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Integral Equation Methods for Pricing Perpetual Bermudan Options*

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Abstract

This paper develops integral equation methods to the pricing problems of perpetual Bermudan options. By mathematical derivation, the optimal exercise boundary of perpetual Bermudan options can be determined by an integral-form nonlinear equation which can be solved by a root-finding algorithm. With the computational value of optimal exercise, the price of perpetual Bermudan options is written by a Fredholm integral equation. A collocation method is proposed to solve the Fredholm integral equation and the price of the options is thus computed. Numerical examples are provided to show the reliability of the method, verify the validity of replacing the early exercise policies with perpetual American options, and explore a simplified computational process using the formulas for perpetual American options.

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1 Introduction

Perpetual American options are American options without expiry date, which means that the options can be exercised at any time in the lifetime. Perpetual Bermudan options are perpetual American options that can be exercised only on the predetermined dates. Perpetual American options and the early exercise boundaries have closed-form formulas (see e.g., Wilmott (1998), Kwok (1998) and Jiang (2005)). While there are no closed-form formulas for value and early exercise boundaries for Perpetual Bermudan options. In the history several papers developed numerical methods to price perpetual Bermudan options and determine the early exercise policies. Boyarchenko and Levendorski (2002) developed a Wiener-Hopf factorization method to price e perpetual Bermudan options. Fatthi (2002) proposed iterated integral methods to price perpetual Bermudan options. Muroi and Yamada (2006) studied finite difference methods for pricing perpetual Bermudan options. Lin and Liang (2007) investigated the binomial tree methods for pricing perpetual American and Bermudan options. Lin (2008) formulated perpetual Bermudan option pricing as a solution of a periodic Black-Scholes partial differential equation and obtained an integral formula for the valuation using contraction mapping theorem. Kay et al. (2009) investigated the early exercise region of perpetual Bermudan options with two underlying assets using iterated integral methods.

In this paper we propose an integral equation method for pricing perpetual Bermudan options. The value of perpetual Bermudan options satisfies a Fredholm integral equation with the early exercise boundary as the parameter. The early exercise boundary can be computed by solving an integral form nonlinear equation. Since perpetual Bermudan options approach to perpetual American options as the exercise time step goes to zero and perpetual American options have explicit valuation and closed-form early exercise policies, one may think if it is possible to replace the early exercise policies for perpetual Bermudan options by those for perpetual American options. We develop collocation methods for solving the Fredholm integral equations. We implement the algorithm and provide a table to verify the validity of replacement for the early exercise policies and investigate a simplified computational process using formulas for perpetual American options.

In the history for integral equation methods for solving American-style options, Kim (1990), Huang et al. (1996), Ju (1998), Detemple and Tian (2002) have studied the implementations of the integral equation methods for pricing American put options. However their approaches for solving the integral equations are based on low-order approximations and the numerical quadratures are used to evaluate the EEP (Early Exercise Premium) representation of the option price (see e.g., Detemple and Tian (2002)). Recently Ma et al. (2010, 2011) developed a high-order collocation method for solving the nonstandard integral equations satisfied by the early exercise boundary.

2 Problem statement

Assume that the underlying asset price follows a diffusion process

$$\frac{dS_t}{S_t} = r dt + \sigma dW_t .$$

where r denotes the interest rate, σ volatility, W_t Brownian motion. Let V be the value of Bermudan put options and θ be the optimal exercise boundary. Then the Bermudan put option pricing problem can be formulated by (see [4])

$$V(S) = \int_0^\theta G(S,\xi,\Delta T)(K-\xi)d\xi + \int_\theta^\infty G(S,\xi,\Delta T)V(\xi)d\xi\ , \quad S > \theta\ .$$

$$V(\theta) = K - \theta\ , \quad S = \theta\ . \tag{1}$$

$$V(\theta) = K - S\ , \quad S < \theta\ .$$

where G is Black-Scholes European Green's function

$$G(S,\xi,\Delta T) = \frac{\exp(-r\Delta T)}{\sigma\xi\sqrt{2\pi\Delta T}}\exp\{\frac{-\left(\ln\frac{S}{\xi}+(r-\frac{\sigma^2}{2})\Delta T\right)^2}{2\sigma^2\Delta T}\}\ ,$$

K is the strike price, and ΔT is Bermudan exercise time-step. Let

$$V^0(S) = \varphi(S,\theta,\Delta T) \equiv \int_0^\theta G(S,\xi,\Delta T)(K-\xi)d\xi\ .$$

Then we construct a sequence $\{V^k(S)\}_{k\geq 1}$, such that

$$V^k(S) = \varphi(S,\theta,\Delta T) + \int_\theta^\infty G(S,\xi,\Delta T)V^{k-1}(\xi)d\xi\ , \quad k = 1,2,.... \tag{2}$$

As derived by Lin (2008), $V^k(S)$ can be represented as

$$V^k(S) = \varphi(S,\theta,\Delta T) + \int_\theta^\infty \sum_{n=1}^k G^n(S,\xi,\Delta T)\varphi(\xi,\theta,\Delta T)d\xi\ , \quad k = 1,2,.... \tag{3}$$

where the sequence $\{G^n(S,\xi,\Delta T)\}_{n\geq 1}$ satisfies

$$G^1(S,\xi,\Delta T) = G(S,\xi,\Delta T)\ . \tag{4}$$

$$G^n(S,\xi,\Delta T) = \int_\theta^\infty G(S,\eta,\Delta T)G^{n-1}(\eta,\xi,\Delta T)d\eta\ , \quad n = 2,3,.... \tag{5}$$

Lin (2008) also proved that the sequence $\{V^k(S)\}_{k\geq 0}$ uniformly converges to $V(S)$ on the set $S \geq \theta$, i.e.,

$$V(S) = \varphi(S,\theta,\Delta T) + \int_\theta^\infty \sum_{n=1}^\infty G^n(S,\xi,\Delta T)\varphi(\xi,\theta,\Delta T)d\xi\ . \tag{6}$$

Taking $S = \theta$ into the above equation and using the second equation in (1), we obtain a nonlinear equation for the optimal exercise boundary θ:

$$K - \theta = \varphi(\theta,\theta,\Delta T) + \int_\theta^\infty \sum_{n=1}^\infty G^n(\theta,\xi,\Delta T)\varphi(\xi,\theta,\Delta T)d\xi\ . \tag{7}$$

Equation (7) will be solved by a root-finding algorithm – secant method (see Press (1992)). Equation (1), which is a Fredholm integral equation with the computed θ, will be solved by collocation methods (see Brunner (2004)).

3 Numerical methods

We first solve equation (7). Since equation (7) contains an infinite series in the integral, we need to truncate it into a finite sum. Denote

$$H(\theta,\xi,\Delta T)=\sum_{n=1}^{\infty}G^n(\theta,\xi,\Delta T), \qquad H_M(\theta,\xi,\Delta T)=\sum_{n=1}^{M}G^n(\theta,\xi,\Delta T).$$

When $M\to\infty$, it is known that $H_M\to H$. Therefore solution of equation (7) can be approximated by solving

$$K-\theta=\varphi(\theta,\theta,\Delta T)+\int_{\theta}^{\infty}H_M(\theta,\xi,\Delta T)\varphi(\theta,\xi,\Delta T)d\xi.$$

This equation is solved by secant method (a root-finding algorithm, see e.g., Press (1992)).

Denote the numerical solution of equation (7) by $\tilde{\theta}$, i.e., $\tilde{\theta}=\theta$. Then the option value can be obtained by solving

$$V(x)=\int_0^{\tilde{\theta}}G(x,\xi,\Delta T)(K-\xi)d\xi+\int_{\tilde{\theta}}^{\infty}G(x,\xi,\Delta T)V(\xi)d\xi, \tag{8}$$

with $x>\tilde{\theta}$ and $V(\tilde{\theta})=K-\tilde{\theta}$. A collocation method will be proposed to solve equation (8). The method is described as follows. Define a mesh:

$$I_h\equiv\{x_i=\tilde{\theta}+ih,\ i=0,1,\ldots,N-1\},$$

where h is predetermined mesh size, N is the number of mesh points, and denote $\sigma_n:=(x_n,x_{n+1}]$. Define a piecewise polynomial space:

$$V_{m-1}^{(-1)}\equiv\{v:v\big|_{\sigma_n}\in\pi_{m-1},\ n=1,2,\ldots,N-1\},$$

where π_{m-1} denotes $m-1$th order polynomial. Collocation method for solving

(8) is defined by

$$V_h(x) = \int_0^{\tilde{\theta}} G(x,\xi,\Delta T)(K-\xi)d\xi + \int_{\tilde{\theta}}^{\infty} G(x,\xi,\Delta T)V_h(\xi)d\xi \ , \qquad (9)$$

where $V_h \in V_{m-1}^{(-1)}$ is the computational solution, i.e.,

$$V_h \approx V \ , \qquad x \in X_h \equiv \{x_n + c_i h_n : \ \ 0 = c_1 < \ldots < c_m < 1; \ \ n = 0,1,2,\ldots,N-1\} \, .$$

Equation (9) is referred as collocation equation. Now we rewrite collocation equation (9) into a matrix form. For ease of exposition and actual computation, we take $m = 4$. Define collocation points

$$x_{i,j} = x_i + \frac{j}{4}(x_{i+1} - x_i) \, , \qquad j = 1,2,3,4 \, , \ \ i = 0,1,\ldots N-1 \, .$$

On the global mesh σ_i , polynomial V_h can be represented by

$$V_h(x) = \sum_{j=1}^{4} V_j^i l_j^i(x) \, , \qquad (10)$$

where $l_j^i(x)$ are the Lagrange basis functions at points $x_{i,j}$, $j = 1,2,3,4$, i.e.,

$$l_j^i(x) = \prod_{k \neq j} \frac{x - x_{i,k}}{x_{i,j} - x_{i,k}} \, .$$

Putting (10) into (9) gives that

$$\sum_{j=1}^{4} V_j^i l_j^i(x) = f(x,\tilde{\theta},\Delta T,K) + \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} G(x,\xi,\Delta T) \sum_{p=1}^{4} V_p^k l_p^k(\xi) d\xi \ , \qquad (11)$$

where $x \in X_h$, $f(x,\tilde{\theta},\Delta T,K) = \int_0^{\tilde{\theta}} G(x,\xi,\Delta T)(K-\xi)d\xi$.

Taking $x = x_{i,j}$, $j = 1,2,3,4$, $i = 1,2,\ldots,N-1$, equation (11) can be rewritten into the form

$$V_j^i = f(x_{i,j},\tilde{\theta},\Delta T,K) + \sum_{k=0}^{N-1} \sum_{p=1}^{4} [\int_{x_k}^{x_{k+1}} G(x_{i,j},\xi,\Delta T) \, l_p^k(\xi) \, d\xi] V_p^k \, . \qquad (12)$$

This can be further simplified by

$$\mathbf{AV} = \mathbf{F} \ , \qquad (13)$$

where

$$\mathbf{A} = \left(A^{i,q} \right)_{i,q=0,1,2,\ldots,N-1},$$

$$\mathbf{V} = \left(V^0, V^1 \ldots, V^{N-1} \right)^T,$$

$$\mathbf{F} = \left(F^0, F^1, \ldots, F^{N-1} \right)^T,$$

$A^{i,q}$ is a 4×4 matrix, V^q, F^q are four-dimensional row vectors. The expressions are given by:

$$A^{i,q} = \left(A_1^{i,q}, A_2^{i,q}, A_3^{i,q}, A_4^{i,q} \right), \quad i \neq q$$

with

$$A_1^{i,q} = \begin{pmatrix} -\int_{x_q}^{x_{q+1}} G(x_{i,1},\xi,\Delta T) l_1^q(\xi) d\xi \\ -\int_{x_q}^{x_{q+1}} G(x_{i,2},\xi,\Delta T) l_1^q(\xi) d\xi \\ -\int_{x_q}^{x_{q+1}} G(x_{i,3},\xi,\Delta T) l_1^q(\xi) d\xi \\ -\int_{x_q}^{x_{q+1}} G(x_{i,4},\xi,\Delta T) l_1^q(\xi) d\xi \end{pmatrix}, \quad A_2^{i,q} = \begin{pmatrix} -\int_{x_q}^{x_{q+1}} G(x_{i,1},\xi,\Delta T) l_2^q(\xi) d\xi \\ -\int_{x_q}^{x_{q+1}} G(x_{i,2},\xi,\Delta T) l_2^q(\xi) d\xi \\ -\int_{x_q}^{x_{q+1}} G(x_{i,3},\xi,\Delta T) l_2^q(\xi) d\xi \\ -\int_{x_q}^{x_{q+1}} G(x_{i,4},\xi,\Delta T) l_2^q(\xi) d\xi \end{pmatrix},$$

$$A_3^{i,q} = \begin{pmatrix} -\int_{x_q}^{x_{q+1}} G(x_{i,1},\xi,\Delta T) l_3^q(\xi) d\xi \\ -\int_{x_q}^{x_{q+1}} G(x_{i,2},\xi,\Delta T) l_3^q(\xi) d\xi \\ -\int_{x_q}^{x_{q+1}} G(x_{i,3},\xi,\Delta T) l_3^q(\xi) d\xi \\ -\int_{x_q}^{x_{q+1}} G(x_{i,4},\xi,\Delta T) l_3^q(\xi) d\xi \end{pmatrix}, \quad A_4^{i,q} = \begin{pmatrix} -\int_{x_q}^{x_{q+1}} G(x_{i,1},\xi,\Delta T) l_4^q(\xi) d\xi \\ -\int_{x_q}^{x_{q+1}} G(x_{i,2},\xi,\Delta T) l_4^q(\xi) d\xi \\ -\int_{x_q}^{x_{q+1}} G(x_{i,3},\xi,\Delta T) l_4^q(\xi) d\xi \\ -\int_{x_q}^{x_{q+1}} G(x_{i,4},\xi,\Delta T) l_4^q(\xi) d\xi \end{pmatrix};$$

$$A^{i,i} = \left(A_1^{i,i}, A_2^{i,i}, A_3^{i,i}, A_4^{i,i} \right),$$

with

$$A_1^{i,i} = \begin{pmatrix} 1-\int_{x_i}^{x_{i+1}} G(x_{i,1},\xi,\Delta T) l_1^i(\xi) d\xi \\ -\int_{x_i}^{x_{i+1}} G(x_{i,2},\xi,\Delta T) l_1^i(\xi) d\xi \\ -\int_{x_i}^{x_{i+1}} G(x_{i,3},\xi,\Delta T) l_1^i(\xi) d\xi \\ -\int_{x_i}^{x_{i+1}} G(x_{i,4},\xi,\Delta T) l_1^i(\xi) d\xi \end{pmatrix}, \quad A_2^{i,i} = \begin{pmatrix} -\int_{x_i}^{x_{i+1}} G(x_{i,1},\xi,\Delta T) l_2^i(\xi) d\xi \\ 1-\int_{x_i}^{x_{i+1}} G(x_{i,2},\xi,\Delta T) l_2^i(\xi) d\xi \\ -\int_{x_i}^{x_{i+1}} G(x_{i,3},\xi,\Delta T) l_2^i(\xi) d\xi \\ -\int_{x_i}^{x_{i+1}} G(x_{i,4},\xi,\Delta T) l_2^i(\xi) d\xi \end{pmatrix},$$

$$A_3^{i,i} = \begin{pmatrix} -\int_{x_i}^{x_{i+1}} G(x_{i,1},\xi,\Delta T) l_3^i(\xi) d\xi \\ -\int_{x_i}^{x_{i+1}} G(x_{i,2},\xi,\Delta T) l_3^i(\xi) d\xi \\ 1-\int_{x_i}^{x_{i+1}} G(x_{i,3},\xi,\Delta T) l_3^i(\xi) d\xi \\ -\int_{x_i}^{x_{i+1}} G(x_{i,4},\xi,\Delta T) l_3^i(\xi) d\xi \end{pmatrix}, \quad A_4^{i,i} = \begin{pmatrix} -\int_{x_i}^{x_{i+1}} G(x_{i,1},\xi,\Delta T) l_4^i(\xi) d\xi \\ -\int_{x_i}^{x_{i+1}} G(x_{i,2},\xi,\Delta T) l_4^i(\xi) d\xi \\ -\int_{x_i}^{x_{i+1}} G(x_{i,3},\xi,\Delta T) l_4^i(\xi) d\xi \\ 1-\int_{x_i}^{x_{i+1}} G(x_{i,4},\xi,\Delta T) l_4^i(\xi) d\xi \end{pmatrix};$$

$$F^q = (F_1^q, F_2^q, F_3^q, F_4^q)^T = \begin{pmatrix} f(x_{i,1},\theta,\Delta T,K) \\ f(x_{i,2},\theta,\Delta T,K) \\ f(x_{i,3},\theta,\Delta T,K) \\ f(x_{i,4},\theta,\Delta T,K) \end{pmatrix}; \quad V^q = (V_1^q, V_2^q, V_3^q, V_4^q)^T .$$

4 Numerical examples

In this section, two examples are implemented using the method in this paper. Numerical tests are carried out to investigate the validity of replacing the early exercise policy for perpetual Bermudan options by that for perpetual American options and explore a simplified computational process using formulas for perpetual American options. In the presentation of the numerical results, we use the following notations:

- $V_A(S)$: Value of perpetual American options at underlying price S;
- $V_B(S)$: Value of perpetual Bermudan options at underlying price S;
- $V_{Ba}(S)$: Value of perpetual Bermudan options with the early exercise policy of perpetual American options at underlying price S;
- θ_A: Early exercise boundary of perpetual American options;
- θ_B: Early exercise boundary of Perpetual Bermudan options.

Perpetual American options have the following closed-form formulas (see e.g., Wilmott (1998) and Jiang (2005))

$$\theta_A = \frac{2rK}{2r+\sigma^2}, \qquad V_A(S) = \frac{\sigma^2}{2r}\left(\frac{2rK}{2r+\sigma^2}\right)^{\frac{2r+\sigma^2}{\sigma^2}} S^{-\frac{2r}{\sigma^2}}. \tag{14}$$

Example 4.1 Consider perpetual Bermudan options with interest rate $r = 10\%$, strike price K=100, exercise time-step $\Delta\text{T=0.25}, 0.5, 1, 1.5$, volatility $\sigma\text{=20\%}$.

Figure 1 shows that fact that perpetual Bermudan options converge to perpetual American options as the exercise time-step $\Delta T \to 0$. Table 1 investigates the validity of simplifying the computation of perpetual Bermudan options using the formulas for perpetual American options. Since perpetual American options have explicit formulas for early exercise boundary and valuation, it will be important in practice to investigate if either the computation of early exercise boundary or the valuation of perpetual Bermudan options can be realized by the formulas for perpetual American options. From Table 1, if the early exercise boundary for Bermudan θ_B is replaced by that for American θ_A, then the computation of equations (7) can be avoided and the value function of Bermudan $V_{Ba}(S)$ can be obtained by computing equation (1). In this case and $\Delta T\text{=0.25}$ (see the 2nd column of Table 1), the value of $V_{Ba}(S)$ at $S = \theta_A$ is $V_{Ba}(\theta_A) = 15.0758$, while the true value of Bermudan at $S = \theta_A$ is $V_B(\theta_A) = 15.3397$. This means that such a replacement is acceptable. In the other case, if the early exercise policy for Bermudan is determined by solving equation (7) and the valuation of Bermudan is computed by formula for American (14), then the value of Bermudan at $S = \theta_B$ by formula (14) is $V_A(\theta_B) = 12.8394$ and the true value of Bermudan is $V_B(\theta_B) = 12.1108$ (see the 2nd column of Table 1). This indicates that such replacement is also acceptable. However Table 1 tells us that it is not acceptable to use all the formulas for American (14) to compute both the early exercise boundary and value of Bermudan.

Table 1: Numerical results for Example 4.1

ΔT	0.25	0.5	1	1.5
θ_A	83.333333333333	83.333333333333	83.333333333333	83.33333333333
θ_B	87.796918308567	89.409109274514	91.448909175584	92.825417152075
$V_{Ba}(\theta_A)$	15.0758	13.8163	11.7972	10.2025
$V_B(\theta_A)$	15.3397	14.1936	12.2617	10.6795
$V_B(\theta_B)$	12.1108	10.5660	8.5262	7.1514
$V_A(\theta_B)$	12.8394	11.7230	10.4726	9.7188
$V_A(\theta_A)$	16.6667	16.6667	16.6667	16.6667

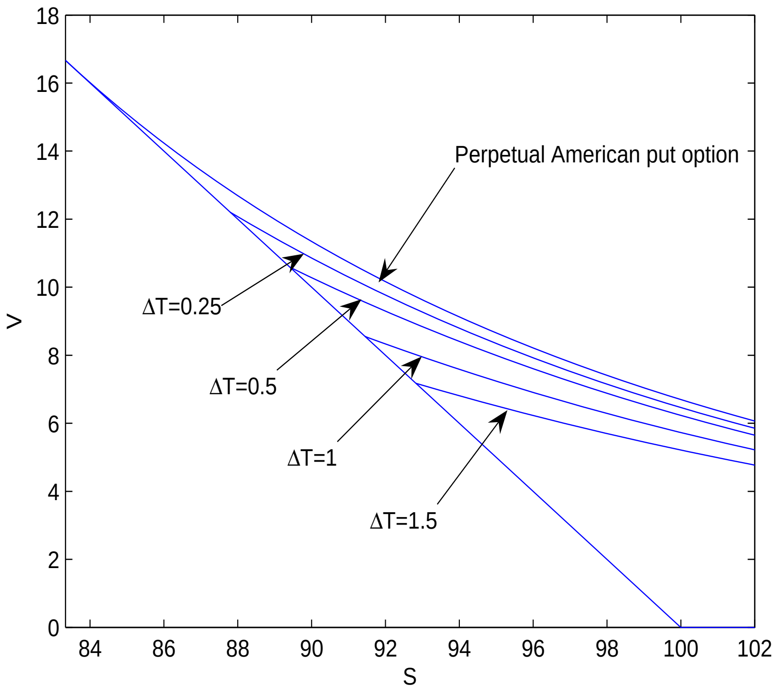

Figure 1: Value of perpetual Bermudan options for Example 4.1

Example 4.2 Consider perpetual Bermudan options with interest rate $r = 1\%$, strike price $K=100$, exercise time-step $\Delta T=1, 5, 10, 20$, volatility $\sigma=20\%$.

Compared to Example 4.1, this example considers a significantly lower interest rate. From the numerics in the 2nd column of Table 2, the values of $V_{Ba}(\theta_A) = 66.0095$, $V_B(\theta_A) = 65.6931$ and $V_A(\theta_A) = 66.6667$ are close. Hence besides the observations made in Example 4.1, it is also concluded that formulas for American (14) can be used to compute both early exercise boundary and valuation of Bermudan in this example.

Table 2: Numerical results for Example 4.2

ΔT	1	5	10	20
θ_A	33.333333333333	33.333333333333	33.333333333333	33.333333333333
θ_B	37.899230264854	43.167372946224	47.320491107509	53.401831505158
$V_{Ba}(\theta_A)$	66.0095	63.4754	60.4555	59.8509
$V_B(\theta_A)$	65.6931	62.9028	59.7904	54.1741
$V_B(\theta_B)$	61.3160	55.4275	51.0110	44.7779
$V_A(\theta_B)$	62.5220	58.5828	55.9530	52.6708
$V_A(\theta_A)$	66.6667	66.6667	66.6667	66.6667

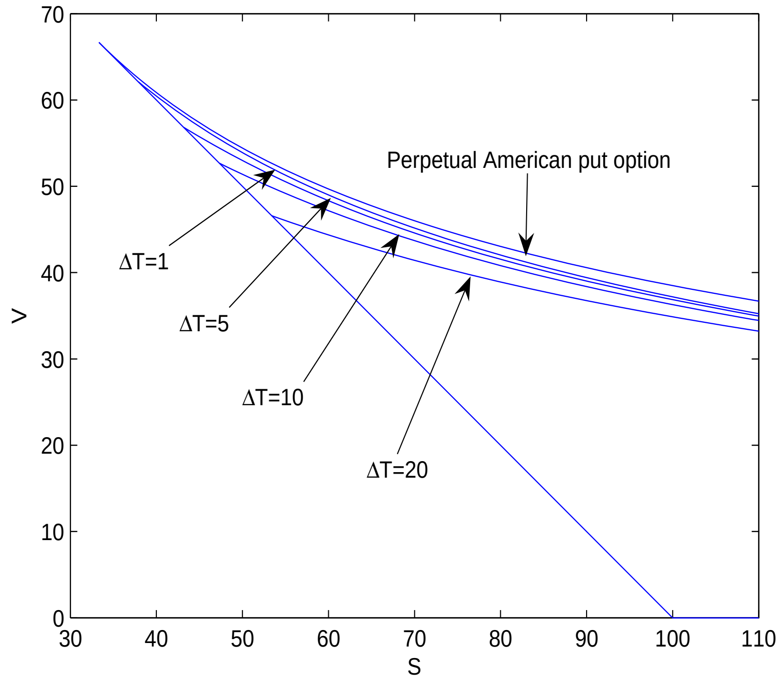

Figure 2: Value of perpetual Bermudan options for Example 4.2

5 Conclusion

In this paper we studied the integral equation methods for valuing perpetual Bermudan options, which are significantly different from the iterated integral methods developed by Fattahi (2002) and Kay et al. (2009). We developed collocation methods to solve the Fredholm integral equations which characterize the value of perpetual Bermudan options. By implementing two examples, we provided numerical tables to investigate a simplified computational process using formulas for perpetual American options and verify the validity of replacing Bermudan with American.

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