recall the definition of $\varepsilon - nets$ in metric spaces. This definition is useful when the problem (10) has more than one solution.

Definition 3.2 (ε **− nets**). Given a metric space $(X; d)$, a subset Y of X is said to be an $\varepsilon - net$ of another subset Y', if for all $x \in Y'$, there exists an $a \in Y$ such that

$$d(x, a) < \varepsilon.$$

Now, we are ready to state the main result.

Theorem 3.3. *Under the assumptions of Theorem 2.2 and if (37) holds, then the problem (11) is equivalent to finding $u_0 \in V$ such that*

$$\int_\Omega A\nabla_{X_2} u_0 \cdot \nabla_{X_2} v + \chi u_0 v dx = \int_\Omega a(l(u_0)) v dx, \qquad \forall v \in V,$$

and the set of solutions of (10) is not empty. Moreover if we consider the metric structure of V corresponding to the norm (36), then for every $r > 0$, there exists $\varepsilon_0 > 0$ such that the set of the solutions of (10) consists a $r - net$ of the set

$$A_{\varepsilon_0} = \{u_\varepsilon \text{ solution to (12) for } \varepsilon < \varepsilon_0\},$$

and we also have

$$\varepsilon \nabla_{X_1} u_\varepsilon \longrightarrow 0 \quad in \quad L^2(\Omega).$$

This theorem has the following immediate corollary that gives the convergence of u_ε.

Corollary 1. *Under the assumptions of Theorem 3.3 and if problem (10) has only one solution u_0 then we have*

$$u_\varepsilon \longrightarrow u_0, \quad \nabla_{X_2} u_\varepsilon \longrightarrow \nabla_{X_2} u_0 \quad \text{and} \quad \varepsilon \nabla_{X_1} u_\varepsilon \longrightarrow 0 \quad \text{in} \quad L^2(\Omega).$$

Remark 1. We will also see in the proof below that from every subsequence of $(u_\varepsilon)_\varepsilon$ there exists another converging subsequence to a solution of (10) in the sense of Corollary 1.

Proof of Theorem 3.3. Let us take $v = u_\varepsilon$ in the weak formulation

$$\int_\Omega \varepsilon^2 \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v + A \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v + \chi u_\varepsilon v \, dx = \int_\Omega a\left(l\left(u_\varepsilon\right)\right) v dx. \tag{40}$$

By the ellipticity assumption we derive

$$\int_\Omega \varepsilon^2 \left|\nabla_{X_1} u_\varepsilon\right|^2 + \lambda \left|\nabla_{X_2} u_\varepsilon\right|^2 + \chi \left(u_\varepsilon\right)^2 dx \le \int_\Omega \left|a(l\left(u_\varepsilon\right)) u_\varepsilon\right| dx. \tag{41}$$

As similarly done before we set

$$\Omega_{r_0} = \{x \in \Omega | \quad \left|l\left(u_\varepsilon\right)\right| \ge r_0\}.$$

Then the right hand side of (41) can be written as

$$\int_\Omega \left|a(l\left(u_\varepsilon\right)) u_\varepsilon\right| dx = \int_{\Omega / \Omega_{r_0}} \left|a(l\left(u_\varepsilon\right)) u_\varepsilon\right| dx ds + \int_{\Omega_{r_0}} \left|a(l\left(u_\varepsilon\right)) u_\varepsilon\right| dx. \tag{42}$$

By (25) we get

$$\begin{aligned} \int_\Omega \left|a(l\left(u_\varepsilon\right)) u_\varepsilon\right| dx \le & \left|\Omega\right|^{1/2} \max_{|r| \le r_0} \left|a\left(r\right)\right| \left|u_\varepsilon\right|_{L^2\left(\Omega / \Omega_{r_0}\right)} \\ & + \frac{\chi}{\left|\omega_1\right| \; \left|h\right|_{L^\infty\left(\omega_1 \times \Omega\right)}} \int_{\Omega_{r_0}} \left|u_\varepsilon\right| \left|l\left(u_\varepsilon\right)\right| dx. \end{aligned}$$

Then using the same argument as in the proof of Theorem 2.2 we deduce that

$$\begin{aligned} & \int_\Omega \left|a(l\left(u_\varepsilon\right)) u_\varepsilon\right| dx \\ \le \; & \left|\Omega\right|^{1/2} \max_{|r| \le r_0} \left|a\left(r\right)\right| \left|u_\varepsilon\right|_{L^2\left(\Omega / \Omega_{r_0}\right)} + \chi \left\{ \int_{\Omega_{r_0}} \left(u_\varepsilon\right)^2 dx ds \right\}^{1/2} \left|u_\varepsilon\right|_{L^2(\Omega)} \\ \le \; & \frac{1}{2\chi} \left|\Omega\right| \left(\max_{|r| \le r_0} \left|a\left(r\right)\right| \right)^2 + \frac{\chi}{2} \left|u_\varepsilon\right|^2_{L^2\left(\Omega / \Omega_{r_0}\right)} + \frac{\chi}{2} \left|u_\varepsilon\right|^2_{L^2\left(\Omega_{r_0}\right)} + \frac{\chi}{2} \left|u_\varepsilon\right|^2_{L^2(\Omega)} \\ \le \; & C + \chi \left|u_\varepsilon\right|^2_{L^2(\Omega)}, \end{aligned}$$

where $C = \frac{1}{2\chi} \left|\Omega\right| \left(\max_{|r| \le r_0} \left|a\left(r\right)\right|\right)^2$. Going back to (41) and using the last estimate we obtain

$$\varepsilon^2 \left|\nabla_{X_1} u_\varepsilon\right|^2_{L^2(\Omega)} + \lambda \left|\nabla_{X_2} u_\varepsilon\right|^2_{L^2(\Omega)} \le C,$$

i.e.

$$u_\varepsilon, \quad \left|\varepsilon \nabla_{X_1} u_\varepsilon\right|, \quad \left|\nabla_{X_2} u_\varepsilon\right| \quad \text{are bounded in } L^2(\Omega), \tag{43}$$

this of course independently of ε. It follows that there exist $u_0 \in L^2(\Omega)$, $u_1 \in (L^2(\Omega))^n$ such that – up to a subsequence

$$u_\varepsilon \rightharpoonup u_0, \qquad \nabla_{X_2} u_\varepsilon \rightharpoonup u_1 \qquad \text{in } L^2(\Omega).$$

(The convergence is meant component by component). Of course the convergence in $L^2(\Omega)$ –weak – implies the convergence in $\mathcal{D}'(\Omega)$ and by the continuity of the derivation in $\mathcal{D}'(\Omega)$ we deduce that

$$u_\varepsilon \rightharpoonup u_0, \qquad \nabla_{X_2} u_\varepsilon \rightharpoonup \nabla_{X_2} u_0 \quad \text{in } L^2(\Omega). \tag{44}$$

Also, since u_ε is bounded in $L^2(\Omega)$ we have

$$\int_\Omega \varepsilon \nabla_{X_1} u_\varepsilon \cdot v \, dx = -\int_\Omega \varepsilon u_\varepsilon div_{X_1} v \, dx \to 0, \quad \forall v \in [\mathcal{D}(\Omega)]^m .$$

It follows since $|\varepsilon \nabla_{X_1} u_\varepsilon|$ is bounded in $L^2(\Omega)$ that

$$\varepsilon \nabla_{X_1} u_\varepsilon \rightharpoonup 0 \quad \text{in } L^2(\Omega).$$

We then go back to the equation satisfied by u_ε and compute the limit of the right hand side, using Lemma 3.1, we obtain - up to a subsequence -

$$l(u_\varepsilon) \to l(u_0) \quad \text{a.e. in } \Omega.$$

The continuity of a gives

$$a(l(u_\varepsilon)) \to a(l(u_0)) \quad \text{a.e. in } \Omega. \tag{45}$$

Together with Lebesgue's theorem and (25) we obtain

$$\int_\Omega a(l(u_\varepsilon)) v dx \to \int_\Omega a(l(u_0)) v dx. \tag{46}$$

(Of course since $l(u_\varepsilon)$ is a converging sequence in $L^2(\Omega)$ (by Lemma 3.1), there exists $g \in L^2(\Omega)$ such that -up to a subsequence-

$$|l(u_\varepsilon)| \le g \quad \text{a.e. in } \Omega.$$

We used this with (7), (45) to get (46).) Then passing to the limit in (35) we derive

$$\int_\Omega A\nabla_{X_2} u_0 \cdot \nabla_{X_2} v + \chi u_0 v \, dx = \int_\Omega a(l(u_0)) v dx. \tag{47}$$

Taking $v = u_\varepsilon$ in (47) and passing to the limit we get

$$\int_\Omega A\nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 \; + \chi u_0 v \, dx = \int_\Omega a(l(u_0)) u_0 dx. \tag{48}$$

Next we expend

$$I_\varepsilon := \int_\Omega \varepsilon^2 \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} u_\varepsilon \, dx + \int_\Omega A\nabla_{X_2}(u_\varepsilon - u_0)\cdot\nabla_{X_2}(u_\varepsilon - u_0) dx + \chi \int_\Omega (u_\varepsilon - u_0)^2 dx,$$

to get

$$\begin{aligned} I_\varepsilon = & \int_\Omega \varepsilon^2 \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} u_\varepsilon + \; A\nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} u_\varepsilon dx + \chi \int_\Omega {u_\varepsilon}^2 dx \\ & - \int_\Omega A\nabla_{X_2} u_0 \cdot \nabla_{X_2} u_\varepsilon \, dx - \int_\Omega A\nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} u_0 \, dx \\ & + \int_\Omega A\nabla_{X_2} u_0 \cdot \nabla_{X_2} u_0 dx - 2\chi \int_\Omega u_\varepsilon u_0 dx + \chi \int_\Omega u_0^2 dx. \end{aligned}$$

Using (40) we derive

$$\begin{aligned} I_\varepsilon = &\int_\Omega a\left(l\left(u_\varepsilon\right)\right) u_\varepsilon dx \\ &-\int_\Omega A\nabla_{X_2} u_0\cdot\nabla_{X_2} u_\varepsilon\, dx - \int_\Omega A\nabla_{X_2} u_\varepsilon\cdot\nabla_{X_2} u_0\, dx \\ &+\int_\Omega A\nabla_{X_2} u_0\cdot\nabla_{X_2} u_0 dx - 2\chi\int_\Omega u_\varepsilon u_0 dx + \chi\int_\Omega u_0^2 dx. \end{aligned}$$

Thanks to Lemma 3.1, (45) and applying the Lebesgue theorem we deduce that

$$a\left(l\left(u_\varepsilon\right)\right)\to a\left(l\left(u_0\right)\right) \quad \text{in } \ L^2\left(\Omega\right).$$

This with the weak convergence of u_ε leads to

$$\int_\Omega a\left(l\left(u_\varepsilon\right)\right) u_\varepsilon dx \to \int_\Omega a\left(l\left(u_0\right)\right) u_0 dx.$$

Using this to pass to the limit in I_ε we get

$$\lim_{\varepsilon\to 0} I_\varepsilon = \int_\Omega a\left(l\left(u_0\right)\right) u_0 dx - \int_\Omega A\nabla_{X_2} u_0\cdot\nabla_{X_2} u_0\, dx - \chi\int_\Omega u_0^2 dx = 0,$$

since we have (48). Using the ellipticity assumption we derive

$$\int_\Omega \varepsilon^2|\nabla_{X_1} u_\varepsilon|^2 + \lambda|\nabla_{X_2}(u_\varepsilon - u_0)|^2\, dx + \chi\int_\Omega (u_\varepsilon - u_0)^2 dx \le I_\varepsilon.$$

It follows that

$$\varepsilon\nabla_{X_1} u_\varepsilon \longrightarrow 0, \qquad \nabla_{X_2} u_\varepsilon \longrightarrow \nabla_{X_2} u_0, \qquad u_\varepsilon \longrightarrow u_0 \quad \text{in } L^2(\Omega).$$

Now we also have

$$\int_{\omega_1}\int_{\omega_2} |\nabla_{X_2}(u_\varepsilon - u_0)|^2\, dX_2\, dX_1 \longrightarrow 0. \tag{49}$$

From this it follows that for a.e. X_1

$$\int_{\omega_2} |\nabla_{X_2}(u_\varepsilon - u_0)|^2\, dX_2 \longrightarrow 0.$$

Since

$$\left\{\int_{\omega_2} |\nabla_{X_2} v|^2\, dX_2\right\}^{\frac{1}{2}}$$

is a norm on $H_0^1(\omega_2)$ and we can easily show that $u_\varepsilon(X_1,\cdot)\in H_0^1(\omega_2)$ (see [2]) we have

$$u_0(X_1,\cdot)\in H_0^1(\omega_2)$$

for a.e. X_1. As a consequence we have

$$u_0 \in V.$$

Using the same argument as in [4] and taking into account (47) we show that u_0 satisfies (11).

Now we have to show that the only possible limits are solutions of (10). For that we suppose that there exist a subsequence of u_ε- still labeled by $u_\varepsilon -$ and a constant $c > 0$ such that

$$|u_\varepsilon - u_0|_V > c, \tag{50}$$

for any solution u_0 to (10). Of course, from the above proof we can extract again a subsequence that converges to a solution of (10), which contradicts (50). By the same argument we deduce that

$$\varepsilon \nabla_{X_1} u_\varepsilon \longrightarrow 0 \quad \text{in } L^2(\Omega).$$

This completes the proof. □

Remark 2. Thanks to the hypothesis (22), we can also consider the Neumann or the mixed boundary conditions.

Acknowledgements. The authors have been supported by the Swiss National Science Foundation under the contracts #20 − 113287/1 and #20 − 117614/1. They are very grateful to this institution.

REFERENCES

[1] B. Brighi and S. Guesmia, *Asymptotic behavior of solutions of hyperbolic problems on a cylindrical domain*, Discrete Contin. Dyn. Syst., suppl. (2007), 160 - 169.
[2] M. Chipot, "Elliptic Equations: An Introductory Course," Birkhäuser, 2009.
[3] M. Chipot, *On some anisotropic singular perturbation problems*, Asymptotic Anal. 55 (3-4) (2007), 125-144.
[4] M. Chipot, S. Guesmia, *On the asymptotic behavior of elliptic, anisotropic singular perturbations problems,* Commun. Pure Appl. Anal. 8 (1) (2009), 179-193.
[5] M. Chipot, S. Guesmia, *Correctors for some asymptotic problems,* (to appear).
[6] R. Dautray, J. L. Lions, "Analyse mathématique et calcul numérique", volume 3, Masson, 1987.
[7] D. Gilbarg, N. S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer Verlag, 1983.
[8] S. Guesmia, Etude du comportement asymptotique de certaines équations aux dérivées partielles dans des domaines cylindriques, Thèse Université de Haute Alsace, December 2006.
[9] S. Guesmia, *On the asymptotic behavior of elliptic boundary value problems with some small coefficients*, Electro. J. Differ. Equ. 59 (2008), 1-13.
[10] E. Lewis, W. Miller Jr., "Computational Methods of Neutron Transport," John Wiley & Sons, London, 1984.
[11] J. L. Lions, "Perturbations Singulières dans les Problèmes aux Limites et en Contrôle Optimal," Lecture Notes in Mathematics # 323, Springer-Verlag, 1973.
[12] V. S. Vladimirov, Mathematical problems in one-speed particle transport theory, Trudy Mat. Inst. Akad. Nauk SSSR, 61 (1961).

E-mail address: m.m.chipot@math.uzh.ch
E-mail address: senoussi.guesmia@math.uzh.ch