ABC Journals
a new era in publishing
ABC Research House
Asian Journal of Applied Science and Engineering
Abbreviated key title: Asian j. appl. sci. eng.
ISSN: 2305-915X(p); ISSN: 2307-9584(e)
Access this article online
www.ajase.weebly.com

Generating a Random Dynamical System

Salah H. Abid[1], & Ihsan J. Kadhim[2]

[1]Professor, Mathematics Department, Al-Mustansirya University, **Iraq**
[2]Mathematics Department, University of Al-Qadysia, **Iraq**

ARTICLE INFO

Volume 3
Number 3/2014
Issue 7
DOI: 10.15590/ajase/2014/v3i7/53570

Received: Mar 25, 2014
Accepted: Apr 14, 2014
Revised: Apr 27, 2014
Published: August 10, 2014
E-mail for correspondence:
abidsalah@gmail.com

ABSTRACT

In this paper we shall create a new random dynamical system from old one. One of the most important of them, is the product of two random dynamical systems, which is also random dynamical system. This technique will be very important to detailed study of the random dynamical system.

Keywords: Dynamical system, Noise, Topological space, Probability space, Semi group

Source of Support: Nil, **Conflict of Interest:** None declared.

How to Cite: Abid SH and Kadhim IJ. 2014. **Generating a Random Dynamical System** *Asian Journal of Applied Science and Engineering,* 3, 7-15. Available at: http://dx.doi.org/10.15590/ajase/2014/v3i7/53570

INTRODUCTION

Random dynamical systems arise in the modeling of many phenomena in physics, biology, economics, climatology, etc., and the random effects often reflect intrinsic properties of these phenomena rather than just to compensate for the defects in deterministic models. The history of study of random dynamical systems goes back to Ulam and von Neumann in 1945 [11] and it has flourished since the 1980s due to the discovery that the solutions of stochastic ordinary differential equations yield a cocycle over a metric dynamical system which models randomness, i.e. a random dynamical system. Through this paper G denote a topological group, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and X a topological space or (Polish) metric space. Let (X, τ) be a topological space. The σ −algebra generated by the topology τ is called ***Borel*** $\boldsymbol{\sigma}$ ***−algebra***[3-6,8]and shall denoted by $\mathcal{B}(X)$; sets in it are called ***Borel sets***.

As a Structure of the paper, In section 2 (Preliminary) we state some definitions and theorems from Measure Theory, Ergodic Theory and Topological Group, that are need in this paper. In section 3 we sate the definition of Random Dynamical System. In section 4

(Induced Random Dynamical Systems) we introduce a new results about induced random dynamical from old one.

PRELIMINARY

In this section we sate some definitions and theorems that are need in our paper.

Definition 2.1 [10,11,15] A function f from a topological space X to an other topological space Y is said to be continuous if the inverse image of every open set in X under f is open Y. And is said to be homeomorphism if f is bijective and f, f^{-1} are continuous. Also f is said to be open if the image under f of every open set in X is open in Y.

Definition2.2.[3-6,8] Let $\mathcal{F}$ be a collection of subsets of a set Ω. Then $\mathcal{F}$ is called a ***field*** (the term ***algebra*** is also used) if $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under complementation and finite union. If the " closed under finite union" replaced by " closed under countable union", $\mathcal{F}$ is called ***σ −filed*** (the term ***σ −algebra*** is also used). If $\mathcal{F}$ is σ −filed , then the pair $(\Omega, \mathcal{F})$ is called ***measurable space*** and the sets of $\mathcal{F}$ are called ***measurable sets***.

Theorem2.2.[3-6]. Let C be a collection of subsets of a set Ω. Then there exists a smallest σ −filed$\mathcal{F}$ containing C. We say that $\mathcal{F}$ is the σ −filed generated by C and write $\mathcal{F} = \sigma(\mathsf{C})$ and C is called a generator for $\mathcal{F}$.

Remarks2.3.

(1) A generator is not unique.

(2) If $\mathsf{C}_1 \subset \sigma(\mathsf{C}_2)$ and $\mathsf{C}_2 \subset \sigma(\mathsf{C}_1)$, then$\sigma(\mathsf{C}_1) = \sigma(\mathsf{C}_2)$.

Theorem2.4.[3-6]. Let $(\Xi, \mathcal{F}^{'})$ be a measurable space. If $f: \Omega \to \Xi$ be any function, then the collection $\mathcal{F} \coloneqq \{f^{-1}(F): F \in \mathcal{F}^{'}\}$is a σ −filed on Ω and consequently $(\Omega, \mathcal{F})$ is a measurable space. The σ −filed $\mathcal{F}$ is called ***σ −filed induced by$f$***.

Corollary2.5.[3-6] Let $(\Omega, \mathcal{F})$ be a measurable space and $\Omega^{'} \subset \Omega$, not necessarily measurable subsets. Consider the inclusion map $i: \Omega^{'} \to \Omega$ where $\omega \longmapsto \omega \in \Omega$. The σ −filed induced by i is given by $\mathcal{F}^* = \{\Omega^{'} \cap F: F \subset \Omega^{'}\}$, called ***the trace σ −filed*** on $\Omega^{'}$.

If $\Omega^{'} \in \mathcal{F}$, then $\mathcal{F}^* = \{F \in \mathcal{F}: F \subset \Omega^{'}\}$.

Theorem2.6.[3-6] Let $f: \Omega \to \Xi$, and let C be a class of subsets of Ξ. Then $\sigma\big(f^{-1}(\mathsf{C})\big) = f^{-1}\big(\sigma(\mathsf{C})\big)$, where $f^{-1}(\mathsf{C}) = \{f^{-1}(A): A \in \mathsf{C}\}$.

Definition2.7.[3-6] A ***measure*** on a σ −filed$\mathcal{F}$ is non-negative, extended real-valued function μ on $\mathcal{F}$ such that whenever $A_1, A_2, \ldots$ form a finite or countably infinite collection of disjoint sets in $\mathcal{F}$,

we have , $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$. If $\mu(\Omega) = 1$, μ is called a ***probability measure***.

Definition2.8.[3-6] A ***measure space*** is a triple $(\Omega, \mathcal{F}, \mu)$ where Ω is a set, $\mathcal{F}$ is a σ −filed of subsets of Ω, and μ is a measure on $\mathcal{F}$. If μ is a probability measure, $(\Omega, \mathcal{F}, \mu)$ is called a ***probability space***. We shall denote to the probability space by $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition2.9.[3-6] Let (X, τ) be a topological space. The σ −filed generated by the topology τ is called ***Borel σ −filed***; sets in it are called ***Borel sets***.

Definition2.10.[3-6]Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces. A function $f: \Omega_1 \to \Omega_2$ is said to be $\mathcal{F}_1, \mathcal{F}_2$ −measurable if $f^{-1}(A) \in \mathcal{F}_1$ for each $A \in \mathcal{F}_2$.

Proposition2.11.[3-6]. (a) It is sufficient that $f^{-1}(A) \in \mathcal{F}_1$ for each $A \in \mathsf{C}$, where C is a class of subsets of Ω_2, such that $\sigma(\mathsf{C}) = \mathcal{F}_2$. **(b)** Theinverse of a bijective measurable function is also measurable.

Definition2.12.[3-6] If $(\Omega, \mathcal{F})$ is measurable space and $f: \Omega \rightarrow \mathbb{R}^n$ (or $\overline{\mathbb{R}}^n$), f is said to be Borel measurable [on $(\Omega, \mathcal{F})$] if f is $\mathcal{F}, \mathcal{B}(\mathbb{R}^n)$ $-$measurable. If Ω is a Borel subset of $\mathbb{R}^k$ (or $\overline{\mathbb{R}}^k$) and we use the term " Borel measurable," we always assume that $\mathcal{F} = \mathcal{B}$.

Definition2.13.[3-6] A continuous map $f: \mathbb{R}^k \rightarrow \mathbb{R}^n$ is Borel measurable; if C is class of open subsets of $\mathbb{R}^n$, then $f^{-1}(A)$ is open, hence belongs to $\mathcal{B}(\mathbb{R}^k)$ for each $A \in C$. If $(\Omega, \mathcal{F}, \mu)$ is a measure space the terminology " is a Borel measurable on $(\Omega, \mathcal{F}, \mu)$ " will mean that f is Borel measurable on $(\Omega, \mathcal{F})$ and μ is a measure on $\mathcal{F}$.

Theorem2.14.[3-6] A composition of measurable functions is measurable; specifically if $f_1: (\Omega_1, \mathcal{F}_1) \rightarrow (\Omega_2, \mathcal{F}_2)$ and $f_2: (\Omega_2, \mathcal{F}_2) \rightarrow (\Omega_3, \mathcal{F}_3)$ are measurable, then so is $f_2 \circ f_1: (\Omega_1, \mathcal{F}_1) \rightarrow (\Omega_3, \mathcal{F}_3)$.

Definition 2.15[3-6] The produce σ $-$algebra $\mathcal{F}$ on $\Omega_1 \times \Omega_2$ is the σ $-$algebragenerated by rectangles of the form$A_1 \times A_2$, with $A_i \in \mathcal{F}_i$ for $i = 1,2$.

We can then define the product measure $\mathbb{P} \coloneqq \mathbb{P}_1 \otimes \mathbb{P}_2$on $\Omega_1 \times \Omega_2$.

Theorem 2.16.[3-6] Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and$(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$be two probability spaces. Thenthere exists a unique probability measure $\mathbb{P}$ on $\Omega_1 \times \Omega_2$equipped with the product σ $-$algebra F,such that for all $A_1 \in \mathcal{F}_1$ and $A_2 \in \mathcal{F}_2$, we have $\mathbb{P}(A_1 \times A_2) \coloneqq \mathbb{P}_1 \otimes \mathbb{P}_2(A_1 \times A_2) \coloneqq \mathbb{P}_1(A_1)\mathbb{P}_2(A_2)$.

Definition 2.17.[12] A subgroup H is **normal** in the group G if $gH = Hg$ for every $g \in G$.

Definition 2.18.[12] A function f from a group G to an other group $G^{'}$ is said to be homomorphism if $f(gh) = f(g)f(h)$ for every $g, h \in G$,and is said to be isomorphism if it is bijective and homomorphism.

Theorem 2.19. [12]. If H is a normal subgroup of the group$G$, then the system G/H form a group, known as **the quotient group** of G by H.

Theorem2.20[1,7,9]Let G be a topological group and H be a normal subgroup of G. Let G/H be the quotient space, endowed with the quotient topology, and nat_H be the natural map from G intoG/H. Then

(1) nat_H is onto;(2) nat_H is continuous;(3) nat_H is open and (4) nat_H homomorphism.

Definition 2.21[1,7,9]A topological group G is said to be **locallycompact** if there exists a relatively compact neighborhood of the identity element e of G.

Theorem 2.22[1,7,8,9]Let Λ be an index set. For each $\lambda \in \Lambda$, let G_λ be a topological group. Then $G \coloneqq \prod_{\lambda \in \Lambda} G_\lambda$, endowed with the product topology, is a topological group.

Theorem 2.23[1,7,8,9]Let $G \coloneqq \prod_{\lambda \in \Lambda} G_\lambda$ be the direct product of topological groups, endowed with the product topology. Then G is locally compact if all G_λ are compact topological groups except for a finite number of $\lambda_i (1 \leq i \leq n)$, say, and if for every those λ_i is a locally compact topological groups.

Definition2.24[1,7,8,9] A ***Haar measure*** is a Borel measure μ on a locally compact topological group G, such that

(a) $\mu(U) > 0$ for every non-empty Borel open set U, and

(b) $\mu(gE) = \mu(E)$ for every Borel set E.

Theorem2.25[1,7,8,9] In every locally compact topological group, there exists at least one regular Haar measure.

Note. Up to now all topological groups are locally compact with Haar measure.

RANDOM DYNAMICAL SYSTEMS

In this section we sate the definition of random dynamical system [5]. Also we introduce some new concepts that are need in section 3.

Definition3.1.[2]The 4-tuple $(\Omega,\mathcal{F},\mathbb{P},\theta)$ is called a ***model of the noise*** or a ***metric dynamical system*** if $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space and $\theta: G\times\Omega\to\Omega$ is $(\mathcal{B}(G)\otimes\mathcal{F},\mathcal{F})$–measurable, $\theta e,\omega=Id\Omega$, $\theta g*q,\omega=\theta(g,\theta q,\omega)$ and $\mathbb{P}(\theta g,F)=\mathbb{P}(F)$ for every $F\in\mathcal{F}$ and every $g\in G$ where G is a measurable group.

Definition3.2.[2]A ***continuous random dynamical system (RDS)***, shortly denote φon$X$ over a topological dynamical system $(\Omega,\mathcal{F},\mathbb{P},\theta)$ is a mapping

$\varphi: G\times\Omega\times X\to X,(g,\omega,x)\to\varphi(g,\omega,x)$

which is $(\mathcal{B}(G)\otimes\mathcal{F}\otimes\mathcal{B}(X),\mathcal{B}(X))$ −measurable and satisfies, for $\mathbb{P}$ −a.e. $\omega\in\Omega$,

(i) $\varphi(e,\omega,\cdot)$ is the identity on X;

(ii)$\varphi(g*q,\omega,\cdot)=\varphi(g,\theta(q,\omega),\varphi(q,\omega,\cdot))$;

(iii) $\varphi(g,\omega,\cdot): X\to X$ is continuous for all $g\in G$.

Where X is a topological space and G is a locally compact topological group with Haar measure.

Random systems are generated by systems of differential equations with random stationary coefficients or with a white noise.

We de not assume the maps $\varphi(g,\omega,\cdot)$ to be invertible a priori. By the cocycle property, $\varphi(g,\omega)$ is automatically invertible (for all $g\in G$ and for $\mathbb{P}$ −almost all ω) with $\varphi(g,\omega,\cdot)-1=\varphi(g-1,\theta g,\omega,\cdot)$ for $g\in G$.

In the following we define " ***fiber preserving action***" and the map that "***acts constantly***" given in [12] on a metric dynamical systems.

Definition3.3. The action of the metric dynamical system θ is said to be a ***fiber preserving action*** with respect to surjective measurable mapping ξ of a measurable space Ω onto the measurable space Ξ if the following condition satisfies : if $\xi(\omega_1)=\xi(\omega_2)$, then $\xi(\theta(g,\omega_1))=\xi(\theta(g,\omega_2))$ for every $\omega_1,\omega_2\in\Omega$ and $g\in G$.

Definition3.4. Let θ be a metric random dynamical system , H be a locally compact topological group and $\alpha: G\to H$ be a surjective map. Then $\alpha^{-1}(h)$ is said to be ***acts constantly*** on every point of Ω if $\theta(g_1,\omega)=\theta(g_2,\omega)$ for every $g_1,g_2\in\alpha^{-1}(h)$ and $\omega\in\Omega$.

GENERATING RANDOM DYNAMICAL SYSTEMS

In this section we create new random dynamical systems from the other one. This is an important tools in study the random dynamical systems.

Theorem4.1. Let (θ,φ) be a random dynamical system , H be a locally compact topological group ,Ξ be measurable space,$\Phi: X\to Y\alpha: G\to H$ be surjective open continuous maps , $\xi:\Omega\to\Xi$ be surjective measurable function. If $\alpha^{-1}(h)$ is acts constantly on every point of X and θ is fiber preserving action with respect to ξ, then (σ,ψ) is random dynamical system where

(1) (Ξ,Σ,Q) is probability space with $Q:\Sigma\to[0,1]$ defined by

$\mathbb{Q}(B):=\mathbb{P}(\xi^{-1}(B))$ for every $B\in\Sigma$.

(2) The action $\sigma: H\times\Xi\to\Xi$ is defined by

$\sigma(h,\varpi):=\xi(\theta(g,\omega))$ for every $(h,\varpi)\in H\times\Xi$.

(3)The mapping $\psi: H\times\Xi\times Y\to Y$ is defined by

$\psi(h,\varpi,y) \coloneqq \Phi(\varphi(g,\omega,x))$ for every $(h,\varpi,y) \in H \times \varXi \times Y$.
Proof. It is easy to see that the triple $(\varXi,\Sigma,Q)$ is probability space.
Now, we need to show that σ group action preserved the probability. First, note that σ is well define. For, let $\omega_1,\omega_2 \in \xi^{-1}(\varpi)$ and $g_1,g_2 \in \alpha^{-1}(h)$. Since $\alpha^{-1}(h)$ is acts constantly on every point of Ω, then
$\xi(\theta(g_1,\omega_1)) = \xi(\theta(g_2,\omega_1))$ …………………(1)
Also, since θ is fiber preserving action with respect to ξ, then
$\xi(\theta(g_2,\omega_1)) = \xi(\theta(g_2,\omega_2))$ …………………..(2)
From (1) and (2) we get $\sigma(h,\varpi) = \xi(\theta(g_1,\omega_1)) = \xi(\theta(g_2,\omega_2))$. Thus σ is well define. Second,Second, since the composition of two measurable function is measurable, then σ is $(\mathcal{B}(H)\otimes\Sigma,\Sigma)$ −measurable. Now, let e be the identity element of , e' be the identity element of H , $\varpi \in \varXi$ and $\omega \in \xi^{-1}(\varpi)$, then
$\sigma(e',\varpi) = \xi(\theta(e,\omega)) = \xi(\omega) = \varpi$.
To show that σ is associative. Let $h_1,h_2 \in H$, $\varpi \in \varXi$, $g_1 \in \alpha^{-1}(h_1)$, $g_2 \in \alpha^{-1}(h_2)$, and $\omega \in \xi^{-1}(\varpi)$.Then

$$\begin{aligned}\sigma(h_1,\sigma(h_2,\varpi)) &= \sigma(h_1,\xi(\theta(g_2,\omega)))\\ &= \xi(\theta(g_1,\theta(g_2,\omega)))\\ &= \xi(\theta(g_1 * g_2,\omega))\\ &= \sigma(\alpha(g_1 * g_2),\xi(\omega))\end{aligned}$$

$= \sigma(h_1 *' h_2,\xi(\omega))$.
To show that σ is preserved the probability. Let B be the measurable subset of $\varXi$ then $\xi^{-1}(B)$ is measurable subset of Ω. From the fact that $\sigma(h,\varpi) \coloneqq \xi(\theta(g,\omega))$, where $\omega \in$ *ξ−1(ϖ), g∈α−1(h)* we have *ξ−1σh,ϖ=θg,ω*. Thus

$$\begin{aligned}\mathbb{Q}(\sigma(h,B)) &= \mathbb{P}\left(\xi^{-1}(\sigma(h,B))\right)\\ &= \mathbb{P}\left(\theta(g,\xi^{-1}(B))\right)\end{aligned}$$

$= \mathbb{P}(\xi^{-1}(B)) = \mathbb{Q}(B)$.
Thus σ is preserved the probability. Thus σ is metric dynamical system.
Now, we need to show that the mapping $\psi: H \times \varXi \times Y \to Y$ is defined by $\psi(h,\varpi,y) \coloneqq \Phi(\varphi(g,\omega,x))$ for every$(h,\varpi,y) \in H \times \varXi \times Y$is cocycle. For $x \in \Phi^{-1}(y)$ and $\omega \in \xi^{-1}(\varpi)$,we have
$\psi(e',\varpi,y) = \Phi(\varphi(e,\omega,x)) = \Phi(x) = y$.
Let $h_1,h_2 \in H$, $\varpi \in \varXi$, $x \in \Phi^{-1}(y)$, $g_1 \in \alpha^{-1}(h_1)$, $g_2 \in \alpha^{-1}(h_2)$, and $\omega \in \xi^{-1}(\varpi)$.Then

$$\begin{aligned}\psi(h_1,\sigma(h_2,\varpi),\psi(h_2,\varpi,y)) &= \psi(h_1,\sigma(h_2,\varpi),\Phi(\varphi(g_2,\omega,x)))\\ &= \Phi(g_1,\theta(g_2,\omega),\Phi(\varphi(g_2,\omega,x)))\\ &= \Phi(\varphi(g_1 * g_2,\omega,x))\end{aligned}$$

$= \psi(h_1 *' h_2,\varpi,y)$.
Finally, we need to show that $\psi(.,\varpi,\cdot): Y \to Y$ is continuous for all $h \in H$. If $(h,\varpi,y) \in H \times \varXi \times Y$, then $\psi(h,\varpi,y) \in Y$. Let W be a neighborhood of $\psi(h,\varpi,y)$ in Y. We have $\psi(h,\varpi,y) \coloneqq \Phi(\varphi(g,\omega,x))$ where $\in \Phi^{-1}(y)$, $g \in \alpha^{-1}(h)$ and $\omega \in \xi^{-1}(\varpi)$. Since Φ is continuous, there exists a neighborhood W' of $\varphi(g,\omega,x)$ such that $\Phi(W') \subset W$. Since φ is continuous, there exist neighborhoods U' of x and V' of g such that $\varphi(V' \times \Omega \times U') \subset W'$. If $U = \Phi(U')$and $V = \alpha(V')$ then
$\psi(V \times \varXi \times U) \subset \Phi(\varphi(V' \times \Omega \times U')) \subset \Phi(W') \subset W$.

Thus ψ is continuous. Therefore (σ,ψ) is random dynamical system.■

Corollary4.2. Let (θ,φ) be a random dynamical system , S be a subgroup of G , H be a topological group and $(\Omega,\mathcal{F})$ be a measurable space. If $\alpha: G\to H$ be a surjective continuous open homomorphism such that every fiber with respect to α acts constantly on every point of X, then $(\alpha(M),X,\Omega,\theta_1,\varphi_1)$ is random dynamical system.

Proof. Define $\theta_1:\alpha(M)\times\Omega\to\Omega$ by $\theta_1(\alpha(m),\omega):=\theta(m,\omega)$ such definition is not well-define unless α acts constantly on every points of fiber $\alpha^{-1}(\alpha(m))$. Define $\varphi_1:\alpha(M)\times\Omega\times X\to X$ by

$\varphi_1(\alpha(m),\omega,x):=\varphi(m,\omega,x)$.

Suppose that $\Phi: X\to X$ and $\xi:\Omega\to\Omega$ be the identity maps. Then by above theorem we get the result.■

Corollary4.3. Let (θ,φ) be a random dynamical system , Y be a topological space . If $\Phi: X\to Y$ be a surjective continuous open map, $\xi:\Omega\to\Xi$ be surjective measurable function and (θ,φ) is a fiber preserving with respect to (ξ,Φ), then $(G,Y,\Xi,\theta_1,\varphi_1)$ is random dynamical system.

Proof. Define $\theta_1: G\times\Xi\to\Xi$ by $\theta_1(g,\varpi):=\theta(g,\omega)$, where $\omega\in\xi^{-1}(\varpi)$. This definition is well- define since θ is fiber preserving with respect to ξ. Also, define $\varphi_1: G\times\Xi\times Y\to\Xi$ by $\varphi_1(g,\varpi,y):=\varphi(g,\omega,x)$, where $\omega\in\xi^{-1}(\varpi)$ and $x\in\Phi^{-1}(y)$ This definition is well- define since φ is fiber preserving with respect to Φ. Consider $\alpha: G\to G$ as the identity function in above theorem we get the result.■

Corollary4.4. Let (θ,φ) be a random dynamical system , H be a normal subgroup of G. If every fiber with respect to nat_H acts constantly on every point of X, then $(G/H,X,\Omega,\theta_1,\varphi_1)$ is random dynamical system.

Proof. This follows from Theorem(2.17) and Corollary(4.4).■

Corollary4.5. Let (θ,φ) be a random dynamical system , S be a subgroup of G and H be a subgroup of G has action which is acts constantly on every point of X, then $(S*H,X,\Omega,\theta_1,\varphi_1)$ is random dynamical system.

Proof. Since $S*H$ is a subgroup of G and H is a normal subgroup of $S*H$, then the natural map $nat_{S*H}: S*H\to S*H/H$ satisfy the required properties.■

Corollary4.6. Let (θ,φ) be a random dynamical system , S be a subgroup of G and H be a subgroup of G has action which is acts constantly on every point of X, then $(S/S\cap H,X,\Omega,\theta_1,\varphi_1)$ is random dynamical system.

Proof. Since $S\cap H$ is a normal subgroup of S satisfy the required properties of Theorem(4.1).■

Corollary4.7. Let (θ,φ) be a random dynamical system , S and H be normal subgroups G such that $H\subset S$ and S has action which is acts constantly on every point of X, then $((G/H)/(S/H),X,\Omega,\theta_1,\varphi_1)$ and $(G/S,X,\Omega,\theta_2,\varphi_2)$ random dynamical systems.

Proof. Clear.■

Theorem4.8. Let (θ,φ) be a random dynamical system. Let $\Phi: X\to X$ be a homeomorphism and $\xi:\Omega\to\Omega$ be a bijective measurable function. Then (σ,ψ) is random dynamical system, where $\sigma: G\times\Omega\to\Omega$ defined by $\sigma(g,\omega):=\xi^{-1}(\theta(g,\xi(\omega))$ and $\psi: G\times\Omega\times X\to X$ defined by

$\psi(g,\omega,x):=\Phi^{-1}(\varphi(g,\omega,\Phi(x))$.

Proof. First we need to construct a metric dynamical system. Since $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space, then we can define a new probability space as follows: Define $\mathbb{Q}:\mathcal{F}\to[0,1]$ as follows $\mathbb{Q}(A)\coloneqq\mathbb{P}(\xi(A))$. It is easy to see that $\mathbb{Q}$ is well-define and it is probability function and consequently $(\Omega,\mathcal{F},\mathbb{Q})$ is probability space. Since ξbe a bijective measurable function, then so is its inverse ξ^{-1}. But the composition of two measurable functions is measurable, then $\sigma$is also measurable. Now, let e be the identity element of the group G and $\omega\in\Omega$. Then

$\sigma(e,\omega)=\xi^{-1}(\theta(e,\xi(\omega))=\xi^{-1}(\xi(\omega))=\omega.$

Let $g_1,g_2\in G$ and $\omega\in\Omega$. Then

$$\sigma(g_1,\sigma(g_2,\omega))=\sigma(g_1,\xi^{-1}(\theta(g_2,\xi(\omega)))$$
$$=\xi^{-1}(\theta(g_1,\xi(\xi^{-1}(\theta(g_2,\xi(\omega))))))$$
$$=\xi^{-1}(\theta(g_1,\theta(g_2,\xi(\omega))))$$

$=\xi^{-1}\left(\theta(g_1*g_2,\xi(\omega))\right)=\sigma(g_1*g_2,\omega).$

To show that σ preserved the probability, let $A\in\mathcal{F}$, then

$$\mathbb{Q}(\sigma(g,A))=\mathbb{P}(\xi(\xi^{-1}(\theta(g,\xi(A))))$$
$$=\mathbb{P}(\theta(g,\xi(A)))$$

$=\mathbb{P}(\xi(A))=\mathbb{Q}(A).$

Now, Let $g_1,g_2\in G$, $\omega\in\Omega$ and $x\in X$. Then

$$\psi(g_1,\sigma_{g_2}\omega,\psi(g_2,\omega,x))=\psi(g_1,\sigma_{g_2}\omega,\Phi^{-1}(\varphi(g_2,\omega,\Phi(x)))$$
$$=\psi(g_1,\omega',x')$$

where $x'=\Phi^{-1}(\varphi(g_2,\omega,\Phi(x))$ and $\omega'=\theta_{g_2}\omega$.

$$=\Phi^{-1}(\varphi\left(g_1,\omega',\Phi(x')\right))$$
$$=\Phi^{-1}(\varphi\left(g_1,\theta_{g_2}\omega,\varphi(g_2,\omega,\Phi(x)\right))$$
$$=\Phi^{-1}(\varphi(g_1*g_2,\omega,\Phi(x)))$$

$=\psi(g_1*g_2,\omega,x).$

Finally, since φ and Φ are continuous, then so is ψ. Thus (σ,ψ) is random dynamical system.■

Theorem4.9. If $(G_1,\Omega_1,X,\theta_1,\varphi_1),(G_2,\Omega_2,Y,\theta_2,\varphi_2)$are two random dynamical systems, then their product is a random dynamical system.

Proof. Wedefine the product of the given dynamical systems as follows:

$$(G_1\times G_2,\Omega_1\times\Omega_2,X\times Y,\theta_1\times\theta_2,\varphi_1\times\varphi_2),$$

where $\mathbb{P}\coloneqq\mathbb{P}_1\otimes\mathbb{P}_2:\mathcal{F}_1\otimes\mathcal{F}_2\longrightarrow[0,1]$

defined by $\mathbb{P}(A_1\times A_2)\coloneqq\mathbb{P}_1\otimes\mathbb{P}_2(A_1\times A_2)\coloneqq\mathbb{P}_1(A_1)\mathbb{P}_2(A_2)$,

$$\theta\coloneqq\theta_1\times\theta_2:G_1\times G_2\times\Omega_1\times\Omega_2:\to\Omega_1\times\Omega_2$$

define by $\theta((g,q),(\omega,\varpi))\coloneqq(\theta_1(g,\omega),\theta_2(q,\varpi))$,

and $\varphi\coloneqq\varphi_1\times\varphi_2:G_1\times G_2\times\Omega_1\times\Omega_2\times X\times Y\to X\times Y$,

defined by $\varphi((g,q),(\omega,\varpi),(x,y))\coloneqq(\varphi_1(g,\omega,x),\varphi_2(q,\varpi,y))$.

Proof. First, the triple $(\Omega_1\times\Omega_2,\mathcal{F}_1\otimes\mathcal{F}_2,\mathbb{P}_1\otimes\mathbb{P}_2)$ is probability space by Theorem2.16. To show that θ is a metric dynamical system. Note that by Theorems (2.22) and (2.23) we get $G_1\times G_2$ is locally compact topological groups. And a consequence of Theorem 2.25 there exists at least one regular Haar measure on $G_1\times G_2$.

(i) $\theta\left((e,e'),(\omega,\varpi)\right)=\left(\theta_1(e,\omega),\theta_2(e',\varpi)\right)=(\omega,\varpi)=Id_{\Omega_1\times\Omega_2}$.

(ii) $\theta\left((g,q)*(g',q'),(\omega,\varpi)\right)=\theta\left((g*g',q*q'),(\omega,\varpi)\right)$

$$= \left(\theta_1(g * g', \omega), \theta_2(q * q', \varpi)\right)$$
$$= (\theta_1(g, \theta_1(g', \omega)), \theta_2(q, \theta_2(q', \varpi)))$$
$$= \theta\left((g, q), (\theta_1(g', \omega), \theta_2(q', \varpi))\right)$$
$$= \theta\left((g, q), \theta((g', q'), (\omega, \varpi))\right).$$

(iii) Since $\mathcal{B}(G_1)\otimes\mathcal{F}_1\otimes\mathcal{B}(G_2)\otimes\mathcal{F}_2 = \mathcal{B}(G_1 \times G_2)\otimes\mathcal{F}_1\otimes\mathcal{F}_2$, then θ is $(\mathcal{B}(G_1 \times G_2)\otimes\mathcal{F}_1\otimes\mathcal{F}_2, \mathcal{F}_1\otimes\mathcal{F}_2)$ −measurable.

(iv) Let $\mathbb{P} \coloneqq \mathbb{P}_1\otimes\mathbb{P}_2$, then

$\mathbb{P}(\theta(h, A)) = \mathbb{P}_1(\theta_1(g, A_1))\mathbb{P}_2(\theta_2(q, A_2))$

$= \mathbb{P}_1(A_1)\mathbb{P}_2(A_2) = \mathbb{P}(A)$.

Then θ is a metric dynamical system. Now, to show that φ is cocycle.

(i') $\varphi(\tilde{e}, \tilde{\omega}, \tilde{x}) = \varphi\left((e, e'), (\omega, \varpi), (x, y)\right)$.

$$= (\varphi_1(e, \omega, x), \varphi_2(e', \varpi, y))$$
$$= (x, y) = id_{X\times Y}.$$

(ii') $\varphi\left(\tilde{g}, \theta_{\tilde{q}}\tilde{\omega}, \varphi(\tilde{h}, \tilde{\omega}, \tilde{x})\right) = \varphi(\tilde{g}, \theta_{\tilde{h}}\tilde{\omega}, (\varphi_1(g, \omega, x), \varphi_2(q, \varpi, y)))$

$= \left(\varphi_1\left(g, \theta_1(g', \omega), \varphi_1(g', \omega, x)\right), \varphi_2\left(q, \theta_2(q', \varpi), \varphi_2(q', \varpi, y)\right)\right)$

$= \left(\varphi_1(g * g', \omega, x), \varphi_2(q *' q', \varpi, y)\right)$

$= \varphi(\tilde{g} \tilde{*} \tilde{h}, \tilde{\omega}, \tilde{x})$.

(iii') Since φ_1 and φ_2 are continuous, then so is their product $\varphi_1 \times \varphi_2$. Thus from theabove discussion we get that the pair (θ, φ) form a random dynamical system and this complete the proof.

Remark4.10. Let (θ, φ) be a random dynamical system and let

$G_\varphi \coloneqq \{\varphi(g, \omega) : (g, \omega) \in G \times \Omega\}$,

then G_φ is a group. The closure of G_φ in function space X^X with point wise converge topology is semigroup and which is denoted by $E \coloneqq \overline{G_\varphi}$, i.e., $E \coloneqq \overline{\{\varphi(g, \omega) : (g, \omega) \in G \times \Omega\}}$

.

Proposition4.11. If (θ, φ) is a random dynamical system, then $(E, \circ)$ is semigroup.

Proof. Since G_φ is non-empty , then so is E. Now, let $f, h \in E$, then there exist nets $f_\alpha, h_\alpha \in G_\varphi$ such that $f_\alpha \to f$ and $h_\alpha \to h$. Then $h_\alpha \circ f_\alpha \to h_\alpha \circ f$. Since h_α is homeomorphism, then $h_\alpha \circ f$ is a net in E and $h_\alpha \circ f \to h \circ f$ and consequently $h \circ f \in E$. Since "$\circ$" is always associative , then E is semigroup.

Theorem4.12. If (θ, φ) is a random dynamical system, then so is (θ, ψ), where $\psi: G \times \Omega \times E \to E$ defined by $\psi(g, \omega, f) \coloneqq \varphi(g, \omega) \circ f$, and E be the semigroup defined in Remark 4.10.

Proof. It is sufficient to see that $\psi: G \times \Omega \times E \to E$ is cocycle. Let $e$be the identity element of the group$G$, then $\psi(e, \omega, f) = \varphi(e, \omega) \circ f$.

Since $\varphi(e, \omega) \circ f(x) = \varphi(e, \omega, f(x)) = f(x)$,

then $\varphi(e, \omega) \circ f = f$, i.e., $\psi(e, \omega, f) = f$.

Now,

$\psi(g, \theta_h\omega, \psi(h, \omega, f)) = \psi(g, \theta_h\omega, \varphi(h, \omega) \circ f)$

$= \psi(g, \theta_h\omega, f')$, $\quad f' \coloneqq \varphi(h, \omega) \circ f$

$= \varphi(g, \theta_h\omega) \circ f'$

$= \varphi(g, \theta_h \omega) \circ \varphi(h, \omega) \circ f$
$= \varphi(g * h, \omega) \circ f = \psi(g * h, \omega, f)$.

Finally, we need to show that $\psi(\cdot, \omega, \cdot): G \times E \to E$ is continuous. Let $\{g_\alpha\}$ and $\{f_\alpha\}$ be two nets in G and E repectively. Then $\{f_\alpha(x)\}$ be a net in X. Thus $\{(g_\alpha, f_\alpha(x))\}$ be a net in $G \times X$. Since $\varphi(\cdot, \omega, \cdot): G \times X \to X$ is continuous, then $\varphi(g_\alpha, \omega, f_\alpha(x)) \to \varphi(g, \omega, f(x))$ for every $x \in X$. Hence $\varphi(g_\alpha, \omega) \circ f_\alpha \to \varphi(g, \omega) \circ f$. Therefore
$\psi(g_\alpha, \omega, f_\alpha) \to \psi(g, \omega, f)$.

This means that $\psi(\cdot, \omega, \cdot): G \times E \to E$ is continuous. Thus$\psi: G \times \Omega \times E \to E$ is cocycle and consequently (θ, ψ) is a random dynamical system.

REFERENCES

[1] A. Arhangel'skii, and M.Tkachenko," Topological Groups and Related Struct-ures", Atlantis Press/ World Scientific, (2008)
[2] J.J. Rotman, " Advanced Modern Algebra", Prentice Hall; 2nd edition (2003)
[3] J.L.Kelley," General Topology", Springer-Verlag New York Beelin (1955)
[4] J.R. Munkres, " Topology", Prentice Hall, upper saddle Rever (2000)
[5] L. Arnold, " Random Dynamical Systems", Springer-Verlag Berlin Heidelberg (1998)
[6] P. Billingsley, " Probability and Measure" John Wiley and Sons,Inc.(1986)
[7] P. R.Halmos, " Measure Theory", Springer-Verlag New York Inc.(1974)
[8] R.B. Ash," Probability and Measure Theory", Academic Press, Inc. New York.(1978)
[9] R.B. Ash,"Real Analysis and Probability", Academic Press,Inc. NewYork.(2000)
[10] R.M. Dudley ,"Real Analysis and Probability", Cambridge University Press (2004)
[11] S.M. Ulam and J. von Neumann, Random ergodic theorems, Bull. Amer. Math. Soc. 51 (1945)
[12] S. Warner, " Topological Fields", Leopoldo Nachbin (1989)
[13] S.Willard," General Topology", Addison-Westy Pub.co., Inc.(1970)
[14] T. Husain,"Introduction toTopological groups" , W.B. Saunders Company Philadelphia And London (1966)
[15] W.H. Gottschalk, and G.A.Hedlund, "Topological Dynamics" , Amer, Math, Soci(1955)