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A Note on the Symmetric Properties for the q-Twisted Tangent Polynomials

C. S. Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

Abstract

In [4], we studied the twisted q-tangent numbers and polynomials. By using these numbers and polynomials, we give some interesting relations between the power sums and the the twisted Tangent polynomials.

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1 Introduction

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. For

$$g \in UD(\mathbb{Z}_p) = \{g|g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{0\le x<p^N} g(x)(-1)^x, \quad \text{(see[1])}. \tag{1.1}$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0), \text{ (see [1-4])}. \tag{1.2}$$

Let $T_p = \cup_{N\geq 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \{\omega | \omega^{p^N} = 1\}$ is the cyclic group of order p^N. For $\omega \in T_p$, we denote by $\phi_\omega : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \omega^x$.

In [4], we introduced the twisted q-tangent numbers $T_{n,q,\omega}$ and polynomials $T_{n,q,\omega}(x)$ and investigate their properties. Let us define the twisted q-tangent numbers $T_{n,q,\omega}$ and polynomials $T_{n,q,\omega}(x)$ as follows:

$$\int_{\mathbb{Z}_p} \phi_\omega(y) q^y e^{2yt} d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_{n,q,\omega} \frac{t^n}{n!}, \tag{1.3}$$

$$\int_{\mathbb{Z}_p} \phi_\omega(y) q^y e^{(2y+x)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_{n,q,\omega}(x) \frac{t^n}{n!}. \tag{1.4}$$

The following elementary properties of the twisted q- tangent numbers $E_{n,q,\omega}$ and polynomials $T_{n,q,\omega}(x)$ are readily derived form (1.1), (1.2), (1.3) and (1.4)(see, for details, [4]). We, therefore, choose to omit details involved.

Theorem 1.1 *For $\omega \in T_p$, we have*

$$\int_{\mathbb{Z}_p} \phi_\omega(x) q^x (2x)^n d\mu_{-1}(x) = T_{n,q,\omega},$$

$$\int_{\mathbb{Z}_p} \phi_\omega(y) q^y (2y+x)^n d\mu_{-1}(y) = T_{n,q,\omega}(x).$$

Theorem 1.2 *For any positive integer n, we have*

$$T_{n,q,\omega}(x) = \sum_{k=0}^{n} \binom{n}{k} T_{k,q,\omega} x^{n-k}.$$

In this paper, by using same method of [2], expect for obvious modifications, we obtain recurrence identities the twisted q-tangent polynomials and the alternating sums of powers of consecutive integers.

2 Alternating sums of powers of consecutive even integers

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. Let ω be the p^N-th root of unity. By using (1.4), we give the alternating sums of powers of consecutive

(ω, q)-even integers as follows:

$$\sum_{n=0}^{\infty} T_{n,q,\omega} \frac{t^n}{n!} = \frac{2}{\omega q e^{2t} + 1} = 2 \sum_{n=0}^{\infty} (-1)^n \omega^n q^n e^{2nt}.$$

From the above, we obtain

$$-\sum_{n=0}^{\infty} (-1)^n \omega^n q^n e^{(2n+2k)t} + \sum_{n=0}^{\infty} (-1)^{n-k} \omega^{n-k} q^{n-k} e^{2nt} = \sum_{n=0}^{k-1} (-1)^{n-k} \omega^{n-k} q^{n-k} e^{2nt}.$$

By using (1.3)and (1.4), we obtain

$$\sum_{n=0}^{k-1} (-1)^n \omega^n q^n (2n)^j = \frac{(-1)^{k+1} \omega^k q^k T_{j,q,\omega}(2k) + T_{j,q,\omega}}{2}.$$

By using the above equation we arrive at the following theorem:

Theorem 2.1 *Let k be a positive integer and $q \in \mathbb{C}$ with $|q| < 1$ and ω be the p^N-th root of unity. Then we obtain*

$$\mathcal{T}_{j,q,\omega}(k-1) = \sum_{n=0}^{k-1} (-1)^n \omega^n q^n (2n)^j = \frac{(-1)^{k+1} \omega^k q^k T_{j,q,\omega}(2k) + T_{j,q,\omega}}{2}. \quad (2.1)$$

Remark 2.2 *For the alternating sums of powers of consecutive even integers, we have*

$$\lim_{q \to 1} \mathcal{T}_{j,q,\omega}(k-1) = \sum_{n=0}^{k-1} (-1)^n \omega^n (2n)^j = \frac{(-1)^{k+1} \omega^k T_{j,\omega}(2k) + T_{j,\omega}}{2},$$

where $T_{j,\omega}(x)$ and $T_{j,\omega}$ denote the twisted tangent polynomials and the twisted tangent numbers, respectively(see [3]).

3 Symmetric properties for the twisted tangent polynomials

In this section, we assume that $q \in \mathbb{C}_p$ and $\omega \in T_p$. By using (1.1), we have

$$I_{-1}(g_n) + (-1)^{n-1} I_{-1}(g) = 2 \sum_{k=0}^{n-1} (-1)^{n-1-k} g(k),$$

where $n \in \mathbb{N}, g_n(x) = g(x+n)$. If n is odd from the above, we obtain

$$I_{-1}(g_n) + I_{-1}(g) = 2 \sum_{k=0}^{n-1} (-1)^{n-1-k} g(k) \text{ (see [1-4])}. \quad (3.1)$$

It will be more convenient to write (3.1) as the equivalent integral form

$$\int_{\mathbb{Z}_p} g(x+n)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = 2\sum_{k=0}^{n-1}(-1)^{n-1-k}g(k). \tag{3.2}$$

Substituting $g(x) = \omega^x q^x e^{2xt}$ into the above, we obtain

$$\omega^n q^n \int_{\mathbb{Z}_p} \omega^x q^x e^{(2x+2n)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^x e^{2xt} d\mu_{-1}(x) = 2\sum_{j=0}^{n-1}(-1)^j \omega^j q^j e^{(2j)t}. \tag{3.3}$$

After some elementary calculations and by substituting Taylor series of e^{2xt} into (3.3), we obtain

$$\begin{aligned} &\sum_{m=0}^{\infty}\left(\int_{\mathbb{Z}_p} \omega^{x+n} q^{x+n}(2x+2n)^m d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^x (2x)^m d\mu_{-1}(x)\right)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(2\sum_{j=0}^{n-1}(-1)^j \omega^j q^j (2j)^m\right)\frac{t^m}{m!}. \end{aligned}$$

By comparing coefficients $\frac{t^m}{m!}$ in the above equation and (2.1), we obtain

$$\begin{aligned} &\omega^n q^n \sum_{k=0}^{m}\binom{m}{k}(2n)^{m-k}\int_{\mathbb{Z}_p} \omega^x q^x (2x)^k d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \omega^x q^x (2x)^m d\mu_{-1}(x) \\ &= 2\mathcal{T}_{m,q,\omega}(n-1). \end{aligned}$$

Therefore, we arrive at the following theorem:

Theorem 3.1 *Let n be odd positive integer. Then we obtain*

$$\frac{2\int_{\mathbb{Z}_p} \omega^x q^x e^{2xt} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} \omega^{nx} q^{nx} e^{2ntx} d\mu_{-1}(x)} = \sum_{m=0}^{\infty}\left(2\mathcal{T}_{m,q,\omega}(n-1)\right)\frac{t^m}{m!}. \tag{3.4}$$

Let w_1 and w_2 be odd positive integers. By using (3.4), we have

$$\begin{aligned} a &= \frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} \omega^{(w_1x_1+w_2x_2)} q^{(w_1x_1+w_2x_2)} e^{(w_1 2x_1 + w_2 2x_2 + w_1w_2x)t} d\mu_{-1}(x_1)d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} \omega^{w_1w_2x} q^{w_1w_2x} e^{2w_1w_2xt} d\mu_{-1}(x)} \\ &= \frac{2e^{w_1w_2xt}(\omega^{w_1w_2} q^{w_1w_2} e^{2w_1w_2t}+1)}{(\omega^{w_1} q^{w_1} e^{2w_1t}+1)(\omega^{w_2} q^{w_2} e^{2w_2t}+1)} \end{aligned} \tag{3.5}$$

By using (3.4) and (3.5), after elementary calculations, we obtain

$$a = \left(\frac{1}{2}\sum_{m=0}^{\infty} T_{m,q^{w_1},\omega^{w_1}}(w_2x) w_1^m \frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty} \mathcal{T}_{m,q^{w_2},\omega^{w_2}}(w_1-1) w_2^m \frac{t^m}{m!}\right).$$

By using Cauchy product in the above, we have

$$a=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}T_{j,q^{w_1},\omega^{w_1}}(w_2x)w_1^j\mathcal{T}_{m-j,q^{w_2},\omega^{w_2}}(w_1-1)w_2^{m-j}\right)\frac{t^m}{m!}. \quad (3.6)$$

By using the symmetry in (3.5), we have

$$a=\left(\frac{1}{2}\sum_{m=0}^{\infty}T_{m,q^{w_2},\omega^{w_2}}(w_1x)w_2^m\frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty}\mathcal{T}_{m,q^{w_1},\omega^{w_1}}(w_2-1)w_1^m\frac{t^m}{m!}\right).$$

Thus we have

$$a=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}T_{j,q^{w_2},\omega^{w_2}}(w_1x)w_2^j\mathcal{T}_{m-j,q^{w_1},\omega^{w_1}}(w_2-1)w_1^{m-j}\right)\frac{t^m}{m!} \quad (3.7)$$

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (3.6) and (3.7), we arrive at the following theorem:

Theorem 3.2 *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$\begin{aligned}&\sum_{j=0}^{m}\binom{m}{j}w_1^{m-j}w_2^jT_{j,q^{w_2},\omega^{w_2}}(w_1x)\mathcal{T}_{m-j,q^{w_1},\omega^{w_1}}(w_2-1)\\&=\sum_{j=0}^{m}\binom{m}{j}w_1^jw_2^{m-j}T_{j,q^{w_1},\omega^{w_1}}(w_2x)\mathcal{T}_{m-j,q^{w_2},\omega^{w_2}}(w_1-1),\end{aligned}$$

where $T_{k,q,\omega}(x)$ and $\mathcal{T}_{m,q,\omega}(k)$ denote the twisted q-tangent polynomials and the alternating sums of powers of consecutive q-even integers, respectively.

By using Theorem 3.2, we have the following corollary:

Corollary 3.3 *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{m-k}w_2^jx^{j-k}T_{k,q^{w_2},\omega^{w_2}}\mathcal{T}_{m-j,q^{w_1},\omega^{w_1}}(w_2-1)\\&=\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^jw_2^{m-k}x^{j-k}T_{k,q^{w_1},\omega^{w_1}}\mathcal{T}_{m-j,q^{w_2},\omega^{w_2}}(w_1-1).\end{aligned}$$

By using (3.5), we have

$$a = \left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{w_1x_1}e^{2x_1w_1t}d\mu_{-1}(x_1)\right)\left(2\sum_{j=0}^{w_1-1}(-1)^j\omega^{w_2j}q^{w_2j}e^{2jw_2t}\right)$$
$$= \sum_{j=0}^{w_1-1}(-1)^j\omega^{w_2j}q^{w_2j}\int_{\mathbb{Z}_p}\omega^{w_1x_1}q^{w_1x_1}e^{\left(2x_1+w_2x+\frac{2jw_2}{w_1}\right)(w_1t)}d\mu_{-1}(x_1)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_1-1}(-1)^j\omega^{w_2j}q^{w_2j}T_{n,q^{w_1},\omega^{w_1}}\left(w_2x+\frac{2jw_2}{w_1}\right)w_1^n\right)\frac{t^n}{n!}. \tag{3.8}$$

By using the symmetry property in (3.8), we also have

$$a = \left(\frac{1}{2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{w_2x_2}e^{2x_2w_2t}d\mu_{-1}(x_2)\right)\left(2\sum_{j=0}^{w_2-1}(-1)^j\omega^{w_1j}q^{w_1j}e^{2jw_1t}\right)$$
$$= \sum_{j=0}^{w_2-1}(-1)^j\omega^{w_1j}q^{w_1j}\int_{\mathbb{Z}_p}\omega^{w_2x_2}q^{w_2x_2}e^{\left(2x_2+w_1x+\frac{2jw_1}{w_2}\right)(w_2t)}d\mu_{-1}(x_1)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_2-1}(-1)^j\omega^{w_1j}q^{w_1j}T_{n,q^{w_2},\omega^{w_2}}\left(w_1x+\frac{2jw_1}{w_2}\right)w_2^n\right)\frac{t^n}{n!}. \tag{3.9}$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (3.8) and (3.9), we have the following theorem.

Theorem 3.4 *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$\begin{aligned}&\sum_{j=0}^{w_1-1}(-1)^j\omega^{w_2j}q^{w_2j}T_{n,q^{w_1},\omega^{w_1}}\left(w_2x+\frac{2jw_2}{w_1}\right)w_1^n\\&=\sum_{j=0}^{w_2-1}(-1)^j\omega^{w_1j}q^{w_1j}T_{n,q^{w_2},\omega^{w_2}}\left(w_1x+\frac{2jw_1}{w_2}\right)w_2^n.\end{aligned} \tag{3.10}$$

Substituting $w_1 = 1$ into (3.10), we arrive at the following corollary.

Corollary 3.5 *Let w_2 be odd positive integer. Then we obtain*

$$T_{n,q,\omega}(x) = w_2^n\sum_{j=0}^{w_2-1}(-1)^j\omega^jq^jT_{n,q^{w_2},\omega^{w_2}}\left(\frac{x+2j}{w_2}\right).$$

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