PARTIAL REGULARITY AT THE FIRST SINGULAR TIME FOR HYPERSURFACES EVOLVING BY MEAN CURVATURE

KLAUS ECKER

ABSTRACT. In this paper, we consider smooth, properly immersed hypersurfaces evolving by mean curvature in some open subset of $\mathbb{R}^{n+1}$ on a time interval $(0, t_0)$. We prove that p - integrability with $p \geq 2$ for the second fundamental form of these hypersurfaces in some space-time region $B_R(y) \times (0, t_0)$ implies that the $\mathcal{H}^{n+2-p}$ - measure of the first singular set vanishes inside $B_R(y)$. For $p = 2$ and $n = 2$, this was established by Han and Sun. Our result furthermore generalizes previous work of Xu, Ye and Zhao and of Le and Sesum for $p \geq n + 2$, in which case the singular set was shown to be empty. By a theorem of Ilmanen, our integrability condition is satisfied for $p = 2$ and $n = 2$ if the initial surface has finite genus. Thus, the first singular set has zero $\mathcal{H}^2$- measure in this case. This is the conclusion of Brakke's main regularity theorem for the special case of surfaces, but derived without having to impose the area continuity and unit density hypothesis. It follows from recent work of Head and of Huisken and Sinestrari that for the flow of closed, k - convex hypersurfaces, that is hypersurfaces whose sum of the smallest k principal curvatures is positive, our integrability criterion holds with exponent $p = n + 3 - k - \alpha$ for all small $\alpha > 0$ as long as $1 \leq k \leq n - 1$. Therefore, the first singular set of such solutions is at most $(k - 1)$ - dimensional, which is an optimal estimate in view of some explicit examples.

1. INTRODUCTION

A family of smooth, properly immersed hypersurfaces $\mathcal{M} = (M_t)_{t \in (0,t_0)}$ in $\mathbb{R}^{n+1}$ *evolves by mean curvature* if

$$\frac{\partial x}{\partial t} = \vec{H}(x)$$

for $x \in M_t$ and $t \in (0, t_0) \subset \mathbb{R}$. Here $\vec{H}(x)$ is the mean curvature vector at $x \in M_t$. Our sign convention is $\vec{H} = -H\nu$, where $H = \operatorname{div} \nu$ is the mean curvature of M_t with respect to some choice ν of unit normal field.

Mean curvature flow was first studied in the framework of geometric measure theory by Brakke ([B]) in 1978 and in the smooth setting by Huisken ([Hu1]) in 1984. We would also like to mention the level set approach due to Chen, Giga and Goto ([CGG]) and to Evans and Spruck ([ES1] - [ES3]) which will, however, not form the topic of the present paper. While Huisken established that convex hypersurfaces asymptotically converge smoothly to round spheres under mean curvature flow, Brakke developed a general regularity theory in his monograph. In the special situation where one considers the flow of smooth hypersurfaces possibly becoming singular in various places at time t_0, his result can be explained without too much notational preparation. We follow here, and often quote from [E2], where a detailed account of Brakke's work in this context can be found. All the results in [E2] are stated for properly embedded solutions, but the ones we need here actually

Date: May 22, 2012.

only require that the solution is properly immersed.

A solution $\mathcal{M} = (M_t)_{t\in(0,t_0)}$ of mean curvature flow *reaches a point* $x_0 \in \mathbb{R}^{n+1}$ *at time* t_0 if there exists a sequence of times $t_j \nearrow t_0$ and points $x_j \in M_{t_j}$ such that $x_j \to x_0$. We say that x_0 *is a singular point of* $\mathcal{M}$ *at time* t_0 if it is reached by the solution at this time, and if $\mathcal{M} = (M_t)_{t\in(0,t_0)}$ has no smooth extension beyond time t_0 in any neighbourhood of x_0. All other points (which include those not reached by the solution) are termed *regular points of* $\mathcal{M}$. The set of singular points of $\mathcal{M}$ at time t_0 we call the *singular set at time* t_0, and denote it by $\mathrm{sing}_{t_0}\mathcal{M}$. If the singular set is nonempty, we call t_0 the *first singular time* since the flow $\mathcal{M} = (M_t)_{t\in(0,t_0)}$ is assumed to be smooth.

An important assumption in Brakke's regularity theory is the *area continuity and unit density hypothesis* (Brakke did not use this terminology), which for smooth flows can be stated as follows: A smooth, properly immersed solution of mean curvature flow $\mathcal{M} = (M_t)_{t\in(0,t_0)}$ inside an open set $U \subset \mathbb{R}^{n+1}$ satisfies the *area continuity and unit density hypothesis at time* t_0 if the hypersurfaces M_t converge for $t \nearrow t_0$ inside U in the sense of Radon measures to an $\mathcal{H}^n$- measurable, countably n - rectifiable set M_{t_0} of locally finite $\mathcal{H}^n$- measure.

For more detailed information on such sets we refer to [S]. In particular, they generalize the notion of a hypersurface as they admit an approximate tangent space at almost every point. Roughly speaking, the above hypothesis ensures that at the first singular time the evolving hypersurfaces do not become too irregular, and form no double or even multiple sheets in a set of positive $\mathcal{H}^n$- measure. Formation of multiple sheets cannot occur in an open set inside the evolving hypersurface due to the strong maximum principle, but multiple sheets could potentially start developing in Cantor type sets of positive $\mathcal{H}^n$- measure without interior points inside these hypersurfaces. There exist generalized surfaces with arbitrarily small mean curvature which contain double sheeted regions, see an example in [B], which is discussed in more detail in Chapter 5 of [E2]. We are now ready to state Brakke's main regularity theorem in the special case where we consider the first singular time.

Theorem (Brakke's Main Regularity Theorem ([B])). *Let* $\mathcal{M} = (M_t)_{t\in(0,t_0)}$ *be a smooth, properly immersed solution of mean curvature flow inside an open set* $U \subset \mathbb{R}^{n+1}$ *which satisfies the area continuity and unit density hypothesis at time* t_0*. Then*

$$\mathcal{H}^n(\mathrm{sing}_{t_0}\mathcal{M} \cap U) = 0.$$

To this day, this is the strongest result about the first singular set in the general case, that is without any additional assumptions on the initial hypersurface. In the special case where the initial hypersurface is embedded and *mean-convex*, that is has positive mean curvature, conditions which are preserved during the flow, White ([W1], [W2]) proved that the dimension of the singular set is at most $n-1$. The embeddedness assumption enters since he considers hypersurfaces which are boundaries of regions in $\mathbb{R}^{n+1}$. White's result is optimal in view of the examples of a shrinking cylinder or a shrinking torus of positive mean curvature. One might conjecture that his estimate also holds in the general case, that is without the assumption of mean-convexity.

In [HaSu], Han and Sun proved Brakke's main regularity theorem in the case $n = 2$ under the additional assumption that the surfaces are closed but without having to impose the area continuity and unit density hypothesis. Their main tool is an ϵ - regularity result involving a space-time integral of the square norm of the second fundamental form, a weaker version of which was first proved in [I1]. We will generalize this ϵ - regularity theorem to a version involving the L^p - norm of the second fundamental form. Moreover, we will derive Brakke's theorem for closed, immersed, mean-convex solutions in general dimensions without the area continuity and unit density hypothesis. For embedded solutions, this result follows of course immediately from [W1] and [W2], but our alternative argument provides an interesting technical criterion from which the conclusion of Brakke's theorem follows quite easily. Our first theorem introduces a p - integrability condition for the second fundamental form which leads to an optimal estimate on the size of the first singular set. This generalizes previous work of Xu, Ye and Zhao ([XYZ]) for the case $p \geq n+2$ which in turn is related to a theorem proved by Le and Sesum, see [LS1].

Theorem 1.1. *Let $\mathcal{M} = (M_t)_{t\in(0,t_0)}$ be a smooth, properly immersed solution of mean curvature flow inside an open ball $B_R(y) \subset \mathbb{R}^{n+1}$ which satisfies*

$$\int_0^{t_0} \int_{M_t \cap B_R(y)} |A|^p < \infty$$

for some $p \geq 2$, where $|A|$ denotes the norm of the second fundamental form of the solution. Then

$$\mathcal{H}^{n+2-p}(\mathrm{sing}_{t_0}\mathcal{M} \cap B_R(y)) = 0$$

for $p \in [2, n+2]$ and $\mathrm{sing}_{t_0}\mathcal{M} \cap B_R(y) = \emptyset$ for $p \geq n+2$.

The validity of our condition with $p = m+2-\alpha$, $0 \leq m \leq n-\alpha$ and all small $\alpha > 0$ implies

$$\mathcal{H}^{n-m+\alpha}(\mathrm{sing}_{t_0}\mathcal{M} \cap B_R(y)) = 0$$

for these $\alpha > 0$. From the definition of Hausdorff dimension we thus obtain

Corollary 1.2. *Let $\mathcal{M} = (M_t)_{t\in(0,t_0)}$ be a smooth, properly immersed solution of mean curvature flow inside an open ball $B_R(y) \subset \mathbb{R}^{n+1}$. Suppose the condition of Theorem 1.1 holds inside this ball with $p = m+2-\alpha$ for some $0 \leq m \leq n$ and all small $\alpha > 0$. Then*

$$\dim(\mathrm{sing}_{t_0}\mathcal{M} \cap B_R(y)) \leq n-m.$$

Remark 1.3. (i) The above integrability condition with $m = 1$, that is $p = 3-\alpha$, would imply

$$\dim(\mathrm{sing}_{t_0}\mathcal{M} \cap B_R(y)) \leq n-1,$$

the dimension estimate known to hold for embedded, mean-convex solutions by [W1] and [W2]. As mentioned earlier, this upper bound on the dimension of the singular set cannot be improved in view of known examples. We do not know yet if closed, embedded mean-convex solutions satisfy the integrability assumption of Theorem 1.1 with $p = 3-\alpha$ for all small $\alpha > 0$. However, the condition of Theorem 1.1 with this exponent is easily verified for the standard examples such as shrinking spheres, cylinders and the symmetric torus of positive mean curvature. An elementary calculation, for instance for the two-dimensional shrinking cylinder with axis containing the origin, also shows that

$$\int_0^{t_0}\int_{M_t\cap B_R(0)}|A|^3=\infty$$

for any $R>0$. Therefore, we cannot conclude that the singular set has vanishing $\mathcal{H}^1$-measure in this case. As expected, this matches the fact that the dimension of the singular set of the two-dimensional shrinking cylinder solution (in this case the cylinder axis) equals 1.

(ii) If the condition of Corollary 1.2 holds with $n=2$ and $m=2$, which corresponds to $p=4-\alpha$, then

$$\dim\left(\mathrm{sing}_{t_0}\mathcal{M}\cap B_R(y)\right)=0.$$

In Chapter 5 of [E2], we conjectured that mean curvature flow of smooth, embedded closed, connected surfaces in $\mathbb{R}^3$ would produce only a finite number of singular points (thus in particular a zero-dimensional set $\mathrm{sing}_{t_0}\mathcal{M}$) if at the first singular time the area of the evolving surfaces has not vanished entirely. The area disappears for instance for the symmetric torus of positive mean curvature, which shrinks to a circle, that is has a one-dimensional first singular set. Our conjecture was partly motivated by the results in [AAG] for embedded closed, connected surfaces of revolution. Although it may turn out to be unrealistic to approach this conjecture by trying to prove that embedded, closed, connected surfaces which do not disappear at time t_0 satisfy

$$\int_0^{t_0}\int_{M_t\cap B_R(y)}|A|^{4-\alpha}<\infty$$

for all $\alpha\in(0,1]$, it would nevertheless be interesting to see if this integrability condition holds under the additional assumption of axial symmetry. Note, that the shrinking sphere solution, which disappears at the first singular time, is special here, since it satisfies the integrability condition anyway. For the example of the symmetric initial torus with positive mean curvature, this integral is infinite for all $\alpha\in[0,1]$. Closed, convex hypersurfaces evolving by mean curvature and all closed, immersed, homothetically shrinking hypersurfaces, even if they do not solve mean curvature flow, satisfy this integrability condition, see Remark 1.6 (ii) below and the next item of this remark.

(iii) One easily checks by a scaling argument that closed, immersed, homothetically shrinking families $(M_t)_{t\in[0,t_0)}$ of hypersurfaces in $\mathbb{R}^{n+1}$, that is where

$$M_t=\sqrt{1-\frac{t}{t_0}}\,M_0$$

for all $t\in[0,t_0)$ (we assume here without loss of generality that they shrink to the origin in $\mathbb{R}^{n+1}$) satisfy the integrability condition of Theorem 1.1 with $p=n+2-\alpha$ for any $\alpha\in(0,1]$. We do not need to assume that these families solve mean curvature flow, which would require in addition that M_t satisfies the equation $H=\frac{x\cdot\nu}{2(t_0-t)}$ for all $t\in[0,t_0)$. For homothetically shrinking solutions of mean curvature flow though, Corollary 1.2 applies, predicting a zero-dimensional singular set. This matches the fact that the singular set of these special solutions consists of the origin only.

(iv) We can also state sufficient conditions on smooth, properly immersed, complete (non-compact) hypersurfaces M_0 which ensure that the associated family $(M_t)_{t\in[0,t_0)}$ defined in (iii) above satisfies the condition of Theorem 1.1 for given $p \geq 2$. The required calculations involve only a scaling argument, the co-area formula applied to M_0 and the evaluation of elementary one-dimensional integrals. Mean curvature flow is not used. We leave the details as a straightforward exercise to the reader. Suppose, for instance, that M_0 satisfies the condition $|A(x)| = O(|x|^{-\beta})$, $|x| \to \infty$ for some $\beta \geq 0$, and $\mathcal{H}^{n-1}(M_0 \cap \partial B_r(0)) = O(r^{s-1})$, $r \to \infty$ for some $s \geq 1$. From this, one can then work out relations between n, s, β and p which imply the p - integrability condition of Theorem 1.1 for an exponent we may want to prescribe a priori or determine a posteriori from given n, s and β. Let us only discuss some special cases here, which correspond to actual explicit examples. These, as well as the previous remarks, illustrate that the integrability condition of Theorem 1.1 is optimal even for $p > 2$.

If for example $\beta = 0$ and $s = l$ where l is an integer between 0 and $n-1$, we obtain integrability with $p = n + 2 - l - \alpha$ for any $\alpha \in (0, 1]$, and therefore, if the family also moves by mean curvature, a singular set of at most l dimensions by Corollary 1.2. These conditions on M_0 are satisfied, for instance, on shrinking solutions of mean curvature flow of the type $S^{n-l} \times \mathbb{R}^l$, $0 \leq l \leq n-1$, which have constant $|A|$, that is $\beta = 0$, and an l - dimensional singular set. The case $s = l = n$ and $\beta = 0$ is not interesting for us as it leads to integrability with exponent $p = 2 - \alpha$, $\alpha \in (0, 1]$ which is less than 2.

On the other hand, we could consider the choice $s = n$ which is a natural one for many hypersurfaces, and ask for which range of exponents β we obtain integrability of $|A|$ with $p = n+2-\alpha$ for any $\alpha \in (0, 1]$. It turns out that $\beta > \frac{n-\alpha}{n+2-\alpha}$ is sufficient, so in particular $\beta = 1$ will do. Corollary 1.2 implies that homothetic solutions starting from a hypersurface M_0 satisfying these conditions have a zero-dimensional singular set. For $n = 2$, the above corresponds to the exponent $p = 4 - \alpha$ for any $\alpha \in (0, 1]$.

The validity of our integrability condition with $p = 4-\alpha$, $\alpha \in (0, 1]$ should be checked for the homothetically shrinking solution found by Chopp in [Ch]. Some of its geometric properties are also described in [I2]. Further examples along these lines are constructed in [ACI]. Chopp's solution is of the form $N_t = \sqrt{-t}\, N$, $t < 0$, where $N \subset \mathbb{R}^3$ is a smooth, properly immersed, complete (non-compact) two-dimensional surface which is asymptotic at infinity to a cone with an isolated singularity at the origin. In particular, the solution (N_t) converges to this cone for $t \nearrow 0$ in a suitable sense. By the above discussion (with time interval $[-1, 0)$ instead of $[0, t_0)$), it is sufficient to check the conditions on the second fundamental form and the boundary area growth of large balls intersecting N in order to verify $(4 - \alpha)$ - integrability for all $\alpha \in (0, 1]$. As the solution is asymptotic to the cone at infinity, we suspect that $s = n = 2$ may be the correct choice. The surface N would then have to satisfy $|A(x)| = O(|x|^{\beta})$, $|x| \to \infty$ for $\beta > \frac{2-\alpha}{4-\alpha}$. In particular, any $\beta > \frac{1}{2}$, such as for example $\beta = 1$, which corresponds to the decay rate for cones, would be sufficient. We could, however, not find this information for $|A|$ in [Ch] or [I2]. The integrability condition with $p = 4$, however, cannot hold for this example, as this would imply by Theorem 1.1 that the singular set is empty, contradicting the fact that the cone which is formed at time 0 has an isolated singularity at the origin.

(v) The integrability condition with $p = 2$ is satisfied for smooth, properly immersed surfaces (that is $n = 2$) with finite genus by results of Ilmanen ([I1]).

Corollary 1.4. *Let $\mathcal{M} = (M_t)_{t\in(0,t_0)}$ be a smooth, properly immersed family of surfaces in $\mathbb{R}^3$ (that is $n = 2$) evolving by mean curvature. Suppose that the initial surface has finite genus. Then*

$$\mathcal{H}^2(\mathrm{sing}_{t_0}\mathcal{M}) = 0.$$

In the case of closed surfaces this theorem was proved in [HaSu].

In [HS1], Huisken and Sinestari proved in the case $n \geq 2$ that if the evolving hypersurfaces M_t are immersed, closed and mean-convex, that is have positive mean curvature (this follows from the mean-convexity of M_0 by the maximum principle), then there is a constant $c_1 > 0$ depending only on the initial hypersurface M_0 (c_1 is actually the supremum of $|A|^2/H^2$ on M_0) such that

$$|A|^2 \leq c_1 H^2$$

holds pointwise on all hypersurfaces M_t. On the other hand, the evolution equation

$$\frac{d}{dt}\mu_t = -H^2\mu_t$$

for the area element derived in [Hu1] yields

$$\int_0^{t_0}\int_{M_t} H^2 \leq \mathcal{H}^n(M_0) < \infty.$$

Combining these, shows that the condition of Theorem 1.1 with $p = 2$ is satisfied for closed, immersed, mean-convex hypersurfaces, and therefore one obtains

Corollary 1.5. *Let $\mathcal{M} = (M_t)_{t\in(0,t_0)}$ be a smooth, immersed solution of mean curvature flow in $\mathbb{R}^{n+1}$ for $n \geq 2$ consisting of closed, mean-convex hypersurfaces. Then*

$$\mathcal{H}^n(\mathrm{sing}_{t_0}\mathcal{M}) = 0.$$

We would like to reiterate that for embedded solutions this is an immediate consequence of White's much stronger dimension estimate ([W1], [W2]), but the curvature integral approach is entirely different.

Remark 1.6. In the case of immersed, closed, *two-convex* hypersurfaces, that is hypersurfaces satisfying the condition $\kappa_1 + \kappa_2 > 0$ for their lowest two principal curvatures (this condition is stronger than mean-convexity in more than two dimensions but weaker than convexity, and it is also preserved by mean curvature flow), Head ([H]) showed that for $n \geq 3$

$$\int_0^{t_0}\int_{M_t} H^{n+1-\alpha} < \infty$$

holds for every $\alpha \in (0, 1]$. An adaptation of Head's calculation in combination with a recent estimate due to Huisken and Sinestrari ([HS3]) shows (see Chapter 3 of this paper) that

$$\int_0^{t_0}\int_{M_t} H^{n+3-k-\alpha} < \infty$$

for all $\alpha \in (0, 1]$ whenever $1 \leq k \leq n-1$ and the solution is closed and k - *convex*, that is satisfies $\kappa_1 + \cdots + \kappa_k > 0$ for its smallest k principal curvatures. Since k - convex hypersurfaces are in particular mean-convex, the discussion preceding Corollary 1.5 yields that the inequality $|A|^2 \leq c_1 H^2$ holds pointwise on every M_t. Thus, for closed, immersed k - convex solutions the condition of Theorem 1.1 is satisfied with $p = m + 3 - k - \alpha$ for any $\alpha \in (0, 1]$ and any integer $k \in [1, n-1]$. In 1984, Huisken ([Hu1]) proved that closed, convex initial hypersurfaces for $n \geq 2$ contract smoothly to a 'round' point in finite time. In particular, the singular set of the solution starting from such a hypersurface consists of just one point. The above discussion in the convex case, that is for $k = 1$, implies that the condition of Theorem 1.1 is satisfied with $p = n + 2 - \alpha$ for any $\alpha \in (0, 1]$.

Remark 1.6 in combination with Corollary 1.2 implies the following dimension estimate for closed, properly immersed k - convex solutions in the case $1 \leq k \leq n - 1$. The calculations leading to the relevant integrability condition are carried out in Chapter 3.

Corollary 1.7. *Let $\mathcal{M} = (M_t)_{t \in (0, t_0)}$ be a smooth, properly immersed solution of mean curvature flow in $\mathbb{R}^{n+1}$. Suppose that $1 \leq k \leq n - 1$ and that the solution hypersurfaces are closed and k - convex, that is $\kappa_1 + \cdots + \kappa_k > 0$ for the first k principal curvatures (these conditions are preserved during mean curvature flow). Then*

$$\dim \left(\mathrm{sing}_{t_0} \mathcal{M} \right) \leq k - 1.$$

This result is optimal in view of the example of a solution with initial data given by a symmetric $S^2_r \times S^1_R$ which is 2 - convex for suitable $R >> r > 0$. Due to the preservation of its S^1-symmetry and since it 'moves inward on itself' due to the fact that its mean curvature is positive, this solution contracts to a circle in finite time, hence resulting in a one-dimensional singular set. Our result also matches the results of Huisken and Sinestrari in [HS2], where it is shown that singularities of closed, two-convex solutions in dimensions $n \geq 3$ asymptotically look like S^n or $S^{n-1} \times \mathbb{R}$. Their result in combination with Head's work on weak solutions in the two-convex case ([H]) can probably also be used to prove our dimension estimate. Note also, that the homothetically shrinking solution of the type $S^{n-k+1} \times \mathbb{R}^{k-1}$ is k - convex and has a singular set of dimension $k - 1$, although we should point out that this is not a closed solution, except in the convex case where $k = 1$. For technical reasons, we cannot yet deal with the mean-convex case, that is with $k = n$.

The proof of Theorem 1.1 is based on the following local regularity result, a weaker form of which was proved by Ilmanen ([I1]) for $p = 2$. In the case $p = 2$, Theorem 1.8 is due to Han and Sun ([HaSu]). For $p = n + 2$, it is due to Le and Sesum ([LS2]). For this exponent, one may simply choose $\rho' = \rho_0$ and $\rho = 0$ below due to the scaling invariance of the double integral. The restriction to $p \in [2, n + 2]$ is not essential but the case $p > n + 2$ is not needed for our applications.

Theorem 1.8. *There exist constants $\epsilon_0 > 0$ and $c_0 > 0$ such that for any smooth, properly immersed solution $\mathcal{M} = (M_t)_{t \in (0, t_0)}$ of mean curvature flow inside an open set $U \subset \mathbb{R}^{n+1}$, every $x_0 \in U$ which the solution reaches at time t_0, every $\rho_0 \in (0, \sqrt{t_0}/2)$ for which $B_{2\rho_0}(x_0) \subset U$ and all $p \in [2, n + 2]$ the assumption*

$$\sup_{0 \leq \rho < \rho' \leq 2\rho_0} \frac{1}{\left(\rho'^2 - \rho^2 \right)^{\frac{n+2-p}{2}}} \int_{t_0 - \rho'^2}^{t_0 - \rho^2} \int_{M_t \cap B_{\sqrt{\rho'^2 - \rho^2}}(x_0)} |A|^p \leq \epsilon_0$$

implies

$$\sup_{(t_0 - \frac{\rho_0^2}{4}, t_0)} \sup_{M_t \cap B_{\frac{\rho_0}{2}}(x_0)} |A|^2 \leq \frac{c_0}{\rho_0^2}.$$

The local smoothness estimates from [EH2] or Chapter 3 of [E2] imply that if $|A|^2$ is bounded on a parabolic cylinder around (x_0, t_0), then the norms of the covariant derivatives of all orders of the second fundamental form (and therefore the entire geometry of the evolving hypersurfaces) are bounded on a smaller parabolic cylinder, but one which includes a space-time neighbourhood of the point (x_0, t_0). By a standard argument, the solution can therefore be extended in some space-time neighbourhood of (x_0, t_0), and therefore points which satisfy the above integral criterion have to be regular points of the flow.

Theorem 1.8 then implies that there exists a constant $\epsilon_0 > 0$ such that for each fixed $p \in [2, n+2]$ and each singular point x_0 at time t_0 there exist sequences $t_k \nearrow t_0$ and $r_k \searrow 0$ with

$$\int_{t_k - r_k^2}^{t_k} \int_{M_t \cap B_{r_k}(x_0)} |A|^p \geq \epsilon_0 \, r_k^{n+2-p}$$

for all $k \in \mathbb{N}$. This, in combination with a straightforward application of Vitali's covering theorem (see [HaSu], [S] or [E2]), implies our main theorem.

Theorem 1.8 in the case $p = 2$, which was proved by Han and Sun in [HaSu], is an improvement of an earlier result established independently by Nakauchi in [N] and the author in [E1], which arrives at the same conclusion as the one in Theorem 1.8, by assuming the stronger condition

$$\sup_{(t_0 - \rho_0^2, t_0)} \int_{M_t \cap B_{\rho_0}(x_0)} |A|^2 \leq \epsilon_0$$

in the special case $n = 2$. It also improves a local regularity result in general dimensions due to Ilmanen, see [I1].

Section 2 contains a standard mean value inequality for subsolutions of the heat operator on hypersurfaces evolving by their mean curvature, which is then applied in the proof of Theorem 1.8. Even in the case $p = 2$, our proof differs from the one in [HaSu]. In Section 3, the proofs of Theorem 1.1 and some of its above-mentioned consequences are presented.

The research for this paper has been partially supported by Project B3 of the Sonderforschungsbereich (Collaborative Research Centre) SFB 647 Raum-Zeit-Materie (Space-Time-Matter) funded by the Deutsche Forschungsgemeinschaft (German Research Foundation). The author would also like to thank John Head and Gerhard Huisken for very useful discussions and Tom Ilmanen for pointing out an error in an earlier version of this paper.

2. Proof of the Local regularity theorem

In this chapter, we present the proof of Theorem 1.8, the partial regularity result discussed in the introduction. In the case $p = 2$, this was established in [HaSu], see also [I1] for an earlier weaker version. Our proof, which differs from the one in [HaSu], uses a scaling argument similar to the one employed in the Appendix of [E1], where a weaker version of this regularity theorem for $p = 2$ was established. The work in [E1] is in turn based on methods used in [CS] and [St].

Since we shall make use of a local mean value inequality (as was also done in [E1], [I1] and [N]), we include its statement and some comments on it here. This mean value inequality was first proved for subsolutions of linear parabolic equations in divergence form by Moser ([M]). A proof can also be found in the monographs [LSU] or [L] .

In the case of mean curvature flow, which is a quasilinear parabolic system, one can either write the solution locally as a graph, and then quote for instance a suitable chapter in [L] or in [LSU] (this was done by Ilmanen in [I1]), or one can adapt Moser's iteration proof directly to the evolving hypersurfaces by using the Michael-Simon Sobolev inequality for hypersurfaces from [MS]. The latter approach was taken in [E1]. As a further alternative, one can replace the Moser iteration scheme by the well-known monotonicity formula of Huisken ([Hu2]), or more precisely a weighted version of it proved in [EH1]. This method was adopted in Chapter 4 of [E2].

Proposition 2.1. *Let $\mathcal{M} = (M_t)_{t\in(0,t_0)}$ be a smooth, properly immersed solution of mean curvature flow inside an open set U. Let $u \geq 0$ be a smooth function defined on these hypersurfaces satisfying the inequality*

$$\left(\frac{d}{dt} - \Delta_{M_t}\right) u \leq 0$$

inside U for all $t \in (0, t_0)$. Suppose $x_0 \in U$ is reached by the solution at time t_0. Then for all $\rho \in (0, \sqrt{t_0})$ for which $B_\rho(x_0) \subset U$ we have the estimate

$$\sup_{(t_0-\frac{\rho^2}{4},t_0)} \sup_{M_t\cap B_{\frac{\rho}{2}}(x_0)} u \leq c(n)\, \rho^{-(n+2)} \int_{t_0-\rho^2}^{t_0} \int_{M_t\cap B_\rho(x_0)} u.$$

The version in [E2] involves the L^2- norm of u (called f therein) on the right hand side, while the proof of Proposition 1.6 in Chapter 1 of [E1] contains a minor error at the end, which results in an L^2- norm inequality rather than the above stated L^1- version also claimed in Proposition 1.6 of [E1]. We take the opportunity to correct this error here. In order to achieve this, we resort to an advanced calculus trick by which the above L^1- mean value inequality is derived from its corresponding L^2- version. We found this argument in a set of handwritten lecture notes due to Schoen [Sch], which were made available to us by Robert Bartnik.

In order to streamline our notation, we will, for the duration of this argument, use the symbol $\mathcal{M}$ to refer both to the smooth, properly immersed solution $(M_t)_{t\in(0,t_0)}$ as well as its space-time track

$$\bigcup_{t=0}^{t_0} M_t \times \{t\}.$$

For $\sigma \in (0, \sqrt{t_0})$, denote the parabolic cylinder $B_\sigma(x_0) \times (t_0 - \sigma^2, t_0)$ by $C_\sigma(x_0, t_0)$. We then use the abbreviations

$$\sup_{\mathcal{M}\cap C_\sigma(x_0,t_0)} f \equiv \sup_{(t_0-\sigma^2,t_0)} \sup_{M_t\cap B_\sigma(x_0)} f$$

and

$$\iint_{\mathcal{M}\cap C_\sigma(x_0,t_0)} f \equiv \int_{(t_0-\sigma^2,t_0)} \int_{M_t\cap B_\sigma(x_0)} f.$$

We start with inequality (i) of Proposition 4.25 in [E2] applied at all points in $C_{\frac{\rho}{2}}(x_0,t_0)$. This implies that under our above conditions (we forgot to write down but used the assumption $u \geq 0$ in [E2]) we have the inequality

$$\sup_{\mathcal{M}\cap C_{\frac{\rho}{2}}(x_0,t_0)} u \leq c(n)\left(\frac{1}{\rho^{n+2}}\iint_{\mathcal{M}\cap C_\rho(x_0,t_0)} u^2\right)^{\frac{1}{2}}. \tag{1}$$

Inequality (1) can also be obtained from the Moser iteration type proof of Proposition 1.6 in [E1] by extracting the supremum of $u^2\eta^\beta$ (rather than $u^3\eta^\beta$ as stated there) on the right hand side of the mean value inequality for u^4 (the third last inequality of the proof), dividing the resulting inequality by this expression and then using the definition of η (by scaling we assumed $\rho = 1$ there).

By an adjustment of the cut-off function used in the derivation of (1) (namely by choosing it to be equal to 1 in $C_\sigma(x_0,t_0)$ and equal to 0 outside $C_{\sigma+r}(x_0,t_0)$) we obtain instead

$$\sup_{\mathcal{M}\cap C_\sigma(x_0,t_0)} u \leq c(n)\, r^{-\frac{n+2}{2}}\left(\iint_{\mathcal{M}\cap C_{\sigma+r}(x_0,t_0)} u^2\right)^{\frac{1}{2}} \tag{2}$$

for all $\sigma, r > 0$ with $\sigma + r \leq \rho$. This leads to

$$\sup_{\mathcal{M}\cap C_\sigma(x_0,t_0)} u \leq c(n)\, r^{-\frac{n+2}{2}}\left(\sup_{\mathcal{M}\cap C_{\sigma+r}(x_0,t_0)} u\right)^{\frac{1}{2}}\left(\iint_{\mathcal{M}\cap C_\rho(x_0,t_0)} u\right)^{\frac{1}{2}}. \tag{3}$$

We set

$$v = \frac{u}{\iint_{\mathcal{M}\cap C_\rho(x_0,t_0)} u},$$

where we may assume without loss of generality that the denominator does not vanish. Inequality (3) implies

$$\sup_{\mathcal{M}\cap C_\sigma(x_0,t_0)} v \leq c(n)\, r^{-\frac{n+2}{2}}\left(\sup_{\mathcal{M}\cap C_{\sigma+r}(x_0,t_0)} v\right)^{\frac{1}{2}}. \tag{4}$$

We now define $\sigma_0 = \rho/2$ and $\sigma_{i+1} = \sigma_i + r_i$, where $r_i = \rho \cdot 2^{-i-2}$ for $i \in \mathbb{N} \cap \{0\}$. Inequality (4) then yields for all such i

$$\sup_{\mathcal{M}\cap C_{\sigma_i}(x_0,t_0)} v \leq c(n)\, \rho^{-\frac{n+2}{2}}\left(2^{\frac{n+2}{2}}\right)^{i+2}\left(\sup_{\mathcal{M}\cap C_{\sigma_{i+1}}(x_0,t_0)} v\right)^{\frac{1}{2}}. \tag{5}$$

Iterating (5) for i between 0 and $j-1$ with $j \in \mathbb{N}$ and using that $\sigma_i < \rho$ for all i leads to

$$\sup_{\mathcal{M}\cap C_{\frac{\rho}{2}}(x_0,t_0)} v \leq \left(c(n)\, \rho^{-\frac{n+2}{2}}\right)^{\sum_{i=0}^{j} 2^{-i}} \prod_{i=0}^{j} 2^{\frac{(n+2)(i+2)}{2^{i+1}}}\left(\sup_{\mathcal{M}\cap C_\rho(x_0,t_0)} v\right)^{\frac{1}{2^j}}.$$

Letting $j \to \infty$ and allowing for a larger constant $c(n)$, we obtain

$$\sup_{\mathcal{M}\cap C_{\frac{\rho}{2}}(x_0,t_0)} v \leq c(n)\, \rho^{-(n+2)}$$

in view of the identity $\sum_{i=0}^{\infty} 2^{-i} = 2$. By the definition of v, we thus arrive at

$$\sup_{\mathcal{M}\cap C_{\frac{\rho}{2}}(x_0,t_0)} u \leq c(n)\,\rho^{-(n+2)} \iint_{\mathcal{M}\cap C_\rho(x_0,t_0)} u.$$

$\square$

Proof of Theorem 1.8. We will prove the Theorem in the following form: There exist constants $\epsilon_0 > 0$ and $c_0 > 0$ such that for any smooth, properly immersed solution $\mathcal{M} = (M_t)_{t\in(0,t_0)}$ of mean curvature flow inside an open set $U \subset \mathbb{R}^{n+1}$, every $x_0 \in U$ which the solution reaches at time t_0, every $\rho_0 \in (0, \sqrt{t_0}/2)$ for which $B_{2\rho_0}(x_0) \subset U$ and all $p \in [2, n+2]$ the assumption

$$\sup_{0\leq\rho<\rho'\leq 2\rho_0} \frac{1}{\left(\rho'^2-\rho^2\right)^{\frac{n+2-p}{2}}} \int_{t_0-\rho'^2}^{t_0-\rho^2} \int_{M_t\cap B_{\sqrt{\rho'^2-\rho^2}}(x_0)} |A|^p \leq \epsilon_0$$

implies

$$\sigma^2 \sup_{(t_0-(\rho_0-\sigma)^2,t_0)} \sup_{M_t\cap B_{\rho_0-\sigma}(x_0)} |A|^2 \leq c_0$$

for all $\sigma \in (0, \rho_0]$. The conclusion stated in the introduction (with c_0 repaced by $4c_0$) then follows directly from the one above by choosing $\sigma = \rho_0/2$.

We shall modify the scaling argument used in the proof of Theorem 5.6 in [E2], a different type of local regularity theorem, in fact we will partly even copy the text of the relevant section of the latter proof as we did not see any need for improving our presentation in [E2]. The proof in [HaSu] follows different lines.

By scaling the solution if necessary and by a translation in space-time we may assume without loss of generality that $\rho_0 = 1, x_0 = 0$ and $t_0 = 0$. Scaling about a point (x_1, t_1) in space-time with $t_1 \in (0, t_0]$ is done as follows: For fixed $\lambda > 0$ consider a new solution defined by

$$M_s^\lambda = \frac{1}{\lambda}\,(M_{\lambda^2 s+t_1} - x_1)$$

for $s \in (-\frac{t_1}{\lambda^2}, \frac{t_0-t_1}{\lambda^2})$, where we have changed variables by $x = \lambda\, y + x_1$ and $t = \lambda^2 s + t_1$. The second fundamental form A_λ of M_s^λ then satisfies $|A(x)| = \lambda^{-1}|A_\lambda(y)|$, and the area element of M_t is λ^n times the one of M_s^λ. Scaling the time-interval contributes a factor of λ^2. Therefore, for a given $r \in (0, \sqrt{t_1})$

$$\int_{t_1-r^2}^{t_1} \int_{M_t\cap B_r(x_1)} |A|^p = \lambda^{n+2-p} \int_{-\frac{r^2}{\lambda^2}}^{0} \int_{M_s^\lambda \cap B_{\frac{r}{\lambda}}(0)} |A_\lambda|^p.$$

Since our result is a local one, we may also assume for simplicity that $U = \mathbb{R}^{n+1}$. Without loss of generality, we may therefore prove the following statement:

There exist constants $\epsilon_0 > 0$ and $c_0 > 0$ such that for any smooth, properly immersed solution $\mathcal{M} = (M_t)_{t\in(-4,0)}$ of mean curvature flow in $\mathbb{R}^{n+1}$ the following holds. Suppose

$0 \in \mathbb{R}^{n+1}$ is reached by the solution at time 0. Then for any $p \in [2, n+2]$ the assumption

$$\sup_{0 \leq \rho < \rho' \leq 2} \frac{1}{(\rho'^2 - \rho^2)^{\frac{n+2-p}{2}}} \int_{-\rho'^2}^{-\rho^2} \int_{M_t \cap B_{\sqrt{\rho'^2-\rho^2}}(0)} |A|^p \leq \epsilon_0$$

implies

$$\sigma^2 \sup_{(-(1-\sigma)^2,0)} \sup_{M_t \cap B_{1-\sigma}(0)} |A|^2 \leq c_0$$

for all $\sigma \in (0, 1]$.

We may also assume without loss of generality that $\mathcal{M} = (M_t)_{t \in (-4,0]}$ is smooth up to and including time 0, because we can first prove the theorem with 0 replaced by $-\delta$ for fixed small $\delta > 0$ and then (since c_0 is independent of δ) let $\delta \searrow 0$ afterwards.

We now follow parts of the proof of Theorem 5.6 in [E2] almost verbatim: Suppose the above statement is not correct. Then for every $j \in \mathbb{N}$ one can find a smooth, properly immersed solution $\mathcal{M}^j = (M_t^j)_{t \in (-4,0]}$ which satisfies $0 \in M_0^j$ and

$$\sup_{0 \leq \rho < \rho' \leq 2} \frac{1}{(\rho'^2 - \rho^2)^{\frac{n+2-p_j}{2}}} \int_{-\rho'^2}^{-\rho^2} \int_{M_t \cap B_{\sqrt{\rho'^2-\rho^2}}(0)} |A|^{p_j} \leq \frac{1}{j} \tag{6}$$

for some exponent $p_j \in [2, n+2]$ but

$$\gamma_j^2 \equiv \sup_{\sigma \in (0,1]} \left(\sigma^2 \sup_{(-(1-\sigma)^2,0)} \sup_{M_t^j \cap B_{1-\sigma}(0)} |A|^2 \right) \to \infty \tag{7}$$

as $j \to \infty$ (note that $\gamma_j^2 < \infty$ since $\mathcal{M}^j$ is smooth up to $t = 0$ by assumption). In particular, one can find $\sigma_j \in (0, 1]$ for which

$$\gamma_j^2 = \sigma_j^2 \sup_{(-(1-\sigma_j)^2,0)} \sup_{M_t^j \cap B_{1-\sigma_j}(0)} |A|^2$$

and a point

$$y_j \in M_{\tau_j}^j \cap \bar{B}_{1-\sigma_j}(0)$$

at a time $\tau_j \in [-(1-\sigma_j)^2, 0]$ so that

$$\gamma_j^2 = \sigma_j^2 |A(y_j)|^2. \tag{8}$$

Since

$$\sigma_j^2 \sup_{(-(1-\sigma_j/2)^2,0)} \sup_{M_t^j \cap B_{1-\sigma_j/2}(0)} |A|^2 \leq 4\gamma_j^2,$$

we obtain

$$\sup_{(-(1-\sigma_j/2)^2,0)} \sup_{M_t^j \cap B_{1-\sigma_j/2}(0)} |A|^2 \leq 4|A(y_j)|^2$$

and therefore

$$\sup_{(\tau_j - \sigma_j^2/4, \tau_j)} \sup_{M_t^j \cap B_{\sigma_j/2}(y_j)} |A|^2 \leq 4|A(y_j)|^2$$

because

$$B_{\sigma_j/2}(y_j) \times (\tau_j - \sigma_j^2/4, \tau_j) \subset B_{1-\sigma_j/2}(0) \times (-(1-\sigma_j/2)^2, 0)$$

as one checks easily. Let now

$$\lambda_j = |A(y_j)|^{-1}$$

and define

$$\tilde{M}^j_s = \frac{1}{\lambda_j}\left(M^j_{\lambda_j^2 s+\tau_j} - y_j\right)$$

for $s \in (-\lambda_j^{-2}\sigma_j^2/4, 0]$, where we have changed variables by setting $x = \lambda_j y + y_j$ and $t = \lambda_j^2 s + \tau_j$. Then $\tilde{\mathcal{M}}^j = (\tilde{M}^j_s)$ is a smooth solution of mean curvature flow satisfying

$$0 \in \tilde{M}^j_0, \qquad |A(0)| = 1 \tag{9}$$

and

$$\sup_{(-\lambda_j^{-2}\sigma_j^2/4,0)} \sup_{\tilde{M}^j_s \cap B_{\lambda_j^{-1}\sigma_j/2}(0)} |A|^2 \leq 4$$

for every $j \in \mathbb{N}$. Since

$$\lambda_j^{-2}\sigma_j^2 = \gamma_j^2 \to \infty$$

by (7) and (8), we conclude that for sufficiently large j

$$\sup_{(-1,0)} \sup_{\tilde{M}^j_s \cap B_1(0)} |A|^2 \leq 4. \tag{10}$$

Scaling inequality (6) gives

$$\sup_{0 \leq \rho < \rho' \leq 2} \left(\frac{\lambda_j^2}{\rho'^2 - \rho^2}\right)^{\frac{n+2-p_j}{2}} \int_{-\frac{\rho'^2+\tau_j}{\lambda_j^2}}^{-\frac{\rho^2+\tau_j}{\lambda_j^2}} \int_{\tilde{M}^j_s \cap B_{\frac{\sqrt{\rho'^2-\rho^2}}{\lambda_j}}(-y_j)} |A|^{p_j} \leq \frac{1}{j}. \tag{11}$$

We now choose $0 \leq \rho < \rho'$ in (11) such that $\rho^2 + \tau_j = 0$ and $\rho'^2 - \rho^2 = \rho'^2 + \tau_j = 4\lambda_j^2 > 0$. Note that since $\sigma_j \in (0,1]$ and $\tau_j \in [-(1-\sigma_j)^2, 0]$ we have $0 \leq -\tau_j < 1$ so that $\rho < 1$. Since $\rho'^2 = 4\lambda_j^2 - \tau_j < 4\lambda_j^2 + 1$ and as $\lambda_j \to 0$ for $j \to \infty$ we can achieve the above with $0 \leq \rho < \rho' \leq 2$, that is within the admissible range of radii, as long as we choose $j \in \mathbb{N}$ sufficiently large. This leads to

$$\int_{-4}^{0} \int_{\tilde{M}^j_s \cap B_2(-y_j)} |A|^{p_j} \leq \frac{2^{n+2-p_j}}{j} \tag{12}$$

for those $j \in \mathbb{N}$. Since $-y_j \in \bar{B}_{1-\sigma_j}(0) \subset B_1(0)$ we have $B_1(0) \subset B_2(-y_j)$ and therefore

$$\int_{-1}^{0} \int_{\tilde{M}^j_s \cap B_1(0)} |A|^{p_j} \leq \frac{c(n)}{j} \tag{13}$$

for large enough $j \in \mathbb{N}$.

We now invoke the evolution equation for the second fundamental form derived in [Hu1], which for our scaled solution states that

$$\left(\frac{d}{ds} - \Delta_{\tilde{M}^j_s}\right)|A|^2 = 2|A|^4 - 2|\nabla A|^2,$$

where ∇ denotes covariant differentiation on the rescaled solution. A straightforward calculation using the chain rule and the well-known inequality $|\nabla|A|| \leq |\nabla A|$ then implies

$$\left(\frac{d}{ds} - \Delta_{\tilde{M}^j_s}\right)|A|^{p_j} \leq p_j |A|^{p_j+2}. \tag{14}$$

This uses also that $p_j \geq 2$. Combining (10) and (14) implies

$$\left(\frac{d}{ds} - \Delta_{\tilde{M}_s^j}\right) |A|^{p_j} \leq 4\, p_j |A|^{p_j}$$

inside the parabolic cylinder $B_1(0) \times (-1, 0]$ for sufficiently large $j \in \mathbb{N}$. We can therefore apply the local mean value inequality of Proposition 2.1 to the functions $u_j = e^{-4p_j s}|A|^{p_j}$ inside $B_1(0) \times (-1, 0]$ for all $j \in \mathbb{N}$ chosen so large such that inequalities (10) and (13) are satisfied. In view of (9) and (13), this yields

$$1 = |A(0)|^{p_j} \leq c(n) \int_{-1}^{0} \int_{\tilde{M}_s^j \cap B_1(0)} |A|^{p_j} e^{-4p_j s} \leq c(n, p_j)\, \frac{1}{j}$$

with a new constant $c(n, p_j)$ which is a product of the constant in (13), the factor e^{4p_j} and the constant which appears in the mean value theorem. Since $p_j \in [2, n+2]$ the final constant only depends on n. Choosing now $j \in \mathbb{N}$ large enough so that inequalities (10) and (13) are satisfied and such that the right hand side of the last inequality is smaller than 1 leads to a contradiction and completes our proof. $\square$

3. Proof of the Partial Regularity Results

In this chapter, we prove Theorem 1.1 and those of its consequences which were not already derived in the introduction. The techniques involve a minor adaptation of the ones employed in Corollary 1.3 of [HaSu] and in the proof of Lemma 5.12 in [E2].

Proof of Theorem 1.1. In view of the smoothness estimates from [EH2] or Proposition 3.22 in [E2], a bound of the form

$$\sup_{(t_0 - \frac{\rho_0^2}{4}, t_0)} \ \sup_{M_t \cap B_{\frac{\rho_0}{2}}(x_0)} |A|^2 \leq \frac{c_0}{\rho_0^2}$$

implies the higher order estimates

$$\sup_{(t_0 - \frac{\rho_0^2}{16}, t_0)} \ \sup_{M_t \cap B_{\frac{\rho_0}{4}}(x_0)} |\nabla^m A|^2 \leq \frac{c(m, n)}{\rho_0^{2(m+1)}}$$

for the covariant derivatives of all orders $m \in \mathbb{N}$ of the second fundamental form. By the evolution equations for geometric quantities under mean curvature flow (see [Hu1]), these in turn imply local bounds on the space and time derivatives of all orders for the immersions describing the evolving hypersurfaces. It then follows from standard arguments that the solution can be extended beyond time t_0 in some neighbourhood of x_0. Hence, the above local curvature bound implies that x_0 is a regular point at time t_0.

In view of Theorem 1.8, regular points x_0 at time t_0 can therefore be characterized by the condition that for some $p \in [2, n+2]$ and some $\rho_0 \in (0, \sqrt{t_0}/2)$ for which $B_{2\rho_0}(x_0) \subset B_R(y)$, the inequality

$$\int_{t_0 - \rho'^2}^{t_0 - \rho^2} \int_{M_t \cap B_{\sqrt{\rho'^2 - \rho^2}}(x_0)} |A|^p \leq \epsilon_0 \left(\rho'^2 - \rho^2\right)^{\frac{n+2-p}{2}} \tag{15}$$

holds for all $0 \le \rho < \rho' \le 2\rho_0$ for some fixed $\epsilon_0 > 0$. Points which the solution does not reach at time t_0 are regular by definition, so we do not need to consider them here.

Let $p \in [2, n+2]$ be fixed but arbitrary. Then for a general point x in $\mathrm{sing}_{t_0}\mathcal{M} \cap B_R(y)$ condition (15) with this p cannot hold for any $\rho_0 \in (0, \sqrt{t_0}/2)$, that is the reverse inequality must be satisfied for sequences $\rho_k \searrow 0$ and $\rho'_k \searrow 0$ (these are allowed to depend on p) with $0 \le \rho_k < \rho'_k$. In other words, there must exist sequences $t_k \nearrow t_0$ and $r_k \searrow 0$ for which $B_{r_k}(x) \subset B_R(y)$ and such that

$$\int_{t_k - r_k^2}^{t_k} \int_{M_t \cap B_{r_k}(x)} |A|^p \ge \epsilon_0 \, r_k^{n+2-p}$$

for instance by choosing $t_k = t_0 - \rho_k^2$ and $r_k^2 = \rho_k'^2 - \rho_k^2$ for a sequence of radii ρ_k and ρ'_k for which (15) is violated.

In the following we are only interested in exponents $p \in [2, n+2]$ for which

$$\int_0^{t_0} \int_{M_t \cap B_R(y)} |A|^p < \infty.$$

In order to estimate $\mathcal{H}^{n+2-p}(\mathrm{sing}_{t_0}\mathcal{M} \cap B_R(y))$ for such a $p \in [2, n+2]$, we proceed analogously to the proof of Corollary 1.3 in [HaSu] (which was carried out in the case $p = 2$ and $n = 2$) or Lemma 5.12 in [E2] where we considered H^2 instead of $|A|^p$.

Let us fix $\delta > 0$. In view of Theorem 1.8 and the above discussion about singular points, there exists for every $x \in \mathrm{sing}_{t_0}\mathcal{M} \cap B_R(y)$ a time $t(x) \in (0, t_0]$ and a radius $r(x) \in (0, \frac{\delta}{10}]$ such that $B_{r(x)}(x) \subset B_R(y)$ and the inequalities $t(x) - r^2(x) > t_0 - \delta^2$ and

$$\int_{t(x) - r(x)^2}^{t(x)} \int_{M_t \cap B_{r(x)}(x)} |A|^p \ge \epsilon_0 \, r(x)^{n+2-p}$$

hold.

The Vitali covering theorem (see [S] or Appendix C of [E2]) allows us to select a pairwise disjoint family of balls $B_{r_j}(x_j)$ with $x_j \in \mathrm{sing}_{t_0}\mathcal{M} \cap B_R(y)$, $r_j \in (0, \frac{\delta}{10}]$, $B_{r_j}(x_j) \subset B_R(y)$ and

$$\mathrm{sing}_{t_0}\mathcal{M} \cap B_R(y) \subset \bigcup_{j=1}^{\infty} B_{5r_j}(x_j),$$

as well as corresponding times $t_j \in (0, t_0]$ satisfying $t_j - r_j^2 > t_0 - \delta^2$ such that

$$\int_{t_j - r_j^2}^{t_j} \int_{M_t \cap B_{r_j}(x_j)} |A|^p \ge \epsilon_0 \, r_j^{n+2-p}$$

is satisfied.

We then estimate the $\mathcal{H}_\delta^{n+2-p}$- measure of the singular set (recall that we assume $p \in [2, n+2]$ so the 'dimension' of the Hausdorff - measure is not negative) by

$$\begin{aligned}\mathcal{H}_\delta^{n+2-p}\left(\mathrm{sing}_{t_0}\mathcal{M} \cap B_R(y)\right) &\leq 5^{n+2-p}\,\omega_{n+2-p}\sum_{j=1}^{\infty} r_j^{n+2-p} \\ &\leq \frac{c(n,p)}{\epsilon_0}\sum_{j=1}^{\infty}\int_{t_j - r_j^2}^{t_j}\int_{M_t \cap B_{r_j}(x_j)} |A|^p \\ &\leq \frac{c(n,p)}{\epsilon_0}\int_{t_0-\delta^2}^{t_0}\sum_{j=1}^{\infty}\int_{M_t \cap B_{r_j}(x_j)} |A|^p \\ &\leq \frac{c(n,p)}{\epsilon_0}\int_{t_0-\delta^2}^{t_0}\int_{M_t \cap B_R(y)} |A|^p,\end{aligned}$$

where we used the pairwise disjointness of the balls $B_{r_j}(x_j) \subset B_R(y)$ for $j \in \mathbb{N}$ to arrive at the last inequality. The factor ω_{n+2-p} is some appropriate positive normalization constant (for $n+2-p \in \mathbb{N}$ the volume of the unit ball in that dimension) used in the definition of Hausdorff - measure. Since we assumed that

$$\int_0^{t_0}\int_{M_t \cap B_R(y)} |A|^p < \infty,$$

the last expression in the string of inequalities above tends to zero as $\delta \to 0$. This implies our result for $p \in [2, n+2]$. If the condition of Theorem 1.1 holds for $p > n+2$, it is also valid for $p = n+2$ in view of Hölder's inequality. Therefore, by the above estimate, the $\mathcal{H}^0$- measure (which is simply the counting measure) of the singular set inside $B_R(y)$ vanishes and hence this set is empty. $\square$

Proof of Corollary 1.4. The argument combines results from [I1] with Theorem 1.1 for $p = 2$. Let us first discuss the case of closed surfaces to illustrate the main geometric idea. This case was established in [HaSu]: Any smooth surface $M \subset \mathbb{R}^3$ satisfies the Gauß equation

$$H^2 - |A|^2 = 2K,$$

where $K = \kappa_1\kappa_2$, the product of the principal curvatures, is the Gauß curvature of M. The Gauß-Bonnet theorem for closed surfaces states that

$$\int_M K = 4\pi(1 - g(M)),$$

where $g(M)$ is the genus of M. By the evolution equation

$$\frac{d}{dt}\mu_t = -H^2\mu_t$$

for the area element derived in [Hu1], a solution family consisting of closed hypersurfaces satisfies the well-known inequality

$$\int_0^{t_0}\int_{M_t} H^2 \leq \mathcal{H}^n(M_0) < \infty,$$

see also [Hu1]. On the other hand, integrating the Gauß-Bonnet formula for each M_t in time from 0 to t_0 and using the fact that the genus does not change during a smooth evolution, leads to

$$\int_0^{t_0}\int_{M_t}|A|^2 = \int_0^{t_0}\int_{M_t} H^2 - 8\pi(1-g(M_0))\,t_0 \le \mathcal{H}^2(M_0) - 8\pi(1-g(M_0)\,t_0 < \infty.$$

Theorem 1.1 is therefore applicable in this case.

For the general case, we employ a local version of the Gauß-Bonnet formula proved by Ilmanen in [I1], which he also integrated in time and combined with the evolution equation for the area element in the way we outlined above in the case of closed surfaces. Indeed, it follows from his work that smooth, properly immersed solutions starting from a smooth, properly immersed surface M_0 of finite genus satisfy

$$\int_{t_0-r^2}^{t_0}\int_{M_t\cap B_r(x)}|A|^2 < \infty$$

for all $x\in\mathbb{R}^3$ and $r\in(0,\sqrt{t_0})$. Theorem 1.1 applied with $y=x$ and $R=\sqrt{t_0}$ therefore yields

$$\mathcal{H}^2(\operatorname{sing}_{t_0}\mathcal{M}\cap B_{\sqrt{t_0}}(x)) = 0$$

for all $x\in\mathbb{R}^3$. Since the countable union of sets of measure zero has again measure zero, the conclusion of our corollary follows.

A technical remark is in order here: For ease of presentation, Ilmanen had assumed the uniform area ratio estimate

$$\sup_{x\in\mathbb{R}^3}\sup_{\rho>0}\frac{\mathcal{H}^2(M_0\cap B_\rho(x))}{\pi\rho^2} \le D < \infty$$

for the initial surface M_0, and then bounded the above double integral of $|A|^2$ by an expression of the form $C(D,g(M_0))\,r^2$. Since we only require the finiteness of this integral and not the explicit form of Ilmanen's bound, we are able to tolerate a dependence of his constant on x. Therefore, the smoothness of M_0 and the assumption that it is properly immersed are sufficient to derive the finiteness of the double integral for all points in $\mathbb{R}^3$ and all radii $r\in(0,\sqrt{t_0})$, as one easily checks by an adaptation of Ilmanen's argument leading to inequalities (7) and (8) in [I1]. Alternatively, one can apply the methods in Chapter 5 of [E2], in particular the proof of Theorem 5.4 using Theorem 5.3 as well as the conclusion of Lemma 5.10, in order to derive inequalities (7) and (8) in [I1] from the smoothness of M_0 and the properness of its immersion only, but with constants depending on n and the area of M_0 inside $B_{c(n)\sqrt{t_0}}(x)$ for some $c(n)>1$. $\square$

Proof of Remark 1.6 for the k - convex case. We adapt Head's calculation in [H] to k - convex solutions of mean curvature flow. The evolution equation

$$\left(\frac{d}{dt}-\Delta_{M_t}\right)H = |A|^2H$$

derived in [Hu1] in combination with the chain rule implies

$$\left(\frac{d}{dt}-\Delta_{M_t}\right)H^q = q\,|A|^2H^q - q(q-1)H^{q-2}|\nabla H|^2 \tag{16}$$

for any $q \in \mathbb{R}$. Combining this with the evolution equation

$$\frac{d}{dt}\mu_t = -H^2\mu_t$$

for the area element μ_t of the evolving hypersurfaces, yields

$$\frac{d}{dt}\int_{M_t} H^q = \int_{M_t} \left(q\,|A|^2 - H^2\right) H^q - q(q-1)\int_{M_t} H^{q-2}|\nabla H|^2 \tag{17}$$

on closed solutions. For $q \in \mathbb{R}\backslash(0,1)$, we therefore obtain the inequality

$$\frac{d}{dt}\int_{M_t} H^q \leq \int_{M_t} \left(q\,|A|^2 - H^2\right) H^q \tag{18}$$

as long as $H \geq 0$, which holds for any k - convex solution with $1 \leq k \leq n$.

We now set $q = n+1-k-\alpha$ for $\alpha \in (0,1]$. If $1 \leq k \leq n-1$ then $q \geq 1$, so inequality (18) holds. In the mean-convex case, that is for $k = n$, we have $q = 1-\alpha$ with $\alpha \in (0,1]$. Since we are interested in arbitrarily small positive α we would have to treat the case $q \in (0,1)$. At this stage, we do not know how to estimate the integral involving the gradient of H on the right hand side of (17), which in this situation has a non-negative factor. This is the reason why we only consider the case $1 \leq k \leq n-1$ in the sequel.

Theorem 5.1 in [Hu1], Theorem 5.3 (i) in [HS2] (which is actually also valid in the case $n = 2$ as it then agrees with the central estimate in [HS1] for mean-convex solutions) and its generalization to closed, immersed k - convex solutions ([HS3]) implies that for any $\epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such that the inequality

$$|A|^2 - \frac{1}{(n+1-k)}H^2 \leq \epsilon\, H^2 + C(\epsilon)$$

holds pointwise on all hypersurfaces M_t for all $1 \leq k \leq n$. Inserting this into (18) with $q = n+1-k-\alpha$ and using Young's inequality, we arrive at

$$\begin{aligned}\frac{d}{dt}\int_{M_t} H^{n+1-k-\alpha} \leq \left(\frac{-\alpha}{n+1-k} + (n+2-k-\alpha)\,\epsilon\right)\int_{M_t} H^{n+3-k-\alpha}\\ + C(\epsilon,n,k,\alpha)\,\mathcal{H}^n(M_t)\end{aligned}$$

for all $\epsilon > 0$. For any $\alpha \in (0,1]$, we can choose ϵ small depending on n, k and α so that

$$\frac{d}{dt}\int_{M_t} H^{n+1-k-\alpha} \leq -\frac{\alpha}{2(n+1-k)}\int_{M_t} H^{n+3-k-\alpha} + C(n,k,\alpha)\,\mathcal{H}^n(M_0), \tag{19}$$

where we also used the inequality $\mathcal{H}^n(M_t) \leq \mathcal{H}^n(M_0)$ for all $t \in [0, t_0)$, the latter being a direct consequence of the evolution equation for the area element. Integrating (19) with respect to time, implies

$$\int_0^{t_0}\int_{M_t} H^{n+3-k-\alpha} \leq C\left(\int_{M_0} H^{n+1-k-\alpha} + \mathcal{H}^n(M_0)\,t_0\right) < \infty$$

for all $\alpha \in (0,1]$ where C depends on n, k and α and tends to infinity for $\alpha \to 0$. We remind the reader that we had to use $k \leq n-1$ for technical reasons to arrive at this estimate.

In view of the pointwise inequality $|A|^2 \leq c_1 H^2$ proved in [HS1], which holds on all solution hypersurfaces in the closed mean-convex case and hence also for k - convex solutions, we therefore conclude that

$$\int_0^{t_0}\int_{M_t}|A|^{n+3-k-\alpha}<\infty$$

for all $\alpha\in(0,1]$. Corollary 1.2 then implies that the dimension of the singular set is at most $k-1$. $\square$

REFERENCES

[AAG] S. Altschuler, S. Angenent and Y. Giga, Mean curvature flow through singularities for surfaces of rotation, J. Geom. Analysis **5**, No 3, 293-358 (1995)

[ACI] S.B. Angenent, D.L. Chopp and T. Ilmanen, A computed example of nonuniqueness of mean curvature flow in $\mathbb{R}^3$, Comm. Partial Differential Equations **20**, 1937 - 1958 (1995)

[B] K. A. Brakke, *The Motion of a Surface by its Mean Curvature*, Math. Notes Princeton, NJ, Princeton University Press 1978

[CGG] Y. G. Chen, Y. Giga and S. Goto, Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, J.Differ.Geom. **33**, 749-786 (1991)

[CS] H. I. Choi and R. Schoen, The space of minimal embeddings of a surface into a three-dimensional manifold of positive Ricci-curvature, Invent.math. **81**, 387-394 (1985)

[Ch] D. Chopp, Numerical computation of self-similar solutions for mean curvature flow," Exper. Math. 3 1-16 (1994)

[E1] K. Ecker, On regularity for mean curvature flow of hypersurfaces, Calc.Var.**3**, 107-126 (1995)

[E2] K. Ecker, *Regularity Theory for Mean Curvature Flow*, Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser (2004)

[EH1] K. Ecker and G. Huisken, Mean curvature evolution of entire graphs, Annals of Math. **130**, 453-471 (1989)

[EH2] K. Ecker and G. Huisken, Interior estimates for hypersurfaces moving by mean curvature, Invent.math. **105**, 547-569 (1991)

[ES1] L. C. Evans and J. Spruck, Motion of level sets by mean curvature I, J.Differ.Geom. **33**, 635-681 (1991)

[ES2] L. C. Evans and J. Spruck, Motion of level sets by mean curvature II, Trans. of the Amer.Math.Soc. **330**, No 1, 321-332 (1992)

[ES3] L. C. Evans and J. Spruck, Motion of level sets by mean curvature III, J.Geom.Analysis **2**, 121-150 (1992)

[H] J. Head, PhD thesis 2011, Max-Planck Institut für Gravitationsphysik (Albert Einstein Institut), Potsdam, Germany

[HaSu] X. Han and J. Sun, An ε-regularity theorem for the mean curvature flow, arxiv: 1102.4800 (2011)

[Hu1] G. Huisken, Flow by mean curvature of convex surfaces into spheres, J.Differ.Geom. **20**, 237-266 (1984)

[Hu2] G. Huisken, Asymptotic behaviour for singularities of the mean curvature flow, J.Differ. Geom. **31**, 285-299 (1990)

[HS1] G. Huisken and C. Sinestrari, Mean curvature flow singularities for mean-convex surfaces, Calc. Var. PDE **8**, 1-14 (1999)

[HS2] G. Huisken and C. Sinestrari, Mean curvature flow with surgery of two-convex hypersurfaces, Invent. Math. 175, no. 1, 137-221 (2009)

[HS3] G. Huisken and C. Sinestrari, personal communication, preprint in preparation

[I1] T. Ilmanen, Singularities of mean curvature flow of surfaces, 1995, http://www.math.ethz.ch/ ilmanen/papers/pub.html

[I2] T. Ilmanen, Lectures on Mean Curvature Flow and Related Equations, 1995 (revised 1998), http://www.math.ethz.ch/ ilmanen/papers/pub.html

[L] G. M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific (2005)

[LS1] N. Q. Le and N. Sesum, On the extension of the mean curvature flow, Math. Zeitschrift **267**, 583 - 604 (2011)

[LS2] N. Q. Le and N. Sesum, The mean curvature at the first singular time of the mean curvature flow, Annales de l'Institut Henri Poincaré **26**, issue 6, 1441 - 1459 (2010)

[LSU] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Uraltseva, *Linear and Quasilinear Equations of Parabolic*, Type. Transl., II. Ser., Am.Math.Soc. **23** (1968)

[M] J. Moser, A Harnack inequality for parabolic differential equations, Comm. Pure Appl.Math. **17**, 101-134 (1964)

[MS] J. H. Michael and L. M. Simon, Sobolev and mean-value inequalities on generalized submanifolds of $\mathbb{R}^n$, Comm. Pure Appl. Math. **26**, 361-379 (1973)

[N] N. Nakauchi, A concentration phenomenon in mean curvature flow, manuscripta math. **78**, 129 - 147 (1993)

[S] L. M. Simon, *Lectures on Geometric Measure Theory*, Proc. of the CMA, Vol.3 (1983)

[Sch] R. M. Schoen, Handwritten lecture notes, Berkeley

[St] M. Struwe, On the evolution of harmonic maps in higher dimensions, J.Differ.Geom. **28**, 485-502 (1988)

[W1] B. White, Stratification of minimal surfaces, mean curvature flows and harmonic maps, J. Reine und Angewandte Math. **488** , 1-35 (1997)

[W2] B. White, The size of the singular set in mean curvature flow of mean-convex sets, J. Amer. Math.Soc **13**, no.3, 665-695 (2000)

[XYZ] H. W. Xu, F. Ye and E. T. Zhao: Extend mean curvature flow with finite integral curvature, to appear in Asian J. Math.