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The Bounded Approximation Property for Weakly Uniformly Continuous Type Holomorphic Mappings

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Abstract: When U is a balanced open subset of a reflexive Banach space E with $\mathcal{P}(^nE) = \mathcal{P}_w(^nE)$ for every positive integer n, we show that the predual of the space of weakly uniformly continuous holomorphic mappings on U, $G_{wu}(U)$, has the bounded approximation property if and only if E has the bounded approximation property if and only if $\mathcal{P}(^nE)$ has the bounded approximation property for every positive integer n. An analogous result is established for the predual of the space of holomorphic mappings of bounded type also.

Key words: Banach spaces, locally convex spaces, bounded approximation property, holomorphic mappings of bounded type, weakly uniformly continuous functions, bounded holomorphic mappings.

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1. Introduction

If E and F are locally convex spaces, always assumed complex and Hausdorff, let $L(E;F)$ denote the vector space of all continuous linear operators from E into F. A locally convex space E is said to have the approximation property (AP for short) if given a compact set $K \subset E$ and a neighborhood of zero V in E, there is a finite rank operator $T \in L(E;E)$ such that $Tx - x \in V$ for every $x \in K$. Given a locally convex space E, we say that E has the bounded approximation property (BAP for short) if there exists an equicontinuous net of finite rank operators on E which converges pointwise to the identity of E. It is easy to see that the BAP implies the AP. But, in [16] Figiel and Johnson gave an example of a separable Banach space with the AP which fails to have the BAP. Hence, in general the AP does not imply the BAP. (See also Casazza [10].)

Let U be an open subset of a Banach space E, let $G^\infty(U)$ denote the predual of the space of all bounded holomorphic mappings $\mathcal{H}^\infty(U)$ constructed by Mujica in [23], and let $G_b(U)$ denote the predual of the space of all holomorphic mappings of bounded type $\mathcal{H}_b(U)$ constructed by Galindo, Garcia and Maestre in [17]. If U is a bounded balanced open subset of E then Mujica [23] proved that E has the AP if and only if $G^\infty(U)$ has the AP, and in [24] he proved that E has the AP if and only if $G_b(U)$ has the AP whenever U is a balanced open subset of E. In [13] the author proved that a separable Banach space E has the BAP if and only if $G^\infty(U)$ has the BAP, where U is the open unit ball of E.

In this work our main purpose is to obtain characterizations of the BAP for the space of weakly uniformly continuous type holomorphic mappings, a generalization of the class of weakly uniformly continuous holomorphic mappings, and for the preduals of these classes. Also, we obtained characterizations of the BAP for the preduals of some classes of holomorphic mappings and homogeneous polynomials.

The paper is organized as follows: In Section 2 we establish our notation and terminology. In section 3 we show that a Banach space E has the BAP if and only if $G_b(U)$, U balanced open subset of E, has the BAP if and only if the predual of the space of m-homogeneous continuous scalar-valued polynomial on E, $Q(^mE)$, has the BAP for every $m \in \mathbb{N}$ (or, equivalently, for some $m \in \mathbb{N}$).

In Section 4 we extend the work of Boyd, Dineen and Rueda [9] to a more general class of holomorphic mappings, called the space of holomorphic mappings of weakly uniformly continuous type and denoted by $\mathcal{H}_{wu}(\mathcal{U}; F)$, to obtain characterizations for the BAP. We know that, if a Banach space E (even separable and reflexive) has the BAP, in general, the space of n-homogenous polynomials $\mathcal{P}(^nE)$ does not have the BAP (see [5, Proposition 5.2]). Here, giving a linearization theorem for the space $\mathcal{H}_{wu}(\mathcal{U}; F)$, and using this result, we show that the predual of $\mathcal{H}_{wu}(\mathcal{U})$, $G_{wu}(\mathcal{U})$, has the BAP if and only if E has the BAP if and only if, for every positive integer n, $\mathcal{P}(^nE)$ has the BAP, where E is a reflexive Banach space with $\mathcal{P}(^nE) = \mathcal{P}_w(^nE)$, for every positive integer n. As a consequence of this, with the same conditions on E, in particular, we obtain that if U is a balanced open subset of E then the predual of the space of weakly uniformly continuous holomorphic mappings on U, $G_{wu}(U)$, has the bounded approximation property if and only if E has the bounded approximation property if and only if $\mathcal{P}(^nE)$ has the bounded approximation property for every positive integer n.

In the last section we consider the predual of spaces of bounded holomorphic mappings on Banach spaces and study the problem in connection with the following problem posed by Mujica: When U is the open unit ball of E, does

$G^{\infty}(U)$ have the BAP, whenever E has the BAP? (See [23, 5.9 Problem].) Recently in [13] the author answered the problem in positive for separable Banach spaces, but the general case still remains open.

2. Notation and terminology

The symbol $\mathbb{C}$ represents the field of all complex numbers, $\mathbb{N}$ represents the set of all positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Unless stated otherwise E and F denote locally convex spaces, always assumed complex and Hausdorff. The letter U denotes a nonvoid open subset of E.

For a Banach space E the symbol U_E represents the open unit ball of E and the symbol B_E^{λ}, $1 \leq \lambda < \infty$, represents a subset of E consisting of the elements of norm $\leq \lambda$. In case of $\lambda = 1$ we write B_E instead of B_E^1 which is the closed unit ball of E.

The symbol τ_c denotes the compact-open topology, and given a subset A of E, $\overline{A}^{\tau_c}$ will denote the τ_c-closure of A in E. The symbol Λ stands for a directed set.

Let $L(E;F)$ denote the vector space of all continuous linear operators from E into F. When $F = \mathbb{C}$ we write E' instead of $L(E;\mathbb{C})$.

We denote by $E \bigotimes F$ the tensor product of E and F.

An operator T in $L(E;F)$ is said to have a finite rank if $T(E)$ is finite dimensional. Observe that the subspace of all finite rank operators $T \in L(E;F)$ can be identified with the space $E' \bigotimes F$.

Let $\mathcal{P}(E;F)$ denote the vector space of all continuous polynomials from E into F, let $\mathcal{P}(^mE;F)$ denote the subspace of all m-homogeneous members of $\mathcal{P}(E;F)$. Let $\mathcal{P}_w(^mE;F)$ (resp. $\mathcal{P}_{wu}(^mE;F)$) denote the subspace of all members of $\mathcal{P}(^mE;F)$ which are weakly (resp. weakly uniformly) continuous on bounded subsets of E, for every $m \in \mathbb{N}_0$. It is shown in [3] that $\mathcal{P}_w(^mE;F) = \mathcal{P}_{wu}(^mE;F)$ for every Banach space E and F, and for every $m \in \mathbb{N}$. If $F = \mathbb{C}$ then we denote $\mathcal{P}(^mE;\mathbb{C})$ (resp. $\mathcal{P}_w(^mE;\mathbb{C})$) by $\mathcal{P}(^mE)$ (resp. $\mathcal{P}_w(^mE)$).

A polynomial $P \in \mathcal{P}(^mE;F)$ is called of finite type if it is generated by linear combination of functions $\phi^m \bigotimes y$ ($\phi \in E'$, $y \in F$), where $\phi^m \bigotimes y(x) = \phi^m(x)y$ for all $x \in E$. Let $\mathcal{P}_f(^mE;F)$ denote the subspace of all members of $\mathcal{P}(^mE;F)$ which are of finite type, for every $m \in \mathbb{N}_0$.

Let $\mathcal{H}(U;F)$ denote the vector space of all holomorphic mappings from U into F and let $\mathcal{H}_b(U;F)$ denote the vector space of all $f \in \mathcal{H}(U;F)$ such that $f(A)$ is bounded in F for each U-bounded set A. We recall that a set $A \subset U$

is said to be U-bounded if A is bounded and there exists a neighborhood of zero V in E such that $A + V \subset U$. Note that U-bounded sets coincide with the bounded sets when $U = E$. If $F = \mathbb{C}$ then we denote $\mathcal{H}(U; \mathbb{C})$ (resp. $\mathcal{H}_b(U; \mathbb{C})$) by $\mathcal{H}(U)$ (resp. $\mathcal{H}_b(U)$).

Let $\mathcal{H}_{wu}(U; F)$ denote space of all $f \in \mathcal{H}(U; F)$ such that f is weakly uniformly continuous on A for each U-bounded set A. We endow $\mathcal{H}_{wu}(U; F)$ with the topology of uniform convergence on U-bounded sets, which is a Fréchet space whenever U is an open subset of a Banach space E (see [2, Proposition 1.4]). When $F = \mathbb{C}$ we write $\mathcal{H}_{wu}(U)$ instead of $\mathcal{H}_{wu}(U; \mathbb{C})$.

We refer to [15] or [22] for the properties of polynomials and holomorphic mappings on infinite dimensional spaces, to [21] for the theory of Banach spaces, and to [19, 20] for the theory of locally convex spaces. We also refer to [2], [3] and [4] for the properties of $\mathcal{P}_w(^m E; F)$, $\mathcal{P}_{wu}(^m E; F)$ and $\mathcal{H}_{wu}(U; F)$ on infinite dimensional spaces.

3. The bounded approximation property for the predual of the space of holomorphic mappings of bounded type

Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be an increasing countable open cover of an open subset U of a locally convex space E. Let $\mathcal{H}^\infty(\mathcal{U}; F)$ denote the locally convex space

$$\mathcal{H}^\infty(\mathcal{U}; F) = \{f \in \mathcal{H}(U; F) \,:\, f(U_n) \text{ is bounded in } F \text{ for every } n\},$$

endowed with the topology of uniform convergence on all the sets U_n. If $F = \mathbb{C}$ then we denote $\mathcal{H}^\infty(\mathcal{U}; \mathbb{C})$ by $\mathcal{H}^\infty(\mathcal{U})$.

If we take $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ as a fundamental sequence of open U-bounded sets, obviously we have $\mathcal{H}^\infty(\mathcal{U}; F) = \mathcal{H}_b(U; F)$.

In [17] Galindo, Garcia and Maestre constructed a complete (LB)-space $G_b(U)$, U is a balanced open subset of a Banach space E, and a mapping $\delta_U \in \mathcal{H}_b(U; G_b(U))$ with the following universal property: For each Banach space F and each mapping $f \in \mathcal{H}_b(U; F)$, there is a unique mapping $T_f \in L(G_b(U); F)$ such that $T_f \circ \delta_U = f$. The space $G_b(U)$ is called predual of $\mathcal{H}_b(U)$. Now, let U be an open subset of a locally convex space E, and let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be an increasing countable open cover of U. The result of Galindo, Garcia and Maestre was generalized by Mujica in [24] where he constructed a complete (LB)-space $G^\infty(\mathcal{U})$ and a mapping $\delta_\mathcal{U} \in \mathcal{H}^\infty(\mathcal{U}; G^\infty(\mathcal{U}))$ with the following universal property: For each complete locally convex space F and each mapping $f \in \mathcal{H}^\infty(\mathcal{U}; F)$, there is a unique operator $T_f \in L(G^\infty(\mathcal{U}); F)$

such that $T_f \circ \delta_U = f$, where the space $G^\infty(\mathcal{U})$ is defined in the following way: For every sequence $\alpha = (\alpha_n)$ of strictly positive numbers, let

$$B_{\mathcal{U}}^\alpha = \{f \in \mathcal{H}^\infty(\mathcal{U}) \,:\, \|f\|_{U_n} \le \alpha_n \text{ for every } n\}.$$

Then $G^\infty(\mathcal{U})$ is defined as the closed subspace of all linear functionals $u \in \mathcal{H}^\infty(\mathcal{U})'$ such that $u_{|B_{\mathcal{U}}^\alpha}$ is τ_c-continuous for every α, which is called predual of $\mathcal{H}^\infty(\mathcal{U})$.

In an earlier work, when E is a Banach space, Ryan [29] constructed a Banach space $Q(^mE)$, $m \in \mathbb{N}$, and a mapping $\delta_m \in \mathcal{P}(^mE; Q(^mE))$ with the following universal property: For each Banach space F and each $P \in \mathcal{P}(^mE; F)$, there is a unique operator $T_P \in L(Q(^mE); F)$ such that $T_P \circ \delta_m = P$, where the space $Q(^mE)$ is defined as the closed subspace of all linear functionals $v \in \mathcal{P}(^mE)'$ such that $v_{|B_{\mathcal{P}(^mE)}}$ is τ_c-continuous, which is called predual of $\mathcal{P}(^mE)$. (See also [25], or [23, Theorem 2.4 and Theorem 4.1].)

There is a close relation between $Q(^mE)$ and $G^\infty(\mathcal{U})$. In [11], parallelling to [8, Proposition 4] the author gives a result ([11, Proposition 5.1]), without a proof, asserting that $\{Q(^mE)\}_{m=0}^\infty$ is an $\mathcal{S}$-absolute decomposition for $G^\infty(\mathcal{U})$, where $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ is an increasing countable open cover of bounded balanced open subsets of a balanced open subset U of a Banach space E (for the definition and details about Schauder decompositions and $\mathcal{S}$-decompositions see [15, § 3.3]). But in the last section we will observe that these conditions on $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ are not enough to have this conclusion. On the other hand, putting an additional condition on $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$, we see that [11, Proposition 5.1] is true in this case, which we include here a complete proof of it.

For the proof we will need the following lemma. We remark that the proofs of the following lemma and the next proposition follow the patterns of the proofs of the corresponding results by C. Boyd in [8]. In what follows the sum $\sum_{n=0}^\infty P^n f(0)(x)$ will denote the Taylor series expansion of a function $f \in \mathcal{H}(U)$, $U \subset E$ open, about 0 with $P^n f(0) \in \mathcal{P}(^nE)$ for each $n \in \mathbb{N}$, and the symbol $\mathcal{S}$ will denote the set of all scalar sequences $(\alpha_n)_{n=1}^\infty$ such that $\limsup_{n\to\infty} |\alpha_n|^{\frac{1}{n}} \le 1$.

Lemma 1. *Let U be an open balanced subset of a Banach space E and let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a sequence of bounded, balanced, open subsets of U such that $U = \bigcup_{n=1}^\infty U_n$ and $\rho_n U_n \subset U_{n+1}$, with $\rho_n > 1$, for every $n \in \mathbb{N}$. Let $\alpha = (\alpha_n)$ be any sequence of strictly positive numbers, let $(\beta_n)_{n\in\mathbb{N}} \in \mathcal{S}$ and let $g_f := \sum_{j=0}^\infty \beta_j P^j f(0)$ for each $f \in B_{\mathcal{U}}^\alpha$. Then there exists a sequence of* *strictly positive numbers* $\alpha' = (\alpha'_n)$ *such that*

$$\sup_{f \in B^\alpha_{\mathcal{U}}} \|g_f\|_{U_n} \leq \sup_{f \in B^\alpha_{\mathcal{U}}} \sum_{j=0}^{\infty} |\beta_j| \left\|P^j f(0)\right\|_{U_n} \leq \alpha'_n \,,$$

for every $n \in \mathbb{N}$. *In particular,* $g_f \in B^{\alpha'}_{\mathcal{U}}$ *for every* $f \in B^\alpha_{\mathcal{U}}$.

Proof. Let $\alpha = (\alpha_n)$ be any sequence of strictly positive numbers with

$$B^\alpha_{\mathcal{U}} = \{f \in \mathcal{H}^\infty(\mathcal{U}) \,:\, \|f\|_{U_n} \leq \alpha_n \text{ for every } n\}.$$

It is easy to see that $g_f \in \mathcal{H}(U)$ for every $f \in B^\alpha_{\mathcal{U}}$. Let us fix an integer $n \in \mathbb{N}$ and take an element t of U_n. Then by hypothesis $t \in U_{n+1}$ and furthermore $\xi U_n \subset U_{n+1}$, for every $|\xi| \leq \rho_n$. Now applying the Cauchy inequalities (see, e.g., [22, Corollary 7.4]) we get that

$$\|P^m f(0)(\rho_n t)\| \leq \sup_{|\xi| = \rho_n} \|f(\xi t)\| \leq \sup_{t \in U_{n+1}} \|f(t)\| \leq \alpha_{n+1} \,,$$

for every $m \in \mathbb{N}$ and for every $f \in B^\alpha_{\mathcal{U}}$. So

$$\|P^m f(0)\|_{\rho_n U_n} \leq \|f\|_{U_{n+1}} \leq \alpha_{n+1}$$

for every $m \in \mathbb{N}$ and for every $f \in B^\alpha_{\mathcal{U}}$. Since $(\beta_j)_{j \in \mathbb{N}} \in \mathcal{S}$ we can find $C > 0$ such that

$$|\beta_j| \leq C \left(\frac{1+\rho_n}{2}\right)^j$$

for every j. Hence we obtain that

$$\|g_f\|_{U_n} \leq \sum_{j=0}^{\infty} |\beta_j| \left\|P^j f(0)\right\|_{U_n} \leq C\alpha_{n+1} \sum_{j=0}^{\infty} \left(\frac{1+\rho_n}{2\rho_n}\right)^j = \alpha'_n \,,$$

for every $f \in B^\alpha_{\mathcal{U}}$. Since n is an arbitrary integer, we have that

$$\sup_{f \in B^\alpha_{\mathcal{U}}} \|g_f\|_{U_n} \leq \sup_{f \in B^\alpha_{\mathcal{U}}} \sum_{j=0}^{\infty} |\beta_j| \left\|P^j f(0)\right\|_{U_n} \leq \alpha'_n \,,$$

for every $n \in \mathbb{N}$, which shows that $g_f \in B^{\alpha'}_{\mathcal{U}}$ for every $f \in B^\alpha_{\mathcal{U}}$. ∎

Proposition 1. *Let U be an open balanced subset of a Banach space E and let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a sequence of bounded, balanced, open subsets of U such that $U = \bigcup_{n=1}^\infty U_n$ and $\rho_n U_n \subset U_{n+1}$, with $\rho_n > 1$, for every $n \in \mathbb{N}$. Then $\{Q(^nE)\}_{n=0}^\infty$ is an $\mathcal{S}$-absolute decomposition for $G^\infty(\mathcal{U})$.*

Proof. Let $\alpha = (\alpha_n)$ be any sequence of strictly positive numbers with

$$B_{\mathcal{U}}^\alpha = \{f \in \mathcal{H}^\infty(\mathcal{U}) : \|f\|_{U_n} \le \alpha_n \text{ for every } n\}.$$

Since $(1, 2^2, \dots, j^2, \dots) \in \mathcal{S}$ it follows by Lemma 1 that there exists a sequence of strictly positive numbers $\alpha' = (\alpha'_n)$ such that

$$\sum_{j=0}^\infty j^2 \left\|P^j f(0)\right\|_{U_n} \le \alpha'_n\,,$$

for every $f \in B_{\mathcal{U}}^\alpha$ and every $n \in \mathbb{N}$. Therefore for every m and every $f \in B_{\mathcal{U}}^\alpha$ we have

$$\left\|m^2 \sum_{j=m}^\infty P^j f(0)\right\|_{U_n} \le \sum_{j=m}^\infty j^2 \left\|P^j f(0)\right\|_{U_n} \le \alpha'_n \qquad \text{for every } n \in \mathbb{N}\,.$$

Thus for every m and every $f \in B_{\mathcal{U}}^\alpha$, the function $m^2 \sum_{j=m}^\infty P^j f(0)$ belongs to $B_{\mathcal{U}}^{\alpha'}$.

Let $\phi \in G^\infty(\mathcal{U})$ and for each $n \in \mathbb{N}_0$ let $\phi_n\left(\sum_{k=0}^\infty P^k f(0)\right) := \phi(P^n f(0))$, $f \in \mathcal{H}^\infty(\mathcal{U})$. For each $n \in \mathbb{N}$ let us define

$$B^n := \left\{P \in \mathcal{P}(^nE) : \|P\| = \sup_{\|x\|\le 1} |P(x)| \le 1\right\}.$$

Since each U_j is bounded, for every $n \in \mathbb{N}$ there exists a sequence of strictly positive numbers $\gamma^n = (\gamma_j^n)$ such that $B^n \subset B_{\mathcal{U}}^{\gamma^n}$. Then it follows that $\phi_{n|B^n} = \phi_{|B^n}$ is τ_c-continuous and therefore $\phi_n \in Q(^nE)$ for every $n \in \mathbb{N}$. Since ϕ is τ_c-continuous on $B_{\mathcal{U}}^\alpha$ and the Taylor series expansion of $f \in B_{\mathcal{U}}^\alpha$ about 0 converges to f in the τ_c-topology, we have that

$$\begin{aligned}\left\|\phi - \sum_{k=0}^{m-1} \phi_k\right\|_{B_{\mathcal{U}}^\alpha} &= \sup_{f\in B_{\mathcal{U}}^\alpha} \left|\left(\phi - \sum_{k=0}^{m-1} \phi_k\right)(f)\right| \\ &= \sup_{f\in B_{\mathcal{U}}^\alpha} \left|\phi\left(\sum_{j=m}^\infty P^j f(0)\right)\right| \le \frac{1}{m^2}\|\phi\|_{B_{\mathcal{U}}^{\alpha'}} \ \to 0 \ \text{ as } m \to \infty\,.\end{aligned}$$

Thus $\phi = \sum_{n=0}^{\infty} \phi_n \in G^\infty(\mathcal{U})$.

As $(\beta_1, 2^2\beta_2, \ldots, n^2\beta_n, \ldots) \in \mathcal{S}$ for $(\beta_n)_{n\in\mathbb{N}} \in \mathcal{S}$, it follows by Lemma 1 that there exists a sequence of strictly positive numbers $\alpha'' = (\alpha''_n)$ such that

$$\left\| \sum_{j=0}^{\infty} j^2 \beta_j P^j f(0) \right\|_{U_n} \leq \sum_{j=0}^{\infty} j^2 \beta_j \left\| P^j f(0) \right\|_{U_n} \leq \alpha''_n,$$

for every $f \in B_{\mathcal{U}}^{\alpha}$ and every $n \in \mathbb{N}$, where $\beta_0 = 0$. Hence, in particular

$$B^{'} := \left\{ j^2 \beta_j P^j f(0) \,:\, f \in B_{\mathcal{U}}^{\alpha}, j \in \mathbb{N} \right\} \subset B_{\mathcal{U}}^{\alpha''}$$

with $\{ j^2 \beta_j P^j f(0) \,:\, f \in B_{\mathcal{U}}^{\alpha} \} \subset P(^jE)$ for every $j \in \mathbb{N}$. Let $\phi_\eta \to 0$ in $G^\infty(\mathcal{U})$ (for the topology of uniform convergence on the subsets $B_{\mathcal{U}}^{\alpha}$ of $\mathcal{H}^\infty(\mathcal{U})$ for every α). Since

$$\|(\phi_\eta)_j\|_{\{P^j f(0)\}_{\{f \in B_{\mathcal{U}}^{\alpha}\}}} = \sup_{f \in B_{\mathcal{U}}^{\alpha}} \left| \phi_\eta(P^j f(0)) \right| \leq \frac{1}{j^2 \beta_j} \|\phi_\eta\|_{B'}$$

then $(\phi_\eta)_j \to 0$ in $Q(^jE)$ for every $j \in \mathbb{N}$. Hence $\{Q(^nE)\}_{n=0}^{\infty}$ is a Schauder decomposition for $G^\infty(\mathcal{U})$. Furthermore, for $\sum_{n=0}^{\infty} \phi_n \in G^\infty(\mathcal{U})$ and $(\beta_n)_{n\in\mathbb{N}} \in \mathcal{S}$ we have that

$$\begin{aligned}\left\| \sum_{n=k}^{\infty} \beta_n \phi_n \right\|_{B_{\mathcal{U}}^{\alpha}} &\leq \sum_{n=k}^{\infty} |\beta_n| \, \|\phi_n\|_{B_{\mathcal{U}}^{\alpha}} = \sum_{n=k}^{\infty} \sup_{f \in B_{\mathcal{U}}^{\alpha}} |\phi(\beta_n P^n f(0))| \\ &= \sum_{n=k}^{\infty} \frac{1}{n^2} \sup_{f \in B_{\mathcal{U}}^{\alpha}} \left| \phi(n^2 \beta_n P^n f(0)) \right| \leq \|\phi\|_{B'} \sum_{n=k}^{\infty} \frac{1}{n^2}\,.\end{aligned}$$

Therefore $\{Q(^nE)\}_{n=0}^{\infty}$ is an $\mathcal{S}$-decomposition for $G^\infty(\mathcal{U})$, and taking $k = 0$ it is seen that the $\mathcal{S}$-decomposition is absolute. ∎

We remark that the hypothesis of Proposition 1 is sufficient for the proof of [11, Theorem 5.2], so with this result [11, Theorem 5.2] still remains true.

There is also a relation concerning the BAP between $Q(^mE)$ and $G^\infty(\mathcal{U})$, which we prove below. Before let us give a technical lemma whose proof is a modification of the proof of [14, Lemma 9] (see also [7] for related results).

Lemma 2. *Let E and F be Banach spaces, and let $1 \leq \lambda < \infty$. If $B_{\mathcal{P}(^mE;F)} \subset \overline{B^{\lambda}_{\mathcal{P}_f(^mE;F)}}^{\tau_c}$ for some $m \in \mathbb{N}$, then $B_{L(E;F)} \subset \overline{B^{c(\lambda,m)}_{E'\otimes F}}^{\tau_c}$, where $c(\lambda, m) = m - 1 + \frac{m^{m+1}}{m!}\lambda$.*

Proof. Since $c(\lambda,1)=\lambda$ the case $m=1$ is our hypothesis. Then we let $m>1$ and suppose that $B_{\mathcal{P}(^mE;F)} \subset \overline{B^{\lambda}_{\mathcal{P}_f(^mE;F)}}^{\tau_c}$. Let $T \in B_{L(E;F)}$, let K be a compact subset of E and let $\epsilon>0$. Let $a \in K$ with $a \neq 0$ and choose $\varphi \in E'$, $\varphi \neq 0$, such that $\varphi(a)=1$ and $\|\varphi\| = \|a\|^{-1}$. Consider the set

$$K_1 := \bigcup_{\substack{\epsilon_i = \mp 1 \\ i=1,\dots,m}} (\epsilon_1 K + \cdots + \epsilon_m K)\,,$$

which is a compact set in E. Let us define

$$T \overset{\vee}{\circ} \varphi^{m-1}(x_1,\dots,x_m) := \frac{1}{m}\big[T(x_1)\varphi(x_2)\cdots\varphi(x_m) + \cdots + T(x_m)\varphi(x_1)\cdots\varphi(x_{m-1})\big]$$

for every $(x_1,\dots,x_m) \in \overbrace{E\times\cdots\times E}^{m \text{ times}}$. Since $\|a\|^{m-1}\, T\circ\varphi^{m-1} \in B_{\mathcal{P}(^mE;F)}$, by hypothesis there is a $P^m \in B_{\mathcal{P}^{\lambda}_f(^mE;F)}$ such that

$$\left\| P^m(x) - \|a\|^{m-1}\, T\circ\varphi^{m-1}(x)\right\| < \frac{m!}{m}\, \|a\|^{m-1}\, \epsilon$$

for every $x \in K_1$.

Thus, for every $(x_1,\dots,x_m) \in \overbrace{K\times\cdots\times K}^{m \text{ times}}$, we have that

$$\begin{aligned}
&\left\| \overset{\vee}{P^m}(x_1,\dots,x_m) - \left(\|a\|^{m-1} T \overset{\vee}{\circ} \varphi^{m-1}\right)(x_1,\dots,x_m)\right\| \\
&\quad = \left\| \frac{1}{m!2^m} \sum_{\epsilon_i=\mp1} \epsilon_1\cdots\epsilon_m \left[P^m\left(\sum_{i=1}^m \epsilon_i x_i\right) - \|a\|^{m-1}\, T\circ\varphi^{m-1}\left(\sum_{i=1}^m \epsilon_i x_i\right)\right]\right\| \\
&\quad < \frac{1}{m!2^m} \sum_{\epsilon_i=\mp1} \frac{m!}{m}\, \|a\|^{m-1}\, \epsilon \;=\; \|a\|^{m-1}\frac{\epsilon}{m}\,.
\end{aligned}$$

Then, in particular we obtain

$$\left\| \overset{\vee}{P^m}(x,a,\dots,a) - \left(\|a\|^{m-1}\, \overset{\vee}{T}\circ\varphi^{m-1}\right)(x,a\dots,a)\right\| < \|a\|^{m-1}\,\frac{\epsilon}{m}$$

for every $x \in K$. This will imply that

$$\left\| \overset{\vee}{P^m}(x, a \dots, a) - \left(\|a\|^{m-1} \frac{1}{m} T(x) + \|a\|^{m-1} \frac{m-1}{m} \varphi(x) T(a) \right) \right\|$$
$$= \left\| \overset{\vee}{P^m}(x, a \dots, a) - \|a\|^{m-1} \left(\frac{m-1}{m} \right) \varphi(x) T(a) - \|a\|^{m-1} \frac{1}{m} T(x) \right\|$$
$$< \|a\|^{m-1} \frac{\epsilon}{m}$$

for every $x \in K$, or equivalently

$$\left\| \frac{m}{\|a\|^{m-1}} \overset{\vee}{P^m}(x, a \dots, a) - (m-1) T(a) \varphi(x) - T(x) \right\| < \epsilon \qquad (*)$$

for every $x \in K$.

Therefore, if we define a linear operator T_f by

$$T_f(x) := \frac{m}{\|a\|^{m-1}} \overset{\vee}{P^m}(x, a \dots, a) - (m-1) T(a) \varphi(x) , \qquad x \in E,$$

then $T_f \in E' \otimes F$, and since $\left\| \overset{\vee}{P^m}(. , a \dots, a) \right\| \leq \|a\|^{m-1} \frac{m^m}{m!} \|P^m\|$ (see [23, Theorem 2.2]) we have that

$$\|T_f\| = \left\| \frac{m}{\|a\|^{m-1}} \overset{\vee}{P^m}(. , a \dots, a) - (m-1) T(a) \varphi \right\|$$
$$\leq \frac{m^{m+1}}{m!} \lambda + m - 1 .$$

Hence letting $c(\lambda, m) = m - 1 + \frac{m^{m+1}}{m!} \lambda$ we have that $T_f \in B^{c(\lambda,m)}_{E' \otimes F}$, and therefore the proof by (*). ∎

In case of Banach spaces we have another equivalent formulation for the definition of the BAP. We say that a Banach space E has the λ-BAP if given a compact set $K \subset E$ and $\epsilon > 0$, there is a finite rank operator $T \in L(E; E)$ so that $\|T\| \leq \lambda$ and $\|Tx - x\| < \epsilon$ for every $x \in K$. We say that E has the BAP if E has the λ-BAP for some λ. Note that, in case of Banach spaces, this definition coincides with that definition of the BAP given in the introduction for the locally convex spaces.

Considering the above definition and using [13, Proposition 1], from the previous lemma, in particular, we obtain the following useful result.

Corollary 1. *Let E be a Banach space and let $1 \leq \lambda < \infty$. If $B_{\mathcal{P}(^m E;F)} \subset \overline{B^{\lambda}_{\mathcal{P}_f(^m E;F)}}^{\tau_c}$, for every Banach space F and for some $m \in \mathbb{N}$, then E has the $c(\lambda, m)$-BAP, where $c(\lambda, m) = m - 1 + \frac{m^{m+1}}{m!}\lambda$.*

Corollary 1 is applied in the proof of the following result which asserts that a Banach space E has the BAP if and only if, for each $m \in \mathbb{N}$ (or, equivalently for some $m \in \mathbb{N}$), the predual of $\mathcal{P}(^m E)$ has the BAP if and only if the predual of $\mathcal{H}^\infty(\mathcal{U})$ has the BAP.

Proposition 2. *Let U be a balanced open subset of a Banach space E and let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a sequence of bounded, balanced, open subsets of U such that $U = \bigcup_{n=1}^{\infty} U_n$ and $\rho_n U_n \subset U_{n+1}$, with $\rho_n > 1$, for every $n \in \mathbb{N}$. Then the following statements are equivalent:*

(a) *E has the BAP.*
(b) *$Q(^m E)$ has the BAP, for every $m \in \mathbb{N}$.*
(c) *$Q(^m E)$ has the BAP, for some $m \in \mathbb{N}$.*
(d) *$G^\infty(\mathcal{U})$ has the BAP.*

Proof. The equivalence of (a) and (b) follows from [13, Proposition 2], and the equivalence of (b) and (d) follows from Proposition 1 and [12, Proposition 2.7].

Since the implication (b) $\Rightarrow$ (c) is trivial we show that (c) implies (a) to complete the proof. Suppose that, for some $m \in \mathbb{N}$, $Q(^m E)$ has the λ-BAP, for some $\lambda \geq 1$. Then by applying [13, Proposition 1], [23, Theorem 2.4] and [23, Proposition 3.1] we see that $B_{\mathcal{P}(^m E;F)} \subset \overline{B^{\lambda}_{\mathcal{P}_f(^m E;F)}}^{\tau_c}$, for every Banach space F. Now it follows from Corollary 1 that E has the $c(\lambda, m)$-BAP, and hence we have (a). ∎

Observe that if $Q(^m E)$ has the BAP for some $m \in \mathbb{N}$ then, it actually has the BAP for every $m \in \mathbb{N}$ (and hence $G^\infty(\mathcal{U})$ has the BAP) by Proposition 2. Therefore the above proposition improves [13, Proposition 2] in some sense.

Previous proposition, by [24, Proposition 7.1], in particular, yields the following result for the predual of holomorphic mappings of bounded type $\mathcal{H}_b(U)$ constructed by Galindo, Garcia and Maestre in [17].

Corollary 2. *A Banach space E has the BAP if and only if $G_b(U)$ has the BAP, for every balanced open subset U of E.*

We note that the results similar to Proposition 2 and Corollary 2 were obtained by Mujica in [24] for the approximation property.

4. The bounded approximation property for weakly uniformly continuous type holomorphic mappings

Paralleling to a work of Aron and Schottenloher [5], in [9] Boyd, Dineen and Rueda gave some characterizations for the AP in connection with the space of weakly uniformly continuous holomorphic mappings. We now introduce a new class of holomorphic mappings, which will coincide with the space of weakly uniformly continuous holomorphic mappings in some particular cases. Let U be an open subset of a Banach space E, and let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be an increasing countable open cover of U. Let $\mathcal{H}_{wu}(\mathcal{U};F)$ denote the locally convex space

$$\mathcal{H}_{wu}(\mathcal{U};F) = \left\{ f \in \mathcal{H}(U;F) \ : \ \begin{array}{c} f \text{ is weakly uniformly} \\ \text{continuous on } U_n \text{ for every } n \end{array} \right\},$$

endowed with the topology of uniform convergence on all the sets U_n, which we call space of holomorphic mappings of weakly uniformly continuous type. When $F = \mathbb{C}$ we write $\mathcal{H}_{wu}(\mathcal{U})$ instead of $\mathcal{H}_{wu}(\mathcal{U};\mathbb{C})$.

It is clear that $\mathcal{H}_{wu}(\mathcal{U};F) = \mathcal{H}_{wu}(U;F)$ if $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ is a fundamental sequence of open U-bounded sets.

Given a locally convex space E, let $E_c^{'}$ denote the dual of E, endowed with the topology of uniform convergence on all convex, balanced, compact subsets of E. For locally convex spaces E and F the ε-product was introduced by L. Schwartz in [32] as the locally convex space $E\varepsilon F := L_\epsilon(E_c^{'};F)$, endowed with the topology of uniform convergence on equicontinuous subsets of $E^{'}$. There are several results relating the approximation property with the ε-product. We mention [5], [6], [9], [28], [31]. Besides, the notion of ε-product can be used to yield topological isomorphisms which are useful to study of approximation properties in connection with certain classes of holomorphic mappings. The following proposition states a result in this direction (for related results see [9] and [31]).

Proposition 3. *Let E and F be locally convex spaces with F is quasi-complete, let U be an open subset of E, and let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be an increasing countable open cover of bounded, balanced, open subsets of U. Then*

(a) *$\mathcal{H}_{wu}(\mathcal{U};F)$ is topologically isomorphic to $\mathcal{H}_{wu}(\mathcal{U})\varepsilon F$.*
(b) *$\mathcal{P}_w(^nE;F)$ is topologically isomorphic to $\mathcal{P}_w(^nE)\varepsilon F$ for each $n \in \mathbb{N}$.*

The proof of this proposition is an obvious modification of the proof of [9, Theorem 3], and is therefore omitted.

Let $L_{w,s}({}^nE;F)$ denote the space of symmetric continuous n-linear mappings from E into F which are weakly uniformly continuous on the bounded subsets of $E\times\ldots\times E$. The next result is inspired by the ideas of Boyd, Dineen and Rueda in [9], which is established there for the AP.

PROPOSITION 4. *Let E be a Banach space such that E' has the BAP. Then $L_{w,s}({}^nE)$ has the BAP for all $n\in\mathbb{N}$.*

Proof. In the proof of [9, Proposition 8] by using [20, § 43, 3(7), p. 243] and [30, Exercise 4.5] we get the proof. ∎

Let us recall that every complemented subspace of a locally convex space with the BAP has the BAP. Now, as a consequence of Proposition 3 and Proposition 4 we can give the following tensor characterizations of the bounded approximation property for $\mathcal{H}_{wu}(\mathcal{U})$ and $\mathcal{P}_w({}^nE)$, $n\in\mathbb{N}$.

THEOREM 1. *Let U be a balanced, open subset of a Banach space E, and let $\mathcal{U}=(U_n)_{n\in\mathbb{N}}$ be an increasing countable open cover of bounded open subsets of U. The following conditions are equivalent:*

(a) *E' has the BAP.*
(b) *For each $n\in\mathbb{N}$, $\mathcal{P}_w({}^nE)$ has the BAP.*
(c) *For each $n\in\mathbb{N}$, given a polynomial $P\in\mathcal{P}_w({}^nE;F)$ there is an equicontinuous net in $F\bigotimes\mathcal{P}_w({}^nE)$ which converges to P for the topology of uniform convergence on the equicontinuous subsets of F'_c, for every quasi-complete locally convex space F (equivalently for every complete locally convex space F).*
(d) *$\mathcal{H}_{wu}(\mathcal{U})$ has the BAP.*
(e) *Given a mapping $f\in\mathcal{H}_{wu}(\mathcal{U};F)$ there is an equicontinuous net in $F\bigotimes\mathcal{H}_{wu}(\mathcal{U})$ which converges to f for the topology of uniform convergence on the equicontinuous subsets of F'_c, for every quasi-complete locally convex space F (equivalently for every complete locally convex space F).*

Proof. The implications (b) ⇔ (c), (d) ⇔ (e) follow from Proposition 3 and [12, Proposition 2.8]. The implication (d) ⇒ (b) follows from the fact that $\mathcal{P}_w({}^nE)$ is a complemented subspace of $\mathcal{H}_{wu}(\mathcal{U})$ for every $m\in\mathbb{N}$, since each U_n is bounded. And since U is balanced and each U_n is bounded then, by using [15, (3.42)] one can see that $\{\mathcal{P}_w({}^nE)\}_{n=0}^{\infty}$ is an $\mathcal{S}$-absolute decomposition for

$\mathcal{H}_{wu}(\mathcal{U})$ (see [15, Proposition 3.36, p. 197]). Hence by [12, Proposition 2.7] we get the implication (b) $\Rightarrow$ (d).

On the other hand, since $\mathcal{P}_w(^nE)$ and $L_{w,s}(^nE)$ are isomorphic the implication (a) $\Rightarrow$ (b) follows from Proposition 4, and since $E^{'}$ is a complemented subspace of $\mathcal{H}_{wu}(\mathcal{U})$ we have that (d) $\Rightarrow$ (a).

The fact that the condition (e) (resp. (c)) for every complete locally convex space F is equivalent to the corresponding condition for every quasi-complete locally convex space F follows from [12, Proposition 2.8] since $(\mathcal{H}_{wu}(\mathcal{U})^{'}_{c})^{'}_{c} = \mathcal{H}_{wu}(\mathcal{U})$ (resp. $\mathcal{P}_w(^nE)^{'}_{c})^{'}_{c} = \mathcal{P}_w(^nE)$) and $\mathcal{H}_{wu}(\mathcal{U})^{'}_{c}$ (resp. $\mathcal{P}_w(^nE)^{'}_{c}$) is a complete locally convex space (see [19, § 21, 6 (4), p. 265]). ∎

Remark 1. In the above theorem some implications are true for more general settings:

(a) The implications (d) $\Leftrightarrow$ (e), (b) $\Leftrightarrow$ (c) are true, in general, for any increasing countable open cover of an open subset U of a Banach space E.

(b) The implication (d) $\Rightarrow$ (b) is true when each U_n is bounded, and U is need not to be balanced.

By [24, Proposition 7.1], from Theorem 1 in particular we obtain the following result for weakly uniformly continuous holomorphic mappings, which parallels to the aforementioned work of Boyd, Dineen and Rueda given in [9].

Corollary 3. *Let U be a balanced open subset of a Banach space E with $\mathcal{P}(^nE) = \mathcal{P}_w(^nE)$, for every $n \in \mathbb{N}$. The following statements are equivalent:*

(a) *$E^{'}$ has the BAP.*

(b) *$\mathcal{P}(^nE)$ has the BAP for each $n \in \mathbb{N}$.*

(c) *$\mathcal{H}_{wu}(U)$ has the BAP.*

We note that in the preceding corollary, for every $n \in \mathbb{N}$, $\mathcal{P}(^nE) = \mathcal{P}_w(^nE)$ holds if, in particular, E is the Tsirelson space which is a reflexive Banach space with a Schauder basis (see [1]). Also, in [15, Corollary 2.37 and Proposition 2.41] it is given conditions under which $\mathcal{P}(^nE) = \mathcal{P}_w(^nE)$, for every $n \in \mathbb{N}$ (see also [15, Proposition 2.49]).

In a series of papers Mujica [23],[24], and Mujica and Nachbin [26] gave linearization theorems for certain classes of spaces of holomorphic mappings. Using the linearization technique given in [24] (or in [26]) we prove the following result for weakly uniformly continuous type holomorphic mappings.

THEOREM 2. *Let U be an open subset of a locally convex space E, and let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be an increasing countable open cover of U. Then there are a complete, barelled, (DF)-space $G_{wu}(\mathcal{U})$ and a mapping $\delta_{\mathcal{U}} \in \mathcal{H}_{wu}(\mathcal{U}; G_{wu}(\mathcal{U}))$ with the following universal property: For each complete locally convex space F and each mapping $f \in \mathcal{H}_{wu}(\mathcal{U}; F)$, there is a unique mapping $T_f \in L(G_{wu}(\mathcal{U}); F)$ such that $T_f \circ \delta_{\mathcal{U}} = f$. This property characterizes $G_{wu}(\mathcal{U})$ uniquely up to a topological isomorphism.*

Proof. We proceed as in the proof of [26, Theorem 2.1]. If $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ is a countable open cover of U, and $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ is a sequence of strictly positive numbers then we set $B^{\alpha}_{\mathcal{U}wu} = \{f \in \mathcal{H}_{wu}(\mathcal{U}) : \|f\|_{U_n} \le \alpha_n \text{ for every } n\}$, which is a τ_c-compact subset of $\mathcal{H}_{wu}(\mathcal{U})$ by the Ascoli Theorem. Now we let $G_{wu}(\mathcal{U})$ be the space of all linear forms on $\mathcal{H}_{wu}(\mathcal{U})$ which when restricted to each $B^{\alpha}_{\mathcal{U}wu}$ are τ_c-continuous for every α. We endow this space with the topology of uniform convergence on all the sets $B^{\alpha}_{\mathcal{U}wu}$. Observe that $G_{wu}(\mathcal{U})$ is a complete, barelled DF-space. The evaluation mapping $\delta_{\mathcal{U}} : x \in U \to \delta_x \in G_{wu}(\mathcal{U})$ is defined by $\delta_x : f \in \mathcal{H}_{wu}(\mathcal{U}) \to f(x) \in \mathbb{C}$ for every $x \in U$, and we see that $\delta_{\mathcal{U}} \in \mathcal{H}(\mathcal{U}; G_{wu}(\mathcal{U}))$. But actually $\delta_{\mathcal{U}} \in \mathcal{H}_{wu}(\mathcal{U}; G_{wu}(\mathcal{U}))$. To see this we use an argument given in the proof of [9, Theorem 3]. Let n be any positive integer and consider a neighborhood of zero, in $\mathcal{H}_{wu}(\mathcal{U})$,

$$W = \{f \in \mathcal{H}_{wu}(\mathcal{U}) : |f(x)| \le 1 \text{ for all } x \in U_n\}.$$

Then $\delta_{\mathcal{U}}(U_n) \subset W^{\circ}$, where W° denotes the polar of W, and hence the weak topology coincides with the topology induced by $\mathcal{H}_{wu}(\mathcal{U})'_c$ on $\delta_{\mathcal{U}}(U_n)$. Since $(\mathcal{H}_{wu}(\mathcal{U})'_c)' = \mathcal{H}_{wu}(\mathcal{U})$, a fundamental system of weak neighborhoods of zero in $\mathcal{H}_{wu}(\mathcal{U})'_c$ restricted to $\delta_{\mathcal{U}}(U_n)$ has the form

$$V_0 = \{\phi \in \mathcal{H}_{wu}(\mathcal{U})' : |\phi(f_i)| < \varepsilon, i = 1, \dots, n\},$$

for some finite set $(f_i)_{i=1}^n \subset \mathcal{H}_{wu}(\mathcal{U})$ and $\varepsilon > 0$. Since each f_i is weakly uniformly continuous on U_n, there exists a weak neighborhood of zero W_0 in E such that $|f_i(x) - f_i(y)| < \varepsilon$ whenever $x, y \in U_n$, $x - y \in W_0$, for all $i = 1, \dots, n$. But $|f_i(x) - f_i(y)| = |\delta_x(f_i) - \delta_y(f_i)|$ and thus, $\delta_x - \delta_y \in V_0$ for all $x, y \in U_n$ such that $x - y \in W_0$. This proves that $\delta_{\mathcal{U}}$ is weak-weak uniformly continuous on U_n, and consequently $\delta_{\mathcal{U}} \in \mathcal{H}_{wu}(\mathcal{U}; G_{wu}(\mathcal{U}))$.

From this point on the argument used in the proof of [26, Theorem 2.1] works for our case also. ∎

From the preceding theorem, by [24, Proposition 7.1], in particular we obtain the following:

Proposition 5. *Let U be an open subset of a locally convex space E. Then there are a complete, barelled, (DF)-space $G_{wu}(U)$ and a mapping $\delta_U \in \mathcal{H}_{wu}(U; G_{wu}(U))$ with the following universal property: For each complete locally convex space F and each mapping $f \in \mathcal{H}_{wu}(U; F)$, there is a unique mapping $T_f \in L(G_{wu}(U); F)$ such that $T_f \circ \delta_U = f$. This property characterizes $G_{wu}(U)$ uniquely up to a topological isomorphism.*

A useful tool to determine that whether or not a given space has the (bounded) AP is the $\mathcal{S}$-absolute decomposition of spaces. By an easy modification of the proof of Proposition 1, we obtain the following $\mathcal{S}$-absolute decomposition for $G_{wu}(\mathcal{U})$.

Proposition 6. *Let U be an open, balanced subset of a Banach space E with $\mathcal{P}(^n E) = \mathcal{P}_w(^n E)$, for every $n \in \mathbb{N}$, and let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a sequence of bounded, balanced, open subsets of U such that $U = \bigcup_{n=1}^{\infty} U_n$ and $\rho_n U_n \subset U_{n+1}$, with $\rho_n > 1$, for every $n \in \mathbb{N}$. Then $\{Q(^n E)\}_{n=0}^{\infty}$ is an $\mathcal{S}$-absolute decomposition for $G_{wu}(\mathcal{U})$.*

As a consequence of the previous results for the preduals we obtain the main result of this section:

Proposition 7. *Let U be a balanced open subset of a reflexive Banach space E with $\mathcal{P}(^n E) = \mathcal{P}_w(^n E)$, for every $n \in \mathbb{N}$, and let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a sequence of bounded, balanced, open subsets of U such that $U = \bigcup_{n=1}^{\infty} U_n$ and $\rho_n U_n \subset U_{n+1}$, with $\rho_n > 1$, for every $n \in \mathbb{N}$. The following conditions are equivalent:*

(a) *E has the BAP.*
(b) *$\mathcal{P}(^n E)$ has the BAP for each $n \in \mathbb{N}$.*
(c) *$G_{wu}(\mathcal{U})$ has the BAP.*

Proof. Since E is reflexive the implications (a) $\Leftrightarrow$ (b) follows from [20, § 43, 8(4), p. 261] and Theorem 1. On the other hand by Proposition 2 we have that E has the BAP if and only if $Q(^n E)$ has the BAP for each $n \in \mathbb{N}$, and by Proposition 6 and [12, Proposition 2.7], if and only if $G_{wu}(\mathcal{U})$ has the BAP, proving the implications (a) $\Leftrightarrow$ (c). ∎

We remark that there are separable reflexive Banach spaces E with a Schauder basis (hence, with the BAP) such that $\mathcal{P}(^2 E)$ does not have the AP (see [5, Proposition 5.2]) (hence, does not have the BAP). Thus Proposition

7, being a positive result in this direction, shows that the condition that $\mathcal{P}(^nE) = \mathcal{P}_w(^nE)$, for every $n \in \mathbb{N}$, is a necessary condition in order to $\mathcal{P}(^nE)$, $n \in \mathbb{N}$, to have the bounded approximation property. We do not know if this assumption can be replaced by a weaker assumption, or, if this condition is the weakest condition for which $\mathcal{P}(^nE)$, $n \in \mathbb{N}$, has the (bounded) approximation property.

Now, by [24, Proposition 7.1], Proposition 7, in particular, yields the following characterization of the BAP for the predual of spaces of weakly uniformly continuous holomorphic mappings.

Corollary 4. *Let U be a balanced open subset of a reflexive Banach space E with $\mathcal{P}(^nE) = \mathcal{P}_w(^nE)$, for every $n \in \mathbb{N}$. Then E has the BAP if and only if $G_{wu}(U)$ has the BAP if and only if $\mathcal{P}(^nE)$ has the BAP for each $n \in \mathbb{N}$.*

5. The bounded approximation property and the predual of the space of bounded nolomorphic mappings on Banach spaces

Let E and F be complex Banach spaces and let U be an open subset of E. Let $\mathcal{H}^\infty(U;F)$ denote the Banach space of all bounded holomorphic mappings, with the norm of supremum. When $F = \mathbb{C}$ we write $\mathcal{H}^\infty(U)$ instead of $\mathcal{H}^\infty(U;\mathbb{C})$.

Similarly to the constructions established in [24] and [26], Mujica [23] constructed a Banach space $G^\infty(U)$ and a mapping $\delta_U \in \mathcal{H}^\infty(U;G^\infty(U))$ with the following universal property: For each Banach space F and each mapping $f \in \mathcal{H}^\infty(U;F)$, there is a unique operator $T_f \in L(G^\infty(U);F)$ such that $T_f \circ \delta_U = f$, where the space $G^\infty(U)$ is defined as the closed subspace of all linear functionals $u \in \mathcal{H}^\infty(U)'$ such that $u_{|B_{\mathcal{H}^\infty(U)}}$ is τ_c-continuous, which is called predual of $\mathcal{H}^\infty(U)$.

In [23] Mujica asks whether $G^\infty(U_E)$ has the BAP, whenever E has the BAP [23, 5.9 Problem]. We have no answer to this question yet when E is a non-separable Banach space (for the separable case see [13] for a solution). In the previous sections we have used the $\mathcal{S}$-absolute decompositions to obtain characterizations for the BAP concerning the preduals $G^\infty(\mathcal{U})$ and $G_{wu}(\mathcal{U})$. But there is no way to use this method to prove that $G^\infty(U)$ has the BAP whenever E has the BAP as the following observation shows.

Remark 2. Let U be a bounded, balanced, open subset of a separable reflexive Banach space E with $\mathcal{P}(^nE) = \mathcal{P}_w(^nE)$, for every $n \in \mathbb{N}$. Then, the sequence $\{Q(^nE)\}_{n=0}^{\infty}$ can not be an $\mathcal{S}$-absolute decomposition for $G^{\infty}(U)$. Indeed, by hypothesis, for each $n \in \mathbb{N}$, $Q(^nE)$ is a reflexive Banach space (see the proof of [27, Theorem 2.5] or [25, Theorem 3.1]). Now, if $\{Q(^nE)\}_{n=0}^{\infty}$ were an $\mathcal{S}$-absolute decomposition for $G^{\infty}(U)$, by [18, Theorem 3.2] this would imply that $G^{\infty}(U)$ is a reflexive Banach space. But as pointed out in [23, Remark 2.2] $G^{\infty}(U)$ is a separable Banach space also, and since $\mathcal{H}^{\infty}(U) = G^{\infty}(U)'$ [23, Theorem 2.1], this would imply that $\mathcal{H}^{\infty}(U)$ is separable, which is a contradiction (see [33, p. 93]).

Therefore, to prove the question that whether or not $G^{\infty}(U)$ has the BAP whenever E (necessarily non-separable) has the BAP, we can not appeal to $\mathcal{S}$-absolute decompositions used in Propositions 2 and 7. Furthermore, Remark 2 shows that Proposition 1 is false in the case $U = U_E$ and $U_n = U_E$ for all $n \in \mathbb{N}$. Also, the following example, shown to us by the referee, shows that Proposition 1 is false, in general, even if the sequence $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ is strictly increasing.

Example 1. Consider a complex Banach space E of dimension bigger than one. Take $x_0 \in E$ such that $\|x_0\| = 1$ and take $U = U_E$ and let

$$U_n = \frac{n-1}{n}\,\mathbb{D}x_0 \cup (U_E \backslash \mathbb{C}x_0)\,, \qquad n = 1, 2, 3, \dots\ ,$$

where $\mathbb{D}$ is the open unit ball in $\mathbb{C}$. Clearly $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ is strictly increasing countable cover of bounded balanced open subsets of U_E. But, since the closure in norm of U_n coincides with the closed unit ball of E for all n, we have $\mathcal{H}^{\infty}(\mathcal{U}) = \mathcal{H}^{\infty}(U_E)$ algebraically and topologically. Hence $\{Q(^nE)\}_{n=0}^{\infty}$ is not an $\mathcal{S}$-absolute decomposition for $G^{\infty}(\mathcal{U}) = G^{\infty}(U_E)$ by the remark above.

On the other hand, in [13], by following a different way, the author proved that a separable Banach space E has the BAP if and only if $G^{\infty}(U_E)$ has the BAP. Below we will give a slight improvement of this result.

Recall that a subspace of E is said to be 1-complemented if it is complemented subspace of E with the projection of norm 1. If U is a bounded open subset of a Banach space E, then by [23, Proposition 2.3] we know that E is topologically isomorphic to a complemented subspace of $G^{\infty}(U)$, and E is isometrically isomorphic to a 1-complemented subspace of $G^{\infty}(U_E)$. One can easily show that if E has the λ-BAP, $1 \le \lambda < \infty$, then every complemented subspace of E with the projection P has the $\lambda\,\|P\|$-BAP. Therefore, for a separable Banach space E we can summarize the results of [13, Proposition 2 and Proposition 3] and Proposition 2 as follows, in particular, by showing that a separable Banach space E has the λ-BAP if and only if $G^\infty(U_E)$ has the λ-BAP if and only if $Q(^mE)$ has the λ-BAP for every $m \in \mathbb{N}$.

PROPOSITION 8. *Let E be a separable Banach space E and let $1 \le \lambda < \infty$. The following are equivalent.*

(a) *For each Banach space F, $B_{\mathcal{H}^\infty(U_E;F)} \subset \overline{B^\lambda_{\mathcal{H}^\infty(U_E)\bigotimes F}}^{\tau_c}$.*
(b) *$G^\infty(U_E)$ has the λ-BAP.*
(c) *$Q(^mE)$ has the λ-BAP, for every $m \in \mathbb{N}$.*
(d) *For each Banach space F and for every $m \in \mathbb{N}$, $B_{\mathcal{P}(^mE;F)} \subset \overline{B^\lambda_{\mathcal{P}_f(^mE;F)}}^{\tau_c}$.*
(e) *E has the λ-BAP.*
(f) *For each Banach space F and for each open subset $V \subset F$, $B_{\mathcal{H}^\infty(V;E)} \subset \overline{B^\lambda_{\mathcal{H}^\infty(V)\bigotimes E}}^{\tau_c}$.*

Proof. The implications (a) $\Leftrightarrow$ (b), (e) $\Leftrightarrow$ (f) are direct consequences of [13, Proposition 3], and the implications (c) $\Leftrightarrow$ (d) follow from [23, Theorem 2.4 and Proposition 3.1] and [13, Proposition 1]. The implication (b) $\Rightarrow$ (c) follows from [23, Proposition 2.6] while (d) $\Rightarrow$ (e) is trivial.

Now we show that (e) $\Rightarrow$ (b). Suppose that (e) holds. Then it follows from [13, Corollary 3] that $G^\infty(U_E)$ has the λ'-BAP for some $\lambda' \ge 1$. It is easily seen that λ' must satisfy $\lambda' \ge \lambda$. Hence, from the implications (a) $\Rightarrow$ (f) and (e) $\Rightarrow$ (f) we conclude that $\lambda' = \lambda$. ∎

In [13] actually we show that $G^\infty(U_E)$ has the BAP whenever E has the BAP (see [13, Corollary 3]), and in Proposition 2 we see that, for every $m \in \mathbb{N}$, $Q(^mE)$ has the BAP whenever E has the BAP. But, in each of these results we did not give any estimation for the λ. It is just given a result in [13] stating that for every $m \in \mathbb{N}$, $Q(^mE)$ has the λ^m-BAP if and only if E has the BAP. Hence Proposition 8 sharpens [13, Corollary 3], Proposition 2, and also [13, Proposition 2].

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I would like to thank the referee for providing Example 1 and detecting several misprints on the original version. Also I am indebted to the referee for pointing out a mistake in the original statement of Lemma 2. There I had written "If $B_{\mathcal{P}(^mE;F)} = \overline{B^\lambda_{\mathcal{P}_f(^mE;F)}}^{\tau_c}$" instead of "If $B_{\mathcal{P}(^mE;F)} \subset \overline{B^\lambda_{\mathcal{P}_f(^mE;F)}}^{\tau_c}$". I take this opportunity to correct similar

mistakes in [13, Proposition 1, Proposition 2 and Proposition 3]. In each of them the symbol "=" should be replaced by the symbol "$\subset$" at the appropriate places. Clearly we cannot have equality when $\lambda > 1$.

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