EVALUATION OF A FAMILY OF BINOMIAL DETERMINANTS

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Abstract. Motivated by a recent work about finite sequences where the n-th term is bounded by n^2, we evaluate some classes of determinants such as the $(n-2)\times(n-2)$ determinant

$$\Delta_n = \left| \left(\binom{x_n - x_k + h - 1}{n-k-1} \right)_{\substack{2\le k\le n-1\\ 0\le h\le n-3}} \right|, \quad \text{for } n \ge 3,$$

and more generally the $n \times n$ determinant

$$D_n = \left| \left(\binom{x_i + j}{i-1} \right)_{\substack{1\le i\le n\\ 1\le j\le n}} \right|, \quad \text{for } n \ge 1,$$

where n, k, h, i, j are integers, $(x_k)_{1\le k\le n}$ is a sequence of indeterminates over $\mathbb{C}$ and $\binom{A}{B}$ is the usual binomial coefficient. We thus prove that

$$D_n = 1 \qquad \text{and} \qquad \Delta_n = (-1)^{\frac{(n-2)(n-3)}{2}}.$$

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1. Introduction

In a recent work [9] about finite sequences whose n-th term does not exceed n^2, there appeared the determinant

$$\Delta_n^* = \left| \left(\binom{n^2 - k^2 + h - 1}{n-k-1} \right)_{\substack{2\le k\le n-1\\ 0\le h\le n-3}} \right|,$$

with an integer $n \ge 3$. One of the authors of [9], L. Haddad, conjectured after some computations that $\Delta_n^* = \pm 1$. The authors of the present paper first proved that

$$\Delta_n^* = (-1)^{\frac{(n-2)(n-3)}{2}},$$

essentially by a process of row reduction. Then, following a suggestion by G. E. Andrews that this result should be true in a more general context, namely upon replacing n^2 by x_n and k^2 by x_k, where $(x_k)_{1\le k\le n}$ is an arbitrary sequence of indeterminates over $\mathbb{C}$, the proof was extended to this general case. We then realized that the problem can be reduced to the evaluation of a simpler, more general family of determinants, namely

$$D_n = \left| \left(\binom{x_i + j}{i-1} \right)_{\substack{1\le i\le n\\ 1\le j\le n}} \right|,$$

for all integers $n \geq 1$. In what follows, we will establish that $D_n = 1$ and deduce that $\Delta_n = (-1)^{\frac{(n-2)(n-3)}{2}}$.

Results of a similar nature, involving determinants of matrices whose entries involve binomial coefficients, can be found in [1, 2, 3, 4, 6, 10, 12]. In contrast to these papers, we note that our determinant evaluations are strikingly simple and easy to state. In fact, our result is a special case of the results contained in [10], but our proof is more elementary, using only row reduction and induction.

We are thankful to L. Haddad for his conjecture and to G. E. Andrews for his insightful suggestion.

2. The method of proof

First recall (e.g. [7] or [8]) that for an indeterminate x over $\mathbb{C}$ and an integer $n \geq 0$, the binomial coefficient $\binom{x}{n}$ is defined by

$$\binom{x}{n} = \frac{x\,(x-1)\,(x-2)\cdots(x-n+1)}{n!},$$

with the convention that $\binom{x}{0} = 1$. It satisfies the fundamental recurrence relation

$$\binom{x}{n} + \binom{x}{n+1} = \binom{x+1}{n+1}, \quad \text{for all} \ \ n \geq 0. \tag{2.1}$$

Let $(x_k)_{1\le k\le n}$ be a sequence of indeterminates over $\mathbb{C}$, with $n \geq 3$, and consider the $(n-2)\times(n-2)$ determinant

$$\Delta_n = \Delta_n(x_1, \ldots, x_n) = \left| \left(\binom{x_n - x_k + h - 1}{n-k-1} \right)_{\substack{2\le k\le n-1\\ 0\le h\le n-3}} \right|.$$

First, setting $i = k-1$ and $j = h+1$ allows one to rewrite Δ_n as

$$\Delta_n = \left| \left(\binom{x_n - x_{i+1} + j - 2}{n-i-2} \right)_{\substack{1\le i\le n-2\\ 1\le j\le n-2}} \right|. \tag{2.2}$$

Second, the substitution $i' = n-i-1$ transforms Δ_n into

$$\Delta'_n = \Delta'_n(x_1, \ldots, x_n) = \left| \left(\binom{x_n - x_{n-i'} + j - 2}{i'-1} \right)_{\substack{1\le i'\le n-2\\ 1\le j\le n-2}} \right|,$$

which has the same rows as Δ_n but in reverse order. This order reversal consists in respectively swapping each row of Δ_n with all the rows above it. The total number of those row swaps is

$$(n-3)+(n-4)+\cdots+2+1=\frac{(n-2)(n-3)}{2}.$$

Therefore,

$$\Delta_n=(-1)^{\frac{(n-2)(n-3)}{2}}\Delta'_n. \tag{2.3}$$

The problem is thus reduced to the determination of Δ'_n.

Third, setting $x=x_n-2$ gives

$$\Delta'_n=\left|\left(\binom{x-x_{n-i}+j}{i-1}\right)_{\substack{1\le i\le n-2\\1\le j\le n-2}}\right|. \tag{2.4}$$

Fourth, setting $y_i=x-x_{n-i}$ yields

$$\Delta'_n=\left|\left(\binom{y_i+j}{i-1}\right)_{\substack{1\le i\le n-2\\1\le j\le n-2}}\right|. \tag{2.5}$$

Finally, setting $m=n-2$ leads to the equality

$$\Delta'_n=\left|\left(\binom{y_i+j}{i-1}\right)_{\substack{1\le i\le m\\1\le j\le m}}\right|. \tag{2.6}$$

The problem is thus reduced to the evaluation of the family of determinants

$$D_n=D_n(x_1,\ldots,x_n)=\left|\left(\binom{x_i+j}{i-1}\right)_{\substack{1\le i\le n\\1\le j\le n}}\right|, \tag{2.7}$$

for $n\ge 1$, where $x_1,x_2,\ldots,x_n$ are arbitrary indeterminates.

Our primary result is now the following:

Theorem 2.1. *For any positive integer n, we have*

$$D_n=1.$$

The proof proceeds by row reduction and is presented in the next section.

Corollary 2.2. *For any integer $n\ge 3$, we have*

$$\Delta'_n=1.$$

Proof. This follows from (2.6) and Theorem 2.1. □

Corollary 2.3. *For any integer $n\ge 3$, we have*

$$\Delta_n=(-1)^{\frac{(n-2)(n-3)}{2}}.$$

Proof. This follows from (2.3) and Corollary 2.2. □

Remark 2.4. An alternative method for deriving the last result is to set $y_k = x_n - x_{n-k} - 2$, and $i = k-1,\ j = h+1$. Then

$$\Delta_n = \left| \left(\binom{y_{n-i-1}+j}{n-i-2} \right)_{\substack{1\le i\le n-2 \\ 1\le j\le n-2}} \right|.$$

Now, reversing the order of the rows, which consists in replacing i by $n-1-i$, transforms Δ_n into

$$\Delta_n' = \left| \left(\binom{y_i+j}{i-1} \right)_{\substack{1\le i\le n-2 \\ 1\le j\le n-2}} \right|.$$

Moreover, the permutation ρ that reverses the $n-2$ rows of Δ_n has $\lceil \frac{n-2}{2} \rceil$ orbits, namely $\{1, n-2\},\ \{2, n-3\}, \ldots$. It follows that (see, e.g., [11]) the sign of ρ is

$$\epsilon(\rho) = (-1)^{\frac{n-2}{2} - \lceil \frac{n-2}{2} \rceil} = (-1)^{\lfloor \frac{n-2}{2} \rfloor}.$$

Hence

$$\Delta_n = (-1)^{\lfloor \frac{n-2}{2} \rfloor} \Delta_n' = (-1)^{\lfloor \frac{n-2}{2} \rfloor},$$

in view of our main theorem, which yields $\Delta_n' = 1$.

3. The proof of the main Theorem

We start with two results about binomial coefficients that will be used in the proof of Theorem 2.1.

Lemma 3.1. *For any integers $0 \le a \le b$ and $n \ge 1$, and any indeterminate x over $\mathbb{C}$, we have*

$$\binom{x+b}{n} - \binom{x+a}{n} = \sum_{h=a}^{b-1} \binom{x+h}{n-1}.$$

Proof. By the fundamental recurrence relation for binomial coefficients (2.1),

$$\binom{x+1}{n} - \binom{x}{n} = \binom{x}{n-1},$$

for $n \ge 1$. Hence, by iterating this recurrence numerous times, we have

$$\binom{x+b}{n} - \binom{x+a}{n} = \sum_{h=a}^{b-1} \left(\binom{x+h+1}{n} - \binom{x+h}{n} \right) = \sum_{h=a}^{b-1} \binom{x+h}{n-1}.$$

□

Lemma 3.2. *For any integers $0 \le m \le n$, we have*

$$\sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1}.$$

Proof. For a fixed integer $m \geq 0$, the proof can be completed by induction on $n \geq m$. Such an argument can be found in [5, p. 138].

This result also follows from Lemma 3.1 by taking $n = m+1,\ a = 0,\ x = m,\ b = n - m + 1$. □

We now proceed to prove Theorem 2.1 by row reduction. We start with

$$D_n = \left|(d_{ij})_{\substack{1\leq i\leq n\\ 1\leq j\leq n}}\right|, \quad \text{where } d_{ij} = \binom{x_i + j}{i-1} \text{ for } 1 \leq i, j \leq n, \tag{3.1}$$

i.e.

$$D_n = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & \dots & 1 \\ \binom{x_2+1}{1} & \binom{x_2+2}{1} & \binom{x_2+3}{1} & \dots & \binom{x_2+j}{1} & \dots & \binom{x_2+n}{1} \\ \binom{x_3+1}{2} & \binom{x_3+2}{2} & \binom{x_3+3}{2} & \dots & \binom{x_3+j}{2} & \dots & \binom{x_3+n}{2} \\ \binom{x_4+1}{3} & \binom{x_4+2}{3} & \binom{x_4+3}{3} & \dots & \binom{x_4+j}{3} & \dots & \binom{x_4+n}{3} \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ \binom{x_i+1}{i-1} & \binom{x_i+2}{i-1} & \binom{x_i+3}{i-1} & \dots & \binom{x_i+j}{i-1} & \dots & \binom{x_i+n}{i-1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ \binom{x_n+1}{n-1} & \binom{x_n+2}{n-1} & \binom{x_n+3}{n-1} & \dots & \binom{x_n+j}{n-1} & \dots & \binom{x_n+n}{n-1} \end{vmatrix}. \tag{3.2}$$

Denoting the i-th row of D_n by R_i, the first row reduction step consists in replacing R_i by $R_i - d_{i1}R_1$ for $2 \leq i \leq n$. This gives

$$D'_n = \left|(d'_{ij})_{\substack{1\leq i\leq n\\ 1\leq j\leq n}}\right|,$$

where

$$d'_{ij} = \begin{cases} d_{ij} - d_{i1} = \binom{x_i+j}{i-1} - \binom{x_i+1}{i-1} = \sum_{h=1}^{j-1} \binom{x_i+h}{i-2}, & \text{if } 2 \leq i \leq n,\ 1 \leq j \leq n, \\ d_{1j} = 1, & \text{if } i = 1,\ 1 \leq j \leq n, \end{cases} \tag{3.3}$$

the last expression, for $i \geq 2$, is obtained by using Lemma 3.1, with the usual convention that an empty sum is equal to 0. Thus

$$D'_n = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & \dots & 1 \\ 0 & 1 & 2 & \dots & j-1 & \dots & n-1 \\ 0 & \binom{x_3+1}{1} & \sum_{h=1}^{2}\binom{x_3+h}{1} & \dots & \sum_{h=1}^{j-1}\binom{x_3+h}{1} & \dots & \sum_{h=1}^{n-1}\binom{x_3+h}{1} \\ 0 & \binom{x_4+1}{2} & \sum_{h=1}^{2}\binom{x_4+h}{2} & \dots & \sum_{h=1}^{j-1}\binom{x_4+h}{2} & \dots & \sum_{h=1}^{n-1}\binom{x_4+h}{2} \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \binom{x_i+1}{i-2} & \sum_{h=1}^{2}\binom{x_i+h}{i-2} & \dots & \sum_{h=1}^{j-1}\binom{x_i+h}{i-2} & \dots & \sum_{h=1}^{n-1}\binom{x_i+h}{i-2} \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \binom{x_n+1}{n-2} & \sum_{h=1}^{2}\binom{x_n+h}{n-2} & \dots & \sum_{h=1}^{j-1}\binom{x_n+h}{n-2} & \dots & \sum_{h=1}^{n-1}\binom{x_n+h}{n-2} \end{vmatrix}. \tag{3.4}$$

Moreover, we obviously have

$$D_n = D'_n. \tag{3.5}$$

Proposition 3.3. *For $1 \le k \le n$, let*

$$D_n^{(k)} = \left| \left(d_{ij}^{(k)} \right)_{\substack{1\le i\le n \\ 1\le j\le n}} \right|$$

be the $n \times n$ determinant obtained from D_n by applying k row reduction steps, each of which consists in replacing the i-th row $R_i^{(k-1)}$ of the determinant $D_n^{(k-1)}$, obtained after $k-1$ such row reduction steps, by the row

$$R_i^{(k)} = R_i^{(k-1)} - d_{ik}^{(k-1)} R_k^{(k-1)}, \quad \text{for } k+1 \le i \le n,$$

while the first k rows are unchanged, i.e.

$$R_i^{(k)} = R_i^{(k-1)} \quad \text{for } 1 \le i \le k.$$

Then

$$d_{ij}^{(k)} = \begin{cases} \sum_{h=1}^{j-k} \binom{j-h-1}{k-1}\binom{x_i+h}{i-k-1}, & \text{if } k+1 \le i \le n,\ 1 \le j \le n, \\ d_{ij}^{(k-1)}, & \text{if } 1 \le i \le k,\ 1 \le j \le n, \end{cases} \tag{3.6}$$

with the convention that an empty sum (here, when $j \le k$) is equal to 0.

Proof. The proof is by induction on k. The first row reduction step was applied to D_n right before this Proposition, and it gave $D'_n = \left| \left(d'_{ij} \right)_{\substack{1\le i\le n \\ 1\le j\le n}} \right|$, which satisfies the stated equalities for d'_{ij} as shown in (3.3) above. So the property holds for $k = 1$. Assume that it holds for $k-1$, where $2 \le k \le n$, i.e. assume that $D_n^{(k-1)} = \left| \left(d_{ij}^{(k-1)} \right)_{\substack{1\le i\le n \\ 1\le j\le n}} \right|$ satisfies

$$d_{ij}^{(k-1)} = \begin{cases} \sum_{h=1}^{j-(k-1)} \binom{j-h-1}{k-2}\binom{x_i+h}{i-(k-1)-1}, & \text{if } k \le i \le n,\ 1 \le j \le n, \\ d_{ij}^{(k-2)}, & \text{if } 1 \le i \le k-1,\ 1 \le j \le n. \end{cases} \tag{3.7}$$

Now, as for $k+1 \le i \le n$, we have $R_i^{(k)} = R_i^{(k-1)} - d_{ik}^{(k-1)} R_k^{(k-1)}$, i.e.

$$d_{ij}^{(k)} = d_{ij}^{(k-1)} - d_{ik}^{(k-1)} d_{kj}^{(k-1)}, \quad \text{for } 1 \le j \le n,$$

and by the induction assumption, there hold

$$d_{ij}^{(k-1)} = \sum_{h=1}^{j-k+1} \binom{j-h-1}{k-2} \binom{x_i+h}{i-k},$$

$$d_{ik}^{(k-1)} = \sum_{h=1}^{k-k+1} \binom{k-h-1}{k-2} \binom{x_i+h}{i-k} = \binom{x_i+1}{i-k},$$

$$d_{kj}^{(k-1)} = \sum_{h=1}^{j-k+1} \binom{j-h-1}{k-2} \binom{x_k+h}{k-k} = \sum_{h=1}^{j-k+1} \binom{j-h-1}{k-2}.$$

Therefore we get

$$\begin{aligned} d_{ij}^{(k)} &= \sum_{h=1}^{j-k+1} \binom{j-h-1}{k-2} \binom{x_i+h}{i-k} - \sum_{h=1}^{j-k+1} \binom{j-h-1}{k-2} \binom{x_i+1}{i-k} \\ &= \sum_{h=1}^{j-k+1} \binom{j-h-1}{k-2} \left(\binom{x_i+h}{i-k} - \binom{x_i+1}{i-k} \right). \end{aligned}$$

Moreover, by Lemma 3.1,

$$\binom{x_i+h}{i-k} - \binom{x_i+1}{i-k} = \sum_{r=1}^{h-1} \binom{x_i+r}{i-k-1},$$

for $i > k$ and $h \ge 1$. Hence

$$d_{ij}^{(k)} = \sum_{h=1}^{j-k+1} \binom{j-h-1}{k-2} \sum_{r=1}^{h-1} \binom{x_i+r}{i-k-1} = \sum_{r=1}^{j-k} \binom{x_i+r}{i-k-1} \sum_{h=r+1}^{j-k+1} \binom{j-h-1}{k-2}.$$

Furthermore, by Lemma 3.2,

$$\sum_{h=r+1}^{j-k+1} \binom{j-h-1}{k-2} = \sum_{s=k-2}^{j-r-2} \binom{s}{k-2} = \binom{j-r-1}{k-1}.$$

Thus

$$d_{ij}^{(k)} = \sum_{r=1}^{j-k} \binom{j-r-1}{k-1} \binom{x_i+r}{i-k-1},$$

for $k+1 \le i \le n$ and $1 \le j \le n$.

Also, for $1 \le i \le k$, since $R_i^{(k)} = R_i^{(k-1)}$, we have

$$d_{ij}^{(k)} = d_{ij}^{(k-1)}, \quad \text{for } 1 \le i \le k,\ 1 \le j \le n.$$

This shows that the property holds for k, and completes the induction.

□

Corollary 3.4. *For $1 \le k \le n$, the determinant*

$$D_n^{(k)} = \left| \left(d_{ij}^{(k)} \right)_{\substack{1 \le i \le n \\ 1 \le j \le n}} \right|$$

obtained from D_n by applying k row reduction steps as described in Proposition 3.3 is given by

$$d_{ij}^{(k)} = \begin{cases} \binom{j-1}{i-1}, & \text{if } 1 \le i \le k,\ 1 \le j \le n, \\ \sum_{h=1}^{j-k} \binom{j-h-1}{k-1} \binom{x_i+h}{i-k-1}, & \text{if } k+1 \le i \le n,\ 1 \le j \le n, \end{cases} \tag{3.8}$$

where if $0 \le m < n$ are integers then $\binom{m}{n} = 0$, and an empty sum is equal to 0.

Proof. Only the expression of $d_{ij}^{(k)}$ for $1 \le i \le k$ and $1 \le j \le n$ needs to be proved. The rest is contained in Proposition 3.3. This expression holds for $k = 1$ since by (3.3)

$$d'_{1j} = d_{1j} = 1 = \binom{j-1}{1-1}, \quad \text{for } 1 \le j \le n.$$

Assume that the expression holds for $k-1$, where $2 \le k < n$, i.e.

$$d_{ij}^{(k-1)} = \begin{cases} \binom{j-1}{i-1}, & \text{if } 1 \le i \le k-1,\ 1 \le j \le n, \\ \sum_{h=1}^{j-k+1} \binom{j-h-1}{k-2} \binom{x_i+h}{i-k}, & \text{if } k \le i \le n,\ 1 \le j \le n. \end{cases}$$

Then, by Proposition 3.3 and Lemma 3.2, we have

$$d_{ij}^{(k)} = d_{ij}^{(k-1)} = \binom{j-1}{i-1}, \quad \text{for } 1 \le i \le k-1,\ 1 \le j \le n,$$

and

$$\begin{aligned} d_{kj}^{(k)} = d_{kj}^{(k-1)} &= \sum_{h=1}^{j-k+1} \binom{j-h-1}{k-2} \binom{x_k+h}{0} = \sum_{h=1}^{j-k+1} \binom{j-h-1}{k-2} \\ &= \sum_{s=k-2}^{j-2} \binom{s}{k-2} = \binom{j-1}{k-1}, \quad \text{for } 1 \le j \le n. \end{aligned}$$

Hence

$$d_{ij}^{(k)} = \binom{j-1}{i-1}, \quad \text{for } 1 \le i \le k,\ 1 \le j \le n,$$

and the expression holds for k. □

Remark 3.5. *We have*

$$D_n = D_n^{(k)},$$

since the determinant is invariant under the row reduction steps consisting of adding to a row a multiple of another row.

In particular,

$$D_n^{(n)} = \left| \left(\binom{j-1}{i-1} \right)_{\substack{1\le i\le n\\ 1\le j\le n}} \right|$$

is the determinant of an upper triangular $n \times n$ matrix whose diagonal entries are $d_{ii}^{(n)} = \binom{i-1}{i-1} = 1$ for $1 \le i \le n$.

Therefore

$$D_n = D_n^{(n)} = 1.$$

This concludes the proof of Theorem 2.1.

Remark 3.6. As noted in the Introduction, our result is a special case of the results contained in [10]. Indeed, in [10], Proposition 1, taking $p_j(x) = \binom{x_j+x}{j-1}$, which is a polynomial of degree $j-1$ in x, with leading coefficient $a_j = \frac{1}{(j-1)!}$, for $1 \le j \le n$, we get

$$\left| \left(\binom{x_j + X_i}{j-1} \right)_{\substack{1\le i\le n\\ 1\le j\le n}} \right| = \prod_{j=1}^{n} \frac{1}{(j-1)!} \cdot \prod_{1\le i<j\le n} (X_j - X_i)\,,$$

then specializing to $X_i = i$ for $1 \le i \le n$, we get

$$\left| \left(\binom{x_j + i}{j-1} \right)_{\substack{1\le i\le n\\ 1\le j\le n}} \right| = \prod_{j=1}^{n} \frac{1}{(j-1)!} \cdot \prod_{1\le i<j\le n} (j-i) = 1.$$

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