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Instability Analysis of Pressure-Loaded Thin Arches of Arbitrary Shape

The governing differential equations and the virtual work expressions for the large displacement analysis of thin arches of arbitrary shape, subjected to pressure loads, are derived. The virtual work expressions are employed as the basis for formulation of finite element stiffness equations. Classical solutions are obtained, from the differential equations, for the buckling of circular rings under uniform "follower" (hydrostatic) and "dead" (constant direction) pressure loadings. Finite element solutions are calculated for elliptical rings for a wide range of axis ratios.

Introduction

Although basic theoretical principles for the inclusion of pressure-load effects in finite element, elastic instability analysis have been established for some time now [1], there is considerable interest in and need for relationships for specific cases of interest and for the study of the basic properties of these relationships. Thus, Hibbitt [2], Loganathan, et al. [3], and Mang [4] have examined the algebraic form and permissible approximations for finite element stiffness relationships that arise when the effects of follower forces are taken into account. Batoz [5, 6] has studied the formulation of such relationships for the particular case of circular arch finite elements.

Because the finite element method owes its significance to its potentiality for the treatment of structures of rather arbitrary geometry, it is desirable to have available the theoretical basis for formulation of arch elements of any shape. Thus, the purpose of this paper is to derive geometrically nonlinear formulations for arches of arbitrary shape acted on by pressure loads. Both the governing differential equations and the associated virtual work expressions are presented. Generalized stress vectors are defined which are consistent with the definitions of the strains. The interaction of membrane and bending deformations is taken into account.

The governing differential equations derived herein are more general than those that have appeared previously. Frisch-Fay [7] gives basic nonlinear equations for arches of arbitrary shape, but neglects the interaction of bending and membrane deformations. Wang [8] presents a linear static analysis for a class of ring segments. The equilibrium equations, however, are established for the undeformed configuration. The theory can only be used for cycloidal, circular, catenary, and parabolic rings. If the radius of ring segments cannot be expressed by $R = a \sec^n \phi$, it is inapplicable.

In this paper, following the derivation of the equations for general shapes, various aspects of circular arches are studied. Using the hypothesis of small middle-surface strain and moderately small rotation, the governing differential equations for circular rings are obtained from the more general equations. These equations are solved for the eigenvalues for the cases of "follower" (hydrostatic) and "dead" (constant direction) pressures, yielding solutions in accordance with previously derived results. Certain aspects of Batoz's formulations for circular arches are also verified. Finally, the finite element method is used to calculate the critical loads for the elliptical rings of different geometric parameters under two kinds of pressures. In case of "follower" load, symmetrized load stiffness matrices are employed.

Strain-Displacement Relations

The middle surface of an undeformed thin arch of arbitrary shape can be expressed by the parametric equations (see Fig. 1)

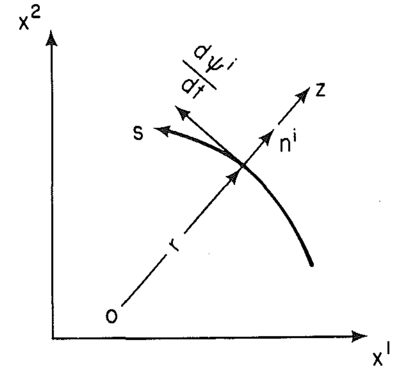

Fig. 1

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$$x^i = \psi^i(t), \quad i=1,2 \tag{1}$$

The displacement of a point on the middle surface, referred to the undeformed shape, is

$$u^i = \frac{d\psi^i}{dt} v^0 + n^i w, \quad i=1,2 \tag{2}$$

where v^0 is the tangent component of displacement, w is the normal component, and n^i is the unit normal to the undeformed middle surface. Lowercase letters are used here to denote the displacements measured from the undeformed state. Subsequently, we will use capital letters to denote displacements referred to the deformed state.

For the deformed middle surface, the expressions for the metric A and the coefficient of the second fundamental form F can be written as follows

$$A = a[(1+e)^2 + \phi^2] \tag{3}$$

and

$$F = \sqrt{\frac{a}{A}}\left\{(1+e)\left[f(1+e) + \frac{d}{dt}(\sqrt{a}\,\phi)\right] - \sqrt{a}\,\phi\left(\frac{de}{dt} - \frac{f}{\sqrt{a}}\phi\right)\right\} \tag{4}$$

where

$$a = \left(\frac{d\psi^1}{dt}\right)^2 + \left(\frac{d\psi^2}{dt}\right)^2 \tag{5}$$

is the metric of the undeformed middle surface, and

$$f = n^1 \frac{d^2\psi^1}{dt^2} + n^2 \frac{d^2\psi^2}{dt^2} \tag{6}$$

is the coefficient of the second fundamental form.
Also, in equations (3) and (4), e represents the membrane deformation and ϕ is the rotation of a normal to the middle surface of the arch. These are, in terms of the displacements

$$e = \frac{dv^0}{dt} - \frac{f}{a} w \tag{7}$$

$$\phi = \frac{1}{\sqrt{a}}\left(fv^0 + \frac{dw}{dt}\right) \tag{8}$$

The strain of the middle surface of the arch is defined as

$$E_m = \frac{1}{2}\,\frac{(d\hat{s})^2 - (ds)^2}{(ds)^2} = e + \frac{1}{2}e^2 + \frac{1}{2}\phi^2 \tag{9}$$

where ds and $d\hat{s}$ are the length of the element of the undeformed and deformed middle surface, respectively.

The strain at a point with coordinate z is

$$E_z = \frac{1}{2}\,\frac{(d\hat{s}_z)^2 - (ds_z)^2}{(ds_z)^2} = \frac{1}{2\left(1 - \frac{f}{a}z\right)^2 a}\left[(A-a) - 2(F-f)z + \left(\frac{F^2}{A} - \frac{f^2}{a}\right)z^2\right] \tag{10}$$

where ds_z and $d\hat{s}_z$ are the length of the element of a fiber that is parallel to and at a distance z from the middle surface.

Virtual Work Equations

In the following, the virtual work equations are derived from which the nonlinear finite element analysis of arches of arbitrary shape under hydrostatic and constant direction pressure can be established on the basis of a consistent theory.

First, the equilibrium equations of the deformed arch (see for example, reference [9]) are

$$\left.\begin{aligned} &\frac{d}{dt}\left(\frac{N_0}{\sqrt{A}}\right) - \frac{Q}{R} + q_0 = 0 \\ &\frac{N_0}{R} + \frac{1}{\sqrt{A}}\frac{dQ}{dt} + p = 0 \\ &\frac{1}{\sqrt{A}}\frac{dM}{dt} + Q = 0 \end{aligned}\right\} \tag{11}$$

where N_0 is the axial force, Q is the shear force, q_0 is the frictional drag, p is the pressure, and R is the radius of curvature. From (A.8), $R = A/F$. According to the principle of virtual work, we can construct the following integral

$$\int_L \left\{\left[\frac{d}{dt}\left(\frac{N_0}{\sqrt{A}}\right) - \frac{Q}{R} + q_0\right]\delta V_0 + \left(\frac{N_0}{R} + \frac{1}{\sqrt{A}}\frac{dQ}{dt} + p\right)\delta W + \left(\frac{1}{\sqrt{A}}\frac{dM}{dt} + Q\right)\delta\theta\right\} ds = 0 \tag{12}$$

δV_0 and δW are the tangent and normal virtual displacements, respectively, referred to the deformed configuration, $\delta\theta$ is the virtual rotation, and $ds = \sqrt{a}\,dt$. L is the length of the middle surface of the arch with boundaries l_1 and l_2.

Integration by parts of equation (12) yields

$$\left[\frac{1}{\sqrt{A}}(N_0\,\delta V_0 + Q\,\delta W + M\,\delta\theta)\right]\Big/_{l_1}^{l_2} + \int_L (q_0\,\delta V_0 + p\,\delta)\,ds = \int_L \left[\left(\frac{1}{\sqrt{A}}\delta\left(\frac{dV_0}{dt}\right) - \frac{1}{R}\delta W\right)N_0 + \left(\frac{1}{R}\delta V_0 + \frac{1}{\sqrt{A}}\delta\left(\frac{dW}{dt}\right) - \delta\theta\right)Q + \frac{1}{\sqrt{A}}\delta\left(\frac{d\theta}{dt}\right)M\right] ds \tag{13}$$

After deformation of the arch, a point on the undeformed middle surface wtih coordinates x^i moves to a new position

$$\hat{x}^i = x^i + u^i \tag{14}$$

where u^i is given by equation (2). Let $\hat{n}^i$ be the unit normal to the deformed middle surface. The virtual displacements can be expressed as follows

$$\delta u^i = \frac{d\hat{x}^i}{dt}\delta V^0 + \hat{n}^i\,\delta W = \frac{d\psi^i}{dt}\,\delta v^0 + n^i\,\delta w \tag{15}$$

where

$$V^0 = \frac{V_0}{\sqrt{A}} \quad \text{and} \quad v^0 = \frac{v_0}{\sqrt{a}}.$$

Differentiating (15) and using (A2), (A3), (A6), (A7), (7), and (8), we obtain

$$\left[\delta\left(\frac{dV^0}{dt}\right) - \frac{F}{A}\,\delta W\right]\frac{d\hat{x}^i}{dt} + \left[F\,\delta V^0 + \delta\left(\frac{dW}{dt}\right)\right]\hat{n}^i = \frac{dx^i}{dt}\,\delta e + n^i\sqrt{a}\,\delta\phi \tag{16}$$

Multiplying both sides of (16) by $d\hat{x}^i/dt$ and $\hat{n}^i$, respectively, and taking account of expressions (9), (A9), and (A5), the following equations are obtained

$$\delta\left(\frac{dV^0}{dt}\right) - \frac{F}{A}\,\delta W = \frac{1}{2A}\,\delta A \tag{17}$$

$$\frac{1}{R}\,\delta V_0 + \frac{1}{\sqrt{A}}\delta\left(\frac{dW}{dt}\right) = -\frac{a}{A}\,[\phi\,\delta e - (1+e)\,\delta\phi] \tag{18}$$

In (13) the coefficient of Q is the virtual shear deformation for the arch. This effect is small and setting it equal to zero we have

$$\frac{1}{R}\,\delta V_0+\frac{1}{\sqrt{A}}\,\delta\left(\frac{dW}{dt}\right)-\delta\theta=0 \tag{19}$$

By using (18), (19), and (A.6)–(A.9), it can be shown that

$$\frac{1}{\sqrt{A}}\,\delta\left(\frac{d\theta}{dt}\right)=\frac{1}{A}\left[\delta F-\frac{1}{2R}\,\delta A+\frac{d}{dt}\left(\frac{1}{\sqrt{A}}\right)A\,\delta\theta\right] \tag{20}$$

In (13) the coefficient of M is $1/\sqrt{A}\ \delta(d\theta/dt)$, which is the variation of the bending curvature. In expression (20) the last term in the square brackets therefore represents the influence of the shear strain on the bending curvature, and can be neglected according to the Love-Kirchoff hypothesis. Thus, equation (20) can be written as

$$\frac{1}{\sqrt{A}}\,\delta\left(\frac{d\theta}{dt}\right)\approx\frac{1}{A}\left(\delta F-\frac{1}{2R}\,\delta A\right) \tag{21}$$

By using (17), (19), and (21), the internal virtual work in expression (13) can be reduced to

$$\delta I_i=\int_L\left[\frac{N_0}{2A}\,\delta A+\frac{M}{A}\left(\delta F-\frac{1}{2R}\,\delta A\right)\right]ds$$

$$=\int_L\ (\tilde{N}\,\delta\,\tilde{E}+\tilde{M}\,\delta\,\tilde{K})\ ds \tag{22}$$

where E and K are defined as the generalized normal and bending strains, respectively. In consideration of (9) and (10)

$$\tilde{E}=\frac{1}{2}\,(A-a)=aE_m \tag{23a}$$

$$\tilde{K}=F-f \tag{23b}$$

and $\tilde{N}$ and $\tilde{M}$ are the corresponding generalized normal stress vector and bending moment:

$$\tilde{N}=\frac{1}{A}\left(N_0-\frac{M}{R}\right) \tag{24a}$$

$$\tilde{M}=\frac{M}{A} \tag{24b}$$

The stress-strain relations can be written in the form

$$\tilde{N}=(E\,\Omega/A^2)\,\tilde{E} \tag{25a}$$

$$\tilde{M}=(EI/A^2)\,\tilde{K} \tag{25b}$$

where $E\,\Omega$ is membrane rigidity and EI is bending rigidity.

Thus, in view of (25a,b) equation (22) can be written as

$$\delta I_i=\int\left[\left(\frac{E\,\Omega}{A^2}\right)\tilde{E}\,\delta\,\tilde{E}+\left(\frac{EI}{A^2}\right)\tilde{K}\,\delta\,\tilde{K}\right]ds \tag{26}$$

Substituting equations (3), (4), (7), and (8) into (23a) and (23b), we obtain

$$\tilde{E}=a\left[\frac{dv^0}{dt}-\frac{f}{a}\,w+\frac{1}{2}\left(\frac{dv^0}{dt}-\frac{f}{a}\,w\right)^2+\frac{1}{2a}\left(fv^0+\frac{dw}{dt}\right)^2\right] \tag{27a}$$

$$\tilde{K}=\sqrt{\frac{a}{A}}\left\{f\left(1+\frac{dv^0}{dt}-\frac{f}{a}\,w\right)^2+\left(1+\frac{dv^0}{dt}-\frac{f}{a}\,w\right)\left[\frac{d}{dt}\,(fv^0)+\frac{d^2w}{dt^2}\right]-\left(fv^0+\frac{dw}{dt}\right)\left[\frac{d^2v^0}{dt^2}-\frac{d}{dt}\left(\frac{f}{a}\,w\right)-\frac{f^2}{a}\,v^0-\frac{f}{a}\,\frac{dw}{dt}\right]-\sqrt{\frac{A}{a}}f\right\} \tag{27b}$$

Substitution of (27a) and (27b) into equation (26) gives the virtual work in terms of the displacements v^0, w, and the metric a, and the coefficient of the second fundamental form f.

The external virtual work is

$$\delta I_h=\int_L p\sqrt{\frac{a}{A}}\left[-\left(fv^0+\frac{dw}{dt}\right)\delta v^0+\left(1+\frac{dv^0}{dt}-\frac{f}{a}\,w\right)\delta w\right]ds, \tag{28}$$

for the hydrostatic pressure, and

$$\delta I_c=\int_L p\,\delta\,w\,d\,s, \tag{29}$$

for the constant direction pressure.

The principle of virtual work then can be expressed as

$$\delta\,(I_i-I_e)=0 \tag{30}$$

where δI_e is the external virtual work, for the hydrostatic pressure $\delta I_e\equiv\delta I_h$ and, for the constant direction pressure $\delta I_e\equiv\delta I_c$.

In accordance with the small middle-surface strain and moderately small rotation hypothesis, i.e.,

$$e<<1,\ \phi^2<<1\quad\text{and}\quad Z\frac{d\phi}{ds}<<1$$

the virtual work expressions (26) and (28) become

$$\delta I_i=\delta\int_L\frac{E\Omega}{2a^2}\left[\sqrt{a}\,\frac{dv}{dt}-\frac{1}{2\sqrt{a}}\,\frac{da}{dt}\,v-fw+\frac{1}{2}\left(\frac{f}{\sqrt{a}}\,v+\frac{dw}{dt}\right)^2\right]^2 ds$$

$$+\delta\int_L\frac{EI}{2a^3}\left[f\frac{dv}{dt}+\left(\frac{df}{dt}-\frac{f}{2a}\,\frac{da}{dt}\right)v+\sqrt{a}\,\frac{d^2w}{dt^2}\right]^2 ds \tag{31}$$

and

$$\delta I_h=\int_L p\left[-\frac{1}{\sqrt{a}}\left(\frac{f}{\sqrt{a}}\,v+\frac{dw}{dt}\right)\delta v+\delta w\right]ds \tag{32}$$

where

$$v=\sqrt{a}\,v^0$$

Formulation of Governing Differential Equations

The internal virtual work of (26) can be written in the alternative form

$$\delta I_i=\int_L\left\{\tilde{N}A\left[\delta\left(\frac{dV^0}{dt}\right)-\frac{F}{A}\,\delta W\right]+\tilde{M}\sqrt{A}\left[\frac{d}{dt}\left(\frac{\sqrt{A}}{R}\,\delta V^0\right)+\frac{d}{dt}\left(\frac{1}{\sqrt{A}}\,\delta\left(\frac{dW}{dt}\right)\right)\right]+\tilde{M}\left[\frac{A}{R}\,\delta\left(\frac{dV^0}{dt}\right)-\frac{F}{R}\,\delta W\right]\right\}ds \tag{33}$$

For the hydrostatic pressure, the external virtual work is

$$\delta I_h=\int_L p\,\delta\,W\,ds \tag{34}$$

and, for the constant direction pressure

$$\delta I_c=\int_L p\left[\sqrt{a}\,\phi\,\delta\,V^0+\sqrt{\frac{a}{A}}\,(1+e)\,\delta\,W\right]ds \tag{35}$$

The external virtual work for the frictional drag q_0 is

$$\delta I_d=\int_L q_0\sqrt{A}\,\delta\,V^0\,ds \tag{36}$$

Integrating (33) by parts and then combining with (34) and (36) we obtain the following form of the equilibrium equations

$$\frac{d}{dt}(\tilde{N}A)+\frac{F}{\sqrt{A}}\frac{d}{dt}(\tilde{M}\sqrt{A})+\frac{d}{dt}(\tilde{M}F)+q_0\sqrt{A}=0 \quad (37a)$$

$$\tilde{N}F-\frac{d}{dt}\left[\frac{1}{\sqrt{A}}\frac{d}{dt}(\tilde{M}\sqrt{A})\right]+\tilde{M}\frac{F}{R}+p=0 \quad (37b)$$

Elimination of $\tilde{N}$ from these two equations gives a single general nonlinear equation for arches of arbitrary form under hydrostatic pressure and frictional drag, with force parameters as unknowns:

$$\frac{d^3\tilde{M}}{dt^3}+c_2\frac{d^2\tilde{M}}{dt^2}+c_1\frac{d\tilde{M}}{dt}+c_0\tilde{M}+D_2\frac{dp}{dt}+D_1p+D_0q_0=0 \quad (38)$$

where

$$c_2=\frac{3}{2}\frac{1}{A}\frac{dA}{dt}-\frac{1}{F}\frac{dF}{dt},\ c_1=\frac{1}{A}\frac{d^2A}{dt^2}-\frac{1}{2A}\frac{dA}{dt}\left(\frac{1}{A}\frac{dA}{dt}+\frac{1}{F}\frac{dF}{dt}\right)+\frac{F}{R}$$

$$c_0=\frac{1}{2A}\frac{d^3A}{dt^3}-\frac{1}{A}\frac{dA}{dt}\left[\frac{1}{A}\frac{d^2A}{dt^2}-\frac{1}{2A^2}\left(\frac{dA}{dt}\right)^2\right]-\frac{1}{2FA}\frac{dF}{dt}\left[\frac{d^2A}{dt^2}-\frac{1}{A}\left(\frac{dA}{dt}\right)^2\right]+\frac{1}{R}\frac{dF}{dt}+\frac{F}{R^2}\frac{dR}{dt}-\frac{1}{2R^2}\frac{dA}{dt},\ D_2=-1,\ D_1=\frac{1}{F}\frac{dF}{dt}-\frac{1}{A}\frac{dA}{dt},\ D_0=\frac{F}{\sqrt{A}}.$$

Substituting (25b) into (38) gives the general nonlinear equation with displacement parameters as unknowns, for hydrostatic pressure and frictional drag

$$\frac{d^3}{dt^3}\left(\frac{EI}{A^2}\tilde{K}\right)+c_2\frac{d^2}{dt^2}\left(\frac{EI}{A^2}\tilde{K}\right)+c_1\frac{d}{dt}\left(\frac{EI}{A^2}\tilde{K}\right)+c_0\frac{EI}{A^2}\tilde{K}+D_2\frac{dp}{dt}+D_1p+D_0q_0=0 \quad (39)$$

Similar expressions can also be obtained for constant direction pressure.

Circular Arch

To confirm the foregoing, we derive the equations of thin circular arches as a special case. The parametric equations of the middle surface of circular arches are

$$x^1=\rho\cos t,\ x^2=\rho\sin t$$

The metric and the coefficient of the second fundamental form are therefore $a=\rho^2, f=-\rho$. Consider a circular ring under hydrostatic pressure and constant direction pressure. For the case of uniformly distributed pressure without frictional drag, $q_0=0$, $dp/dt=0$, and $p=-p_{\rm cr}$, in accordance with the small middle-surface strain and moderately small rotation hypotheses. From equation (39) the following differential equation for the hydrostatic pressure case can be derived.

$$\frac{d^5w}{ds^5}+\frac{2}{\rho^2}\frac{d^3w}{ds^3}+\frac{1}{\rho^4}\frac{dw}{ds}+\frac{p_{\rm cr}\rho}{EI}\left(\frac{d^3w}{ds^3}+\frac{1}{\rho^2}\frac{dw}{ds}\right)=0 \quad (40)$$

Similarly for constant direction pressure the equation is

$$\frac{d^6w}{ds^6}+\frac{2}{\rho^2}\frac{d^4w}{ds^4}+\frac{1}{\rho^4}\frac{d^2w}{ds^2}+\frac{p_{\rm cr}\rho}{EI}\left(\frac{d^4w}{ds^4}+\frac{2}{\rho^2}\frac{d^2w}{ds^2}+\frac{w}{\rho^4}\right)=0 \quad (41)$$

where $ds=\rho\,dt$.

The buckling mode is assumed as $w=w_0\sin ns/\rho$, where n is the number of waves.

When $n=2$, we obtain the well-known critical pressure (for example see [10]), for the hydrostatic pressure

$$p_{\rm cr}=3\frac{EI}{\rho^3}$$

and for the constant direction pressure

$$p_{\rm cr}=4\frac{EI}{\rho^3}$$

Finite Element Formulations

The buckling of rings of different dimensions and shapes under both hydrostatic and constant direction pressures can be investigated by the finite element method. For this we use expressions for the virtual work written in terms of displacements, i.e., equations (29), (31), and (32).

To transform the preceding expressions into algebraic form, we first choose approximations for the displacements and virtual displacements as follows:

$$v=\lfloor N_1\rfloor\{\Delta\}\qquad \delta v=\lfloor N_1\rfloor\{\delta\Delta\}$$
$$w=\lfloor N_2\rfloor\{\Delta\}\qquad \delta w=\lfloor N_2\rfloor\{\delta\Delta\}$$

where $\lfloor N_i\rfloor$ is row vector of expressions that approximate the shape of the displaced state (i.e., "shape functions") and $\{\Delta\}$ is a column vector of displacements (including rotations, as appropriate) of specified points on the element, and $\{\delta\Delta\}$ is the column vector of joint virtual displacements. After differentiation of v, w, δv, and δw and insertion of the foregoing into the left-hand side of the virtual work expression (30), for an element, we obtain,

$$\delta(I_i-I_e)=\lfloor\delta\Delta\rfloor\,[[K^e]+p_{\rm cr}([K_G^e]-[K_L^e])]\,\{\Delta\} \quad (42)$$

where $[K^e]$ is the elastic stiffness matrix which includes the membrane stiffness matrix $[K_m^e]$ and the bending stiffness matrix $[K_b^e]$,

$$[K_m^e]=\int_L(\lfloor d_m\rfloor[N])^T E\,\Omega\,(\lfloor d_m\rfloor[N])\,a^{-2}\,ds$$

and

$$[K_b^e]=\int_L(\lfloor d_b\rfloor[N])^T EI\,(\lfloor d_b\rfloor[N])\,a^{-2}\,ds$$

where

$$\lfloor d_m\rfloor=\left\lfloor\left(\sqrt{a}\frac{d}{dt}-\frac{1}{2\sqrt{a}}\frac{da}{dt}\right)\Big|-f\right\rfloor$$

and

$$\lfloor d_b\rfloor=\left\lfloor\left(\frac{f}{\sqrt{a}}\frac{d}{dt}+\frac{1}{\sqrt{a}}\left(\frac{df}{dt}-\frac{f}{2a}\frac{da}{dt}\right)\right)\Big|\frac{d^2}{dt^2}\right\rfloor$$

$[K_G^e]$ is the "geometric stiffness matrix"

$$[K_G^e]=\int_L\frac{1}{f}(\lfloor d_g\rfloor[N])^T(\lfloor d_g\rfloor[N])\,ds$$

where

$$\lfloor d_g\rfloor=\frac{f}{\sqrt{a}}\quad\frac{d}{dt}\,J$$

The load stiffness matrix is

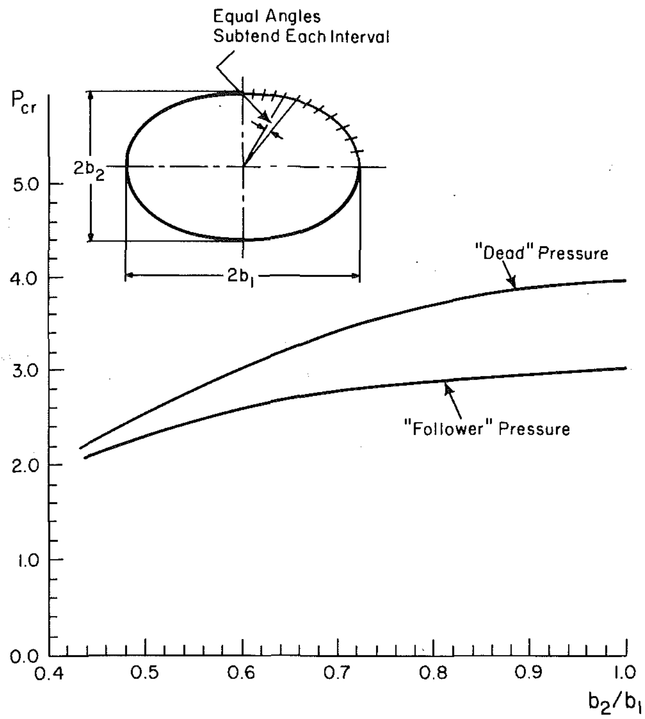

Fig. 2

$$[K_L^e] = \int_L (\lfloor d_{l1} \rfloor [N])^T (\lfloor d_{l2} \rfloor [N])\, ds$$

where

$$\lfloor d_{l1} \rfloor = \lfloor 1\ 0 \rfloor, \quad \lfloor d_{l2} \rfloor = \left\lfloor \frac{f}{a} \ \frac{1}{\sqrt{a}} \frac{d}{dt} \right\rfloor$$

It should be noted that $[K_L^e]$, in general, is an unsymmetric matrix.

Summation of the element virtual work of all elements gives the global virtual work. In accordance with the principle of virtual work, equation (30), from equation (42), we have for the structural system

$$[[K] + P_{cr}([K_G] - [K_L])]\{\Delta\} = 0 \tag{43}$$

The critical pressures are obtained by solving the algebraic eigenvalue equation stemming from the foregoing.

Finite Element Solution for Elliptical Rings

The buckling of elliptical rings of different dimensions under both "follower" and "dead" pressures is now investigated by use of the finite element method. For elliptical arches the middle-surface equations are

$$x^1 = b_1 \cos t, \quad x^2 = b_2 \sin t$$

According to (4) and (5), we have

$$a = b_1^2 \sin^2 t + b_2^2 \cos^2 t$$

and

$$f = \frac{-b_1 b_2}{(b_1^2 \sin^2 t + b_2^2 \cos^2 t)^{1/2}}$$

The hypotheses of small middle-surface strain and moderate rotation are used in the formulation. Both displacement functions, v and w, are approximated by cubic polynomials. The finite element mesh of a quarter of the ring (see inset, Fig. 2) consists of 12 elements with a total of 52 degrees of freedom. For the "follower" pressure the load stiffness matrix is symmetrized.

Table 1

Load	P_{cr} $^*b_1 = b_2 = 100$	$b_1 = 110$, $b_2 = 90$	$b_1 = 120$, $b_2 = 80$	$b_1 = 140$, $b_2 = 60$
"dead"	4.0000	3.7724	3.3128	2.1617
"follower"	3.0232	2.8140	2.7352	2.0333

$^*2b_1$ – long diameter of the ellipse
$2b_2$ – short diameter of the ellipse

$$[K_L^{(s)}] = \frac{1}{2}([K_L] + [K_L]^T)$$

where $[K_L]$ is the unsymmetric load stiffness matrix. The thickness of the arch is taken to be $t = 1.0$, the width $b = 12.0$, and elastic modulus $E = 10^6$. The results of the computation are summarized in Table 1 and Fig. 2.

The results show that the 1.33 ratio between "dead" and "follower" instability pressures for circular rings approaches 1.0 as the axis ratio decreases. Comparison of the finite element and classical solutions for the circular ring discloses a high degree of accuracy for the former. However, there appears to be no available comparison solutions for elliptical rings for the phenomena studied.

Concluding Remarks

The purpose of this paper has been to present the basic relationships, in the form of both differential equations and the virtual work expression for pressure-loaded thin arches of arbitrary shape. Using the virtual work expression and displacement approximations often employed in the finite element representation of circular arches, finite element stiffness equations are constructed for an elliptic arch element. These are employed in analyses of pressure-loaded elliptic arches for the full range of axis ratios.

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APPENDIX

Basic Geometric Relations

For the undeformed middle surface, the relations between the geometric characteristics are

$$n^i = \frac{1}{\sqrt{a}} e_{ij} \frac{d\psi^j}{dt} \quad (A1)$$

$$\frac{dn^i}{dt} = -\frac{f}{a} \frac{d\psi^i}{dt} \quad (A2)$$

$$\frac{d^2\psi^i}{dt^2} = fn^i \quad (A3)$$

$$r = \frac{a}{f} \quad (A4)$$

where e_{ij} is the permutation symbol and r is the radius of curvature of the undeformed middle surface.

Similarly, for the deformed middle surface we have

$$\hat{n}^i = \frac{1}{\sqrt{A}} e_{ij} \frac{d\hat{x}^j}{dt} \quad (A5)$$

$$\frac{d\hat{n}^i}{dt} = -\frac{F}{A} \frac{d\hat{x}^i}{dt} \quad (A6)$$

$$\frac{d^2\hat{x}^i}{dt^2} = F\hat{n}^i \quad (A7)$$

$$R = \frac{A}{F} \quad (A8)$$

where R is the radius of curvature of the deformed middle surface.

According to expressions (2), (7), (8), and (14) the tangent to the middle surface of deformed arches can be written as

$$\frac{d\hat{x}^i}{dt} = \frac{d\psi^i}{dt}(1+e) + n^i \sqrt{a}\, \phi \quad (A9)$$