# Folk Theorem with Communication*

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**Abstract**

This paper proposes an alternative folk theorem for repeated games with private monitoring and communication, extending the technique of Compte (1998) to the case where private signals are correlated.

## 1 Introduction

We have observed a significant progress in repeated games with private monitoring in the last few years. It started with a series of papers which proved a folk theorem with communication, such as Ben-Porath and Kahneman [3], Compte [5], and Kandori and Matsushima [11]. Since there are a variety of Economic environments in which private monitoring is the natural assumption and players can communicate, these results are very important contributions to the theory of long term relationships, especially given that repeated games with private monitoring are very difficult to analyze without communication. These folk theorems, however, do not cover all the interesting cases because of the specific assumptions on private monitoring structure. This paper proves a folk theorem based on a new assumption on private monitoring to expand the range of environments to which the folk theorem result applies.

Some of these papers are based on and extends the technique from the preceding literature on repeated games with public monitoring. For example, Kandori and Matsushima [11] is a natural extension of Fudenberg, Levine, and Maskin [9]. Compte [5] is based on the idea of Abreu, Milgrom and Pearce [1]. These two ideas are distinct and complementary, the

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former emphasizing the use of transfer as costless punishment and the latter emphasizing efficient statistical test which makes the cost of punishment negligible.[1][2] This paper belongs to the second group. It proposes an alternative (Nash-threats) folk theorem for repeated games with private monitoring and communication, extending the technique of Compte [5].

The contribution of this paper is twofold. First, it extends the idea of Compte [5] to the case where private signals are correlated. To employ a better statistical test, players store their private information for T periods and release them at the end of each T-period block. Compte [5] (also Matsushima [14]) assumes that private signals are essentially independent so that players do not learn anything about the other players' private history until the end of each block.[3] This is because otherwise some player may realize in the middle of the T periods that punishment is unlikely to be triggered and start deviating. To avoid this, more harsh punishment needs to be used so that players do not have an incentive to deviate even after such history. But then, the expected punishment in each period becomes too large to approximate efficiency. Even almost independence does not solve this problem. Since the length of each block needs to go to infinity to obtain an exact folk theorem, even a slight correlation of private signals may become a serious obstacle to this type of construction. I provide a way to circumvent this problem, while keeping the spirit of Compte [5]'s original construction.

Second, the number of the players can be two. Ben-Porath and Kahneman [3] assumes that, for each player, there are at least two players who observe the player's action perfectly. These monitors have no incentive to lie because they are severely punished when their messages do not coincide. This logic requires at least three players. In Compte [5] and Kandori and Matsushima [11], punishment is sometimes based on transfer between two players. Transfer from player $i$ to player $j$ is based on other players' messages and not on $i$ or $j$'s. This is to keep the transfer for each player independent of her own message so that revelation constraints are trivially satisfied. This trick also requires at least three players. In this note, transfer

[1] Review strategy (Radner [16] and, more recently, Matsushima [14]) is also based on the same idea.

[2] However, this distinction is not clear-cut. Kandori and Matsushima [11] devotes one section to AMP-type strategy for repeated prisoners' dlemma and Compte [5] also uses transfer in his proof.

[3] However, Compte [5] allows correlation of private signals off the equilibrium path where players deviate from the equilibrium action. Matsushima [14] also allows a correlation between private signals caused by unobserved macro shock.

is based on every player's announcement. In particular, a player's message can affect her transfer. However, it is still optimal for each player to be honest because she expects that her announcement would "look odd" with the other players' announcement and get punished accordingly if she lies.

There are other related folk theorems besides the above three papers. Fudenberg and Levine [8] proves Nash-threat folk theorem when players' private signals are highly correlated ("approximate common knowledge"). It focuses on the two player case for the reason stated above. Ashkenazi-Golan [2] assumes that all the relevant deviations are perfectly observable by some player with positive probability as in Ben-Porath and Kahneman [3]. Her folk theorem is also a Nash-threat folk theorem unlike Ben-Porath and Kahneman's, but she does not need two monitors for each deviation, which allows her folk theorem to apply to the two players case. Finally, McLean, Obara and Postlewaite [15] also proves a folk theorem when private signals are correlated and can be treated like a public signal when they are aggregated. At least three players are required for this result.

I also should mention that many folk theorem results without communication were obtained recently. However, most of them assumes almost perfect monitoring (Bhaskar and Obara [4], Ely and Välimäki [6], Hörner and Olszewski [10], Mailath and Morris [12][4]). One exception is Matsushima [14], which allows private monitoring to be noisy. However it assumes a certain type of independency between private signals as in Compte [5].

The next section presents the model briefly. Section 3 introduces the assumptions on monitoring structure, which are then compared to the assumptions in other literature. Section 4 presents the main result and Section 5 discusses some extension.

## 2 Model

The set of the players is $I = \{1, 2, ..., n\}, n \geq 2$. In each period, player $i$ chooses an action from a finite action set $A_i$ and receives a private signal $s_i$ from a finite set $S_i$. Both $a_i$ and $s_i$ is private information. In the end of each period, players send their message $m = (m_1, ..., m_n) \in \prod_{i=1}^{n} M_i$ simultaneously, which becomes pubic information. Let $p(s|a)$ be a joint signal distribution on $S = \prod_{i=1}^{n} S_i$ given $a \in A$. It is assumed that $p(s|a)$ has a full support on $S_i$ for every $a$.[5] Signal distribution on $S_M = \sum_{j \in M} S_j$ for player

[4] See also Mailath and Morris[13]

[5] This is weaker than the assumption of full support on $S$. These two assumptions are equivalent when private signals are conditionally independent, as assumed in Compte [5].

$i$ given $a \in A$ (and $s_i \in S_i$) is denoted by $p_M(s_M|a)$ $(p_M(s_M|a, s_i))$. In particular, $p_{-i}(s_{-i}|a)$ is used when $M = I/\{i\}$. Let $g_i(a) = \sum u_i(a_i, s_i)\, p(s|a)$ be player $i$'s expected stage-game payoff and $g^*$ be a Nash equilibrium payoff profile. Let $V = co\{g(a), a \in A\}$ be the feasible payoff set, which is assumed to be full dimensional, and define $V^*$ with respect to $g^*$ as follows;

$$V^* = \{v \in V : \forall i, v_i \geq g_i^*\}.$$

Note that $V^*$ would be different for a different choice of $g^*$.

Private history and public history at period $t$ is $(a_{i,1}, s_{i,1}, ..., a_{i,t-1}, s_{i,t-1})$ and $(m_1, ..., m_{t-1})$ respectively. Player $i$'s *strategy* $\sigma_i$ consists of *action strategy* $\sigma_i^a$ and *message strategy* $\sigma_i^m$. Player $i$'s *action strategy* $\sigma_i^a$ is a collection of $t$-period behavior strategies $(\sigma_{i,t}^a, t = 1, 2, ...,)$, which maps $t-$period private and public history into $\triangle A_i$. Player $i$'s *message strategy* $\sigma_i^m$ is a collection of $t$-period behavior strategies $(\sigma_{i,t}^m,\ t = 1, 2, ...,)$, which maps $t+1-$period private history and $t-$period public history into $M_i$. Later I use some public randomization device, thus strategies also need to depend on the realization of a randomization device. But it is not made explicit here to save notations. It will be mentioned when public randomization is necessary.[6] The players discount future by discount factor $\delta \in (0, 1)$ and maximize their discounted average payoff $E\left[(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}g_i(a_t)\right]$ as usual.

Let $\Lambda = \{\boldsymbol{\lambda} \in \Re^n |\, \|\boldsymbol{\lambda}\| = 1\}$ be the space of weights and $\Lambda_{\varepsilon+} = \{\boldsymbol{\lambda} \in \Lambda | \exists i,\ \lambda_i \geq \varepsilon\}$. For each weight $\boldsymbol{\lambda} \in \Lambda$, let $a^{\boldsymbol{\lambda}}$ be an action profile to maximize the weighted sum of the players' payoff with respect to $\boldsymbol{\lambda}$, that is,

$$a^{\boldsymbol{\lambda}} = \arg\max_{a \in A} \boldsymbol{\lambda} \cdot (g_1(a), ..., g_n(a)).$$

For each $\lambda$, let $I^{\lambda} \subset I$ be the subset of the players with a strictly positive weight.

## 3 The Assumptions on Monitoring Structure

In this section, a few conditions on the monitoring structure are introduced. All the conditions are stated with respect to some fixed action profile as they are not required for all the action profiles for the folk theorem.

First, I assume that each player's deviation can be statistically detectable at relevant action profiles. Let $Q_i(a)$ be the convex-hull of probability vectors $\{p_{-i}(\cdot|a_i', a_{-i})\, |a_i' \neq a_i\}$ on $S_{-i}$ given $a \in A$. Then this assumption,

[6] Note, however, that it is possible to generate any public randomization device endogenenously through communication by, for example, redefining $i$'s message space as $M_i \times [0, 1]$ for each $i \in I$

given $a \in A$ and $i \in I$, can be formally stated as follows.

$$p_{-i}(\cdot|a) \notin Q_i(a) \tag{A1}$$

This is standard and satisfied in any paper, including Compte [5] and Kandori and Matsushima [11].

It is possible to use an alternative condition here. Let $R_M(a, s_i)$ be the convex-hull of conditional probability vectors $\{p_M(\cdot|(a_i', a_{-i}), s_i') \,|\, (a_i', s_i') \neq (a_i, s_i)\}$ for $M \subset I$ given $a \in A$ and $s_i \in S_i$. Then this alternative condition for $a \in A$ and $i \in I$ can be stated as follows.

$$\begin{aligned} \text{There exists } s_i &\in S_i \text{ and } M \subset I/\{i\} \text{ such that} \\ p_M(\cdot|a, s_i) &\notin R_M(a, s_i) \end{aligned} \tag{A1'}$$

This is neither stronger nor weaker than A1. This $M$ is called *monitor* of player $i$ at $a \in A$. This is different from the "monitor" in Compte [5], which is based on unconditional probabilities.(See footnote 5 for definition). It is unbiased (monitor of $i$ does not include $i$) as in Compte [5], but not independent, that is, player $i$'s private signal contains some information about the private signals of the players in $M$. Compte [5] assumes that $p_{-i}(\cdot|a) = p_{-i}(\cdot|a, s_i)$, which cannot be satisfied here. Thus this condition is complementary to Compte's.

Both assumption guarantees that $a$ is enforceable, through payoffs contingent on $s_{-i}$ for A1 and through payoffs contingent on $s$ for A1', while the truth-telling constraints for $i$ are satisfied.

**Lemma 1** *(1) If (A1) is satisfied at $a \in A$ for $i \in I$, then there exists $x_i : S_{-i} \to \Re$ such that*

$$\begin{aligned} \sum_{s_{-i}} p_{-i}(s_{-i}|a) \cdot x_i(s_{-i}) &> \sum_{s_{-i}} p_{-i}\left(s_{-i}|\left(a_i', a_{-i}\right)\right) \cdot x_i(s_{-i}) \\ \textit{for all } a_i' &\neq a_i \end{aligned}$$

*(2) If (A1') is satisfied at $a \in A$ for $i \in I$, then there exists $x_i : S \to \Re$ such that*

$$\begin{aligned} \sum_{s} p(s|a) \cdot x_i(s) &> \sum_{s} p\left(s|\left(a_i', a_{-i}\right)\right) \cdot x_i(\eta_i(s_i), s_{-i}) \\ \textit{for all } a_i' &\neq a_i \textit{ and } \eta_i \end{aligned} \tag{1}$$

*and*

$$\begin{aligned} \sum_{s} p(s|a) \cdot x_i(s) &\geq p(s|a) \cdot x_i(\eta_i(s_i), s_{-i}) \\ &\textit{for all } \eta_i \end{aligned} \tag{2}$$

*where $\eta_i$ is a mapping from $S_i$ to $S_i$.*

**Proof.** Since (1) is proved in Compte [5] and others, I focus on (2). Suppose that A1' is satisfied at $\widetilde{s}_i$. Set $x_i(s) = 0$ when $s_i \neq \widetilde{s}_i$. When $s_i = \widetilde{s}_i$, $x_i(\widetilde{s}_i, s_{-i})$ is defined so as to satisfy $\sum_{s_{-i}} p(s_{-i}|a, \widetilde{s}_i) \cdot x_i(\widetilde{s}_i, s_{-i}) > 0 > \sum_{s_{-i}} p(s_{-i}|(a'_i, a_{-i}), s'_i) \cdot x_i(\widetilde{s}_i, s_{-i})$ for all $(a'_i, s'_i) \neq (a_i, s_i)$. This can be done by hyperplane theorem. Then (1) is clearly satisfied for all all $a'_i \neq a_i$ and $\eta_i$ because player $i$ can expect a positive expected payoff only when she plays $a_i$ and truthfuly reveals $\widetilde{s}_i$. It is also clear that (2) is satisfied for all $\eta_i$ because player $i$'s expected payoff from lying is at most 0 conditional on any $s_i \in S_i$. ■

To enforce any action profile which satisfies either (A1) or (A1)', $x_i(s_{-i})$ or $x_i(s)$ can be used as continuation payoffs (by scaling them up if necessary). Information revelation constraints are satisfied automatically for the former because it does not depend on $s_i$. For the latter, truthful information revelation *on the equilibrium path* is guaranteed by (2). In the following, *it is assumed that either (A1) or (A1') is satisfied at all the action profiles.*

The next condition, which is a stronger form of (A1'), is the focus of this paper.

**Assumption. Existence of Contingent Monitor**

$$\begin{aligned} \text{For any } s_i &\in S_i, \text{ there exists } M \subset I/\{i\} \text{ such that} \\ p_M(\cdot|a, s_i) &\notin R_M(a, s_i) \end{aligned} \tag{A2}$$

This means that there exists a monitor for every realization of the private signal for $i$ at $a$. Monitor can be different for different private signals. Again it requires that player $i$'s information is correlated with her monitor's private signals. I say that there exists a contingent monitor for $i$ at $a$ or $i$ has a contingent monitor at $a$ when this condition is satisfied for $i$ at $a$.

The following lemma can be derived from this assumption.

**Lemma 2** *Suppose that there exists a contingent monitor for player $i \in I$ at $a \in A$. Then, for any $\pi \in (0,1)$, there exists $q_i(s) \in (0,1)$ such that, for every $s_i \in S_i$,*

$$\begin{aligned} \pi &= \sum_{s_{-i}} q_i(s) p_i(s_{-i}|a, s_i) \\ &< \sum_{s_{-i}} q_i(s) p_i\left(s_{-i}|\left(a'_i, a_{-i}\right), s'_i\right) \text{ for all } \left(a'_i, s'_i\right) \neq (a_i, s_i) \end{aligned}$$

**Proof.** There exists such $q_i(s)$ to satisfy the above strict inequalities by hyperplane theorem. Such $q_i(s)$ can be chosen so that the conditional expectation of $q_i$ is $\pi \in (0,1)$ for each $s_i$ by adding an appropriate constant to $q_i(s)$. If this $q_i(s)$ is not in $[0,1]$, then redefine $q_i(s)$ as $\widetilde{q_i}(s) = (q_i(s) + n\pi)\frac{1}{n+1}$. Then $\widetilde{q_i}(s)$ still satisfies all the equalities and inequalities, and lies between 0 and 1 if $n$ is large enough. ■

This $q_i(s)$ can be interpreted as a probability of "bad signal" given the announced private signal profile $s$. Then this lemma implies that the probability of a bad signal increases for any deviation from $a$ and/or misrepresentation of the private signal by player $i$.

This condition (A2) is not stronger than any existing sufficient condition for the folk theorem. Note first that only one monitor is needed for each player. Compte [5] assumes that there are at least two independent monitors $M$ and $M'$ for each player which are unbiased and do not overlap ($i \notin M, M'$ and $M \cap M' = \emptyset$).[7] This requires at least three players. Kandori and Matsushima [11] also needs at least three players because player $i$'s deviation needs to be statistically distinguished from player $j$'s deviation through the private signals of an independent monitor (= all the other players). Ben-Porath and Kahneman [3] also needs at least three players as explained in the introduction. Also note that these assumptions do not presume any type of almost public monitoring or almost common knowledge (Fudenberg and Levine [8], Mailath and Morris [12], and McLean, Obara and Postlewaite [15]).

On the other hand, A2 is not weaker than the sufficient conditions for the folk theorem with communication in these papers. Thus the folk theorem in the next section is complementary to other folk theorems.

Finally, existence of contingent monitor is similar to Condition IV (Conditional Independndce) in Matsushima [14], but different. Matsushima [14] allows a certain type of correlation between private signals to prove a folk therem for repeated Prisoner's dillemma with private monitoring without communication. He assumes that $p(s|a)$ takes the following form;

$$p(s|a) = \prod_{i=1}^{n} q_i(s_i|a,\theta) f(\theta|a)$$

where $\theta \in \Xi$ is some unobservable macro shock. Thus private signals are conditionally independent given $(a,\theta)$. Then Condition IV at $a$ says that

[7] In Compte, monitor of player $i$ at $a$ is defined as a subset $M$ of the players such that $p_M(\cdot|a) \notin Q_i^M(a)$ where $Q_i^M(a) = co\{p_M(\cdot|a_i', a_{-i}) | a_i' \neq a_i\}$.

$\{\prod_{i=1}^{n} q_i(s_i|a,\theta)|a_i \in A_i, \theta \in \Xi\}$ is linearly independent for every $i$. To compare this assumption with my assumption, consider any two player repeated game. Then $M = I/\{i\}$ for any $s_i \in S_i$ for (A2); that is, the contingent monitor for $i$ must be all the other players. Then it can be checked that Condition IV implies (A2) because independency is stronger than the convexity-type condition like (A2). In particular, $|A_i| \times |\Xi| \leq |S_i|$ is required for Condition IV, but not for (A2). The example in the next section, which satisfies (A2) but not Condition IV, illustrates this point clearly. Another difference is that (A2) needs to be satisfied for only one player, while Condition IV is satisifed for all the players.

In addition to this, (A2) has an additional flexibility in terms of the selection of contingent monitor when there are more than two players.

# 4 Folk Theorem

To prove the folk theorem, *T-public equilibrium* is employed as in Compte [5] and Kandori and Matsushima [11]. It is a sequential equilibrium in which players send a meaningful message only every T periods.[8] Player $i$'s action strategy $\sigma_i^a$ only depends on the public message announced in every $T-$period. Player $i$'s announcement strategy $\sigma_i^m$ depends on both public history and the most "recent" private history after the last meaningful public message is sent. Let $E(\delta, T)$ be the set of all T-public equilibrium payoffs given $\delta$.

The main theorem of this paper is as follows.

**Theorem 3** *Suppose that, for any $\boldsymbol{\lambda} \in \Lambda_{0+}$, there exists $a^{\boldsymbol{\lambda}}$ at which a contingent monitor exists for some $i \in I^{\boldsymbol{\lambda}}$. Then, for any smooth compact set $W$ in the interior of $V^*$, there exists $\underline{\delta}$ and $T$ such that $W \subset E(\delta, T)$ for all $\delta \in (\underline{\delta}, 1)$.*

The following example is covered by this folk theorem. It does not follow from any existing folk theorem.

**Example. Prisoner's Dilemma**

Consider the following standard prisoners' dilemma game.

| | C | D |
|---|---|---|
| C | 1,1 | -l,1+g |
| D | 1+g,-l | 0,0 |

[8] Players announce meaningless random messages until the end of each T-period block.

Suppose that there exists some unobservable random variable $y$, which takes either $\overline{y}$ or $\underline{y}$ and satisfies

$$0 < \pi\left(\overline{y}|DD\right) < \pi\left(\overline{y}|DC \text{ or } DC\right) < \pi\left(\overline{y}|CC\right) < 1$$

Each player's private signal $s_i, i = 1, 2$ is a noisy observation of $y$. Assume that they are conditionally independent given $y$. Also assume that $p_i\left(s_i = y|y, a_i = C\right) = 1 - \varepsilon'$, $p_i\left(s_i = y|y, a_i = D\right) = 1 - \varepsilon''$, and $\varepsilon' < \varepsilon'' < 1/2$. That is, private signal is more informative when $C$ is chosen.

When $\frac{\varepsilon'}{\varepsilon''}$ is small enough, (A2) is satisfied for both players at $(C, C)$. (A2) is satisfied for player 1 (2) at $(C, D)$ $((D, C))$. Thus the folk theorem is obtained for this example.

**Remark.** Note that existence of contingent monitor is required for only one player. If it is satisfied for some player, then no strong assumption is needed for the rest of the players. For example, suppose that there is some player $i$ such that $|A_i| \times |S_i| \leq |S_{-i}|$. Then, for a generic distribution of private signal profile, $\{p_{-i}\left(\cdot|a, s_i\right) : a_i \in A_i, s_i \in S_i\}$ for any $a \in A$ is linearly independent, thus satisfies (A2). Thus the following corollary is obtained.

**Corollary 4** *Suppose that (i) $|A_j| \leq |S_{-j}|$ for all $j \in I$, and (ii) there exists some player $i \in I$ such that $|A_i| \times |S_i| \leq |S_{-i}|$. Then the above folk theorem holds for a generic distribution of private signal profile and any full dimensional stage game payoff matrix.*

**Proof.** (i) implies that (A1) is satisfied for a generic $p\left(\cdot|a\right)$ for all $j \in I$. (ii) implies that (A2) is satisfied for a generic $p\left(\cdot|a\right)$ for $i \in I$ for any action profile $a \in A$. Finally, $V$ is full dimensional for a generic stage game payoff matrix. Thus all the assumptions for the folk theorem above are satisfied. ■

Remember that Compte [5] assumes independent private signals, which is a nongeneric assumption. Kandori and Matsushima [11] assumes that player $i$'s deviation and player $j$'s deviation is distinguishable by the rest of the players. For this condition to be satisfied for generic private signal distributions, they need $|A_i| + |A_j| - 1 \leq |S_{-ij}|$ for all pair of $i, j \neq i$.

Now I prove the folk theorem. The first part of the proof (until Lemma 6) is directly borrowed from Compte [5], thus very sketchy. Interested readers should refer to Compte [5].

Consider the following T-period game with side transfer; stage game $G$ is played $T$ times and each player announces accumulated private signals $m_i =$

$(s_{i,1},...,s_{i,T})$ at the end of the Tth period on which the side transfer $x_i(m)$ is based.[9] Let $\sigma_i^{a,T}$ be $i'$s T-period action strategy, $\sigma_i^{m,T}$ be $i$'s announcement strategy and $\sigma_i^T = \left(\sigma_i^{a,T}, \sigma_i^{m,T}\right)$ be $i$'s T-period strategy. Define player $i$'s payoff from this game given $\delta_0 \in (0,1)$ as

$$g_i^{T,\delta_0}\left(\sigma^T\right) + E\left[x_i(m)\,|\sigma^T\right]$$

where

$$g_i^{T,\delta_0}\left(\sigma^T\right) = \sup_{\delta\in[\delta_0,1]} \frac{(1-\delta)\sum_{t=1}^{T}\delta^{t-1}E\left[g_i(a)\,|\sigma^T\right]}{1-\delta^T}$$

You can regard $g_i^{T,\delta_0}\left(\sigma^T\right)$ as an average payoff within the first T periods and $x_i(m)$ as continuation payoffs in the original repeated game with communication.

A strategy is called *stationary* if it specifies the same action independent of private history during the T periods. Player $i$'s stationary strategy to continue playing $a_i$ is denoted $\sigma_i^T(a_i)$. I say $\mathbf{v} \in \Re^n$ is generated by $\left(\sigma^T(a), x\right)$ if

$$v_i = g_i(a) + E\left[x_i(m)\,|\sigma^T(a)\right]$$

and $\left(\sigma_i^T(a_i)\right)_{i=1}^n$ is a Nash equilibrium, that is

$$\begin{aligned} v_i &= g_i(a) + E\left[x_i(m)\,|\sigma^T(a)\right] \\ &\geq g_i^{T,\delta_0}\left(\sigma_i^{T\prime}, \sigma_{-i}^T(a)\right) + E\left[x_i(m)\,|\sigma_i^{T\prime}, \sigma_{-i}^T(a)\right] \\ &\quad \text{for all T-period strategies } \sigma_i^{T\prime} \text{ and for all } i \end{aligned}$$

Now consider the following linear programming problem for each weight $\boldsymbol{\lambda} \in \Lambda$,

$$\begin{aligned} k(\boldsymbol{\lambda},\delta_0,T) &= \max_{\mathbf{v},a,x,\sigma^T} \boldsymbol{\lambda}\cdot\mathbf{v} \\ &\quad s.t.\ \mathbf{v} \text{ is generated by } \left(\sigma^T(a), x\right) \\ 0 &\geq \sum_{i=1}^n \lambda_i \cdot x_i(m), \forall m \in S^T, \end{aligned}$$

Compte [5] showed that this problem provides the tight upper bound of T-public equilibrium payoffs in the direction of $\boldsymbol{\lambda}$.[10] Let $k(\boldsymbol{\lambda},T) = \lim_{\delta_0\to 1} k(\boldsymbol{\lambda},\delta_0,T)$

[9] Thus player $i$'s message space needs to be at least as large as $S_i^T$.

[10] This techinique was first introduced by Fudenberg and Levine [7] to a variety of dynamic games with public information.

and $D(\boldsymbol{\lambda},T) = \{\mathbf{v}|\boldsymbol{\lambda}\cdot\mathbf{v} \leq k(\boldsymbol{\lambda},T)\}$.[11] Finally, let $Q(T) = \cap_{\boldsymbol{\lambda}\in\Lambda} D(\boldsymbol{\lambda},T)$. It is intuitively clear that $E(\delta,T) \subset Q(T)$. Compte [5] proves that every payoff profile in $Q(T)$ is eventually supported by T-public equilibrium as $\delta \to 1$.

**Proposition 5** *(Compte 1998) If $Q(T)$ is full dimensional, then $Q(T) = \lim_{\delta\to 1} E(\delta,T)$.*

Given this result, I only need to show that $Q(T)$ contains any smooth compact set in the interior of $V^*$ for appropriate $T$. More precisely, I need to show that, for any $\varepsilon > 0$, there exist $T$ and $\underline{\delta}$ such that $k(\boldsymbol{\lambda},\underline{\delta},T)$ is within $\varepsilon$ of $k^*(\boldsymbol{\lambda}) = \max_{v\in V}\sum_{i=1}^{n}\lambda_i v_i$ for any $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_{0+}$. Note that, among all the relevant extreme payoff profiles, the static Nash equilibrium payoff can be easily achieved with 0 transfer without any assumption. Therefore the question is whether it is possible to implement some constant strategy $\sigma^T(a^{\boldsymbol{\lambda}})$ for some $a^{\boldsymbol{\lambda}}$ when $\boldsymbol{\lambda}$ has a strictly positive component ($\boldsymbol{\lambda} \in \Lambda_{0+}$), while keeping the expected "efficiency loss" from the transfer ($|E[\sum_{i=1}^{n}\lambda_i \cdot x_i(m)]|$) negligible to achieve $k(\boldsymbol{\lambda},\delta,T) + \varepsilon > \sum_{i=1}^{n}\lambda_i g_i(a^{\boldsymbol{\lambda}}) = k^*(\boldsymbol{\lambda})$ for any $\delta \in (\underline{\delta},1)$ and $\boldsymbol{\lambda} \in \Lambda_{0+}$. The following lemma does just that.[12]

**Lemma 6** *Suppose that, for any $\boldsymbol{\lambda} \in \Lambda_{0+}$, there exists $a^{\boldsymbol{\lambda}}$ at which a contingent monitor exists for some $i$. Then, for any $\varepsilon > 0$, there exist $T$ and $\underline{\delta}$ such that $k(\boldsymbol{\lambda},\delta,T) + \varepsilon > k^*(\boldsymbol{\lambda})$ for any $\delta \in (\underline{\delta},1)$ and $\boldsymbol{\lambda} \in \Lambda_{\varepsilon+}$.*

***Proof.*** *See Appendix.* ■

Once this lemma is proved, then the proof of the folk theorem is complete. Below I provide a rough sketch of the proof of Lemma 6. For each $\boldsymbol{\lambda}$, $a^{\boldsymbol{\lambda}}$ is played for $T-$periods. Suppose that $\lambda_i \geq \varepsilon$ for $i \in I$ and there exists a contingent monitor for $i$ at $a^{\boldsymbol{\lambda}}$. First consider the incentive of $j \neq i$. Since either (A1) or (A1') is satisfied for $j$, there exists $x_j(s_{-j})$ or $x_j(s)$ to enforce $a^{\boldsymbol{\lambda}}$. For convenience, assume that (A1') is satisfied. Now define $j$'s transfer as follows

$$x_j(m) = \sum_{t=1}^{T} x_j(s_t).$$

Then $a^{\boldsymbol{\lambda}}$ is enforced period by period. In order to make sure that this transfer does not create efficiency loss, transfer this amount to player $i$. That is, add $-\frac{\lambda_j}{\lambda_i}x_j(m)$ to player $i$'s transfer.

[11] Note that $k(\boldsymbol{\lambda},\delta_0,T)$ is monotonically increasing in $\delta_0$.

[12] Nash equilibrium profile is used when $\lambda_i \leq \varepsilon$ for all $i$ even if some $\lambda_i$ is strictly positive. This is not a problem since $V^*$ can be asymptotically covered as $\varepsilon \to 0$.

Next consider player $i'$s incentive. When player $i$ deviates or misrepresents her private signal, she may gain not only from the stage game payoff, but also from the transfers just defined ($-\sum_{j\neq i}\frac{\lambda_j}{\lambda_i}x_j(m)$) period by period.

The similar problem arises in Compte [5]. The difference is that $x_j$ in his paper depends on neither $i$'s signal nor $j$'s signal, thus information revelation constraint is never a problem. This is based on the assumption that there are at least two distinct monitors for each player, which makes it possible to find a monitor of $j$ which does not include $i$ or $j$.

Taking this effect from $x_j(m)$ into account, I can compute player $i$'s maximum deviation gain from her equilibrium behavior with respect to $g_i$ and $x_j(m)$ within each period. Let's call it $\overline{g}$. Note that $\overline{g}$ is in the order of $\frac{1}{T}$ (for large enough $\delta$). This is because one period deviation gain from $g_i$ and $x_j(m)$ decreases in the order of $\frac{1}{T}$ ($x_j(m)$ decreases because $j$'s deviation gain within a period decreases). Note also that, since $\lambda_i \geq \varepsilon$, $\overline{g}$ can be defined as a uniform bound with respect to all $\lambda \in \Lambda_{\varepsilon+}$.

Now I define player $i'$s additional transfer $\widetilde{x}_i(m)$ as follows (thus player $i'$s exact transfer is $\widetilde{x}_i(m)-\sum_{j\neq i}\frac{\lambda_j}{\lambda_i}x_j(m)$). When $s_t$ is announced as a signal from period $t$, 1 is assigned to player $i$ with probability $q_i(s_t)$ and 0 is assigned with probability $1-q_i(s_t)$. This $q_i(\cdot)$ comes from Lemma 2 and satisfies $E\left[q_i(s)|a^{\boldsymbol{\lambda}},s_i\right]=\pi\in(0,1)\in$ for every $s_i\in S_i$. Since this information needs to be shared by all the players, a public randomization device needs to be used here. Let's denote the obtained T-digit code of 1 and 0 by $c$. Player $i$ is disciplined by punishment if and only if $c=(1,...,1)$. With some abuse of notation, this additional transfer for player $i$ $\widetilde{x}_i(c)$ is defined by

$$\begin{aligned}\widetilde{x}_i(c) &= -\triangle_i<0 \text{ when } c=(1,...,1)\\ &= 0 \qquad\qquad \text{otherwise}\end{aligned}$$

The expected probability of punishment on the equilibrium path is

$$\begin{aligned}&E\left[q_i(s_1)...q_i(s_T)|\sigma^T(a(\boldsymbol{\lambda}))\right]\\ =\ &E\left[q_i(s)|a(\boldsymbol{\lambda})\right]^T \text{ (i.i.d. over time)}\\ =\ &\pi^T\end{aligned}$$

Note that player $i$'s belief about the punishment is not affected by her private history because $E\left[q(s)|a^{\boldsymbol{\lambda}},s_i\right]=\pi$ for any $s_i\in S_i$ on the equilibrium path. This is where our Lemma 2 becomes critical. In Compte [5], independence of private signals are invoked here. If this condition is not satisfied, then player $i$ may become confident, after observing a certain sequence of private signals, that punishment is very unlikely to be triggered in the end, and start deviating.

Let's check all the incentive constraints of player $i$. I need to check only one period deviation constraints on and off the equilibrium path. If player $i$ deviates in period $t$, $E\left[q_i\left(s_t\right)|a\left(\boldsymbol{\lambda}\right)\right]$ goes up from $\pi$ to some $\pi' > 0$. Let $\triangle\pi$ be the minimum increment by any kind of profitable deviation (which requires choosing nonequilibrium action). Then the probability of the punishment increases by at least $\pi^{T-1}\left(\pi' - \pi\right)$ whether $i$ is on the equilibrium path or off the equilibrium path. Since $\overline{g}$ is the upperbound of deviation gain in each period, all the one period deviation constraints boil down to

$$\overline{g} \leq \triangle_i \pi^{T-1}\left(\pi' - \pi\right) \tag{3}$$

Now $\triangle_i$ is defined so that this inequality is satisfied with equality. Since $\overline{g}$ is a bound for one period deviation, thus in the order of $\frac{1}{T}$ (for large enough $\delta$), such $\triangle_i$ if $T$ is large enough..

Then it turns out that, since single-period deviation is almost negligible in its impact when $T$ is large, the expected efficiency loss associated with this punishment scheme can be made as small as possible by choosing $T$ and $\delta$ large enough.[13] In fact an upper bound of the efficiency loss can be explicitly computed here. The expected efficiency loss is simply

$$\triangle_i \pi^T$$

but this is bounded above by

$$\frac{\overline{g}\pi}{\pi' - \pi} \tag{4}$$

because (3) holds with equality. This is bounded by $\frac{1}{T}\times$constant as $\delta \to 1$, which can be made as small as possible by choosing large $T$. Thus, for any $\varepsilon > 0$, it is possible to choose $T$ and $\underline{\delta}$ so that (4) is less than $\varepsilon$ uniformly over all $\boldsymbol{\lambda} \in \Lambda_{\varepsilon+}$ and $\delta \in \left(\underline{\delta}, 1\right)$.

# 5 Discussion

It is possible to extend this theorem to (mutual) minmax folk theorem. If Compte [5]'s assumption is satisfied at the minmax action profile or there exists a contingent monitor f*or all the players* at the minmax action profile, then the minmax folk theorem which corresponds to Theorem 1 in Compte

[13] This idea originates in Abreu, Milgrom and Pearce [1] and used extensively in Compte [5].

[5] is obtained. To see why a contingent monitor is needed for every player, let $a^{mm}$ be the mutual minmax profile. Then the minimax payoff for player $i$ is $\underline{v}_i = \max_{a_i} g_i\left(a_i, a_{-i}^{mm}\right)$. The problem is that this maximum payoff given $a^{mm}$ may become larger than $\underline{v}_i$ when transfer is introduced. Such transfer need not to be used in the proof if there exist a contingent monitor for every player and their incentive is based on reward rather than transfer.

# 6 Appendix

**Proof of Lemma 6**

Fix $\varepsilon > 0$. For each $\boldsymbol{\lambda} \in \Lambda_{\varepsilon+}$, there is some player $i$ such that $\lambda_i \geq \varepsilon$. In the following, such player is always called $i$.

For all the players but player $i$, $x_j(m), j \neq i$ can be defined as explained in Section 4 and always playing $a^{\boldsymbol{\lambda}}$ and announcing their private signals truthfully becomes incentive compatible. This $x_j(m) = \sum_{t=1}^{T} x_j(s_t)$ satisfies, for all $a_j \in A_j$,

$$\frac{(1-\underline{\delta})}{1-\underline{\delta}^T}\left(g_j\left(a_j, a_{-j}^{\boldsymbol{\lambda}}\right) - g_j\left(a^{\boldsymbol{\lambda}}\right)\right) \leq E\left[x_j(s)\left|a_j, a_{-j}^{\boldsymbol{\lambda}}\right.\right] - E\left[x_j(s)\left|a^{\boldsymbol{\lambda}}\right.\right]$$

where $\delta^0$ is set to be $\underline{\delta}$ ($\underline{\delta}$ is to be chosen later). Since $\frac{(1-\underline{\delta})}{1-\underline{\delta}^T} \approx \frac{1}{T}$ for large $\underline{\delta}$, these $x_j(s)$, henceforth $x_j(m)$, can be taken to be the same order as $\frac{1}{T}$ as $T \to \infty$.

In the following, I focus on incentive of player $i$. First, The following expression characterizes the maximum deviation gain of player $i$ in each period taking into account the effect of the transfer from $j$ to $i$,

$$\max_{a_i,\eta_i}\left\{\frac{(1-\underline{\delta})}{1-\underline{\delta}^T}.g_i\left(a_i, a_{-i}^{\boldsymbol{\lambda}}\right) - \sum_{j\neq i}\frac{\lambda_j}{\lambda_i}E\left[x_j\left(\eta_i(s_i), s_{-i}\right)\left|a_i, a_{-i}^{\boldsymbol{\lambda}}\right.\right]\right\} \quad (5)$$
$$-\left\{\frac{(1-\underline{\delta})}{1-\underline{\delta}^T}g_i\left(a^{\boldsymbol{\lambda}}\right) - \sum_{j\neq i}\frac{\lambda_j}{\lambda_i}E\left[x_j(s)\left|a^{\boldsymbol{\lambda}}\right.\right]\right\}.$$

This depends on the choice of $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_{\varepsilon+}$ and $a^{\boldsymbol{\lambda}}$. Let $\overline{g}(\underline{\delta}, T)$ be the maximum of (5) with respect to all $\boldsymbol{\lambda}$ and $a^{\boldsymbol{\lambda}}$. This is possible because $\lambda_i \geqq \varepsilon$ and the number of the players and actions are finite. Not that this upper bound is valid for any $\delta \in (\underline{\delta}, 1)$.

Introduce a random variable $c$ as explained in Section 4 and define $\widetilde{x}_i(m)$ as follows

$$\begin{aligned}\widetilde{x}_i(m) &= -\triangle_i < 0 \text{ when } c(m) = (1, ..., 1) \\ &= 0 \qquad\qquad \text{otherwise}\end{aligned}$$

Player $i$'s transfer is the sum of $\widetilde{x}_i(m)$ and $-\sum_{j\neq i}\frac{\lambda_j}{\lambda_i}x_j(m)$.

It is enough to check all the one-period deviations at every history of player $i$. When player $i$ deviates, the probability of punishment increases by at least

$$\begin{aligned}&\min_{a,\eta_i}\left(E\left[q_i(\eta_i(s_i), s_{-i})\,|\left(a_i, a_{-i}^{\boldsymbol{\lambda}}\right)\right] - \pi\right) \times \Pr(\text{the code is 1 in all the other periods}) \\ \geq\ &\min_{a,\eta_i}\left(E\left[q_i(\eta_i(s_i), s_{-i})\,|\left(a_i, a_{-i}^{\boldsymbol{\lambda}}\right)\right] - \pi\right) \times \pi^{T-1} \\ \geq\ &\triangle\pi \times \pi^{T-1}\end{aligned}$$

where $\triangle\pi > 0$ is the minimum of $\min_{a,\eta_i}\left(E\left[q_i(\eta_i(s_i), s_{-i})\,|\left(a_i, a_{-i}^{\boldsymbol{\lambda}}\right)\right] - \pi\right)$ with respect to all $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_{\varepsilon+}$ and $a^{\boldsymbol{\lambda}}$.

Since any deviation gain per period is at most $\overline{g}(\underline{\delta}, T)$, all the one-period deviation constraints are satisfied if

$$\overline{g}(\underline{\delta}, T) \leq \triangle\pi \times \pi^{T-1}\triangle_i$$

is satisfied. Set $\triangle_i$ so that this inequality holds with equality, which is possible if $T$ and $\underline{\delta}$ is large enough.

The expected efficiency loss on the equilibrium path, which arises from $\widetilde{x}_i(m)$, is simply

$$\lambda_i\pi^T\triangle_i = \lambda_i\frac{\pi\overline{g}(\underline{\delta}, T)}{\triangle\pi}.$$

This efficiency loss can be made smaller than $\varepsilon$ because $\overline{g}(\underline{\delta}, T)$ can be made as small as possible by taking $T$ and $\underline{\delta}$ large enough.

Therefore, given $\varepsilon > 0$, I can find $T$, $\underline{\delta}$, $a^{\boldsymbol{\lambda}}$, and $\sigma^T\left(a^{\boldsymbol{\lambda}}\right)$ such that, for any $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_{\varepsilon+}$, all the incentive constraints are satisfied and

$$\begin{aligned}k(\boldsymbol{\lambda}, \delta, T) &\geq k(\boldsymbol{\lambda}, \underline{\delta}, T) \\ &\geq \sum_{j=1}^{n}\lambda_j\left\{g_j\left(a^{\boldsymbol{\lambda}}\right) + E\left[x_j(m)\,|\sigma^T\left(a^{\boldsymbol{\lambda}}\right)\right]\right\} \\ &\geq k^*(\boldsymbol{\lambda}) - \lambda_i\frac{\pi\overline{g}(\underline{\delta}, T)}{\triangle\pi} \\ &\geq k^*(\boldsymbol{\lambda}) - \varepsilon\end{aligned}$$

for all $\delta \in (\underline{\delta}, 1)$. Hence the lemma is proved.■

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