Equilibria and multidimensional quadratic BSDEs: relations, counterexamples and remedies

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"By pursuing his own interest he frequently promotes that of the society more effectually than when he really intends to promote it."
Adam Smith in *The Wealth of Nations* (1776)

Abstract

While trading on a financial market, the agents we consider take the performance of their peers into account. By maximizing individual utility subject to investment constraints, the agents may ruin each other even unintentionally so that no equilibrium can exist. However, when the agents are willing to waive little expected utility, an approximated equilibrium can be established. The study of the associated backward stochastic differential equation (BSDE) reveals the mathematical reason for the absence of an equilibrium. Presenting two illustrative counterexamples, we explain why such multidimensional quadratic BSDEs may not have solutions despite bounded terminal conditions and in contrast to the one-dimensional case.

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1 Introduction

Assuming you have invested in a fund, are you satisfied with the fund manager if she achieved a performance of 4 % in the last year? You may say that the answer depends mainly on two factors: the risk the manager has taken and the development of the markets in the last year. In mathematical finance, the frequently used approach of maximizing expected utility from terminal wealth incorporates simultaneously the performance and the risk related to a trading strategy. However, the relative performance compared to an index or other investors is typically not taken into account, although benchmarking may even be part of human nature and is important for a fund manager who needs to keep the fund competitive. The goal of this paper is to study the impacts of integrating relative-performance considerations into the framework of utility maximization.

The model we consider consists of n agents who can trade in the same market subject to some individual restrictions. Each agent measures her preferences by an exponential utility function and chooses a trading strategy that maximizes the expected utility of a weighted sum consisting of three components: an individual claim, the absolute performance and the relative performance compared to the other $n-1$ agents. The question is whether there exists a Nash equilibrium in the sense that there are optimal strategies simultaneously for all agents. We make the usual assumption that the financial market is big enough so that the trading of our investors does not affect the price of the assets.

A model similar to ours has been recently studied in the PhD thesis of Espinosa [6] but in the absence of individual claims and with assets modeled as Itô processes with deterministic coefficients. These assumptions crucially simplify the analysis and enable Espinosa [6] to show the existence of a Nash equilibrium. He also studies its form, while our focus is on existence questions in a more general setting and interpretations as well as possible alternatives in the absence of a Nash equilibrium. We obtain existence and uniquess in a stochastic framework if all agents are faced to the same trading restrictions. Under different investment constraints, however, an agent may ruin another one by solely maximizing her individual utility. Different investment possibilities may allow an agent to follow a risky and beneficial strategy, and thereby negatively affect another agent who benchmarks her own strategy against the less restricted one. The bankruptcy of the agents can be avoided if agents with more investment possibilities are showing solidarity and willingness to waive some expected utility. This leads to the existence of an approximated equilibrium, in the sense that there exists an ε-equilibrium for every $\epsilon > 0$. In an ε-equilibrium, every agent uses a strategy whose outcome is at most ε away from that of the individual best response. Behind this well-known concept stands the idea that agents may not care about very small improvements. Our setting brings up the additional aspect of solidarity: by accepting a small deduction from the optimum, an agent can help to save the others from failure. Applying freely to our model Adam Smith's citation stated on the first page of this paper, we could say that maximizing individual utilities sometimes leads to an equilibrium. But when one agent can dominate another because of less trading restrictions, the invisible hand of the market has to be accompanied with solidarity to guarantee an acceptable outcome for every agent.

This financial interpretation goes along with an interesting mathematical basis, which is due to the correspondence between an equilibrium of the investment problem and a solution of a certain backward stochastic differential equation (BSDE). BSDEs provide a genuine stochastic approach to control problems which typically find their analytic analogues in the convex duality theory and the Hamilton-Jacobi-Bellman formalism. A BSDE is of the form

$$\mathrm{d}Y_t = f(t, Y_t, Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \quad 0 \le t \le T, \quad Y_T = \xi,$$

where given are a d-dimensional Brownian motion W, an n-dimensional random variable ξ and a generator function f. A solution (Y, Z) consists of an n-dimensional semimartingale Y and an $(n \times d)$-dimensional control process Z predictable with respect to the filtration generated by W.

Existence and uniqueness results have first been shown for BSDEs with generators f satisfying a Lipschitz condition; see for example Pardoux and Peng [14]. However, BSDEs related to mathematical finance, as in our situation, typically involve generators f which are quadratic in the control variable. For such cases, Kobylanski [12] proved existence, uniqueness and comparison results when ξ is bounded and Y is one-dimensional ($n = 1$). Her results were generalized by Briand and Hu [1] and Delbaen et al. [3] to BSDEs with unbounded terminal conditions. While Kobylanski's proof cannot be generalized to $n > 1$, Tevzadze [16] presents an alternative derivation of Kobylanski's results via a fix point argument. This yields as a byproduct an existence and uniqueness result also for $n > 1$ if the generator f is specific (purely quadratic) and ξ is sufficiently small (the L^∞-norm of ξ needs to be tiny). The result is in line with the mantra that partial differential equations (PDEs) can often be solved for sufficiently small data or on a sufficiently small time interval, although the known existence and uniqueness results cover only some types of PDEs. For a multidimensional quadratic BSDE (i.e., $n > 1$ and f is quadratic in the control variable) like that related to our problem, no general existence and uniqueness results are known, even when ξ is bounded. On the other hand, no explicit counterexample is available so far to the best of our knowledge.

The paper is structured as follows. We start by presenting in Section 2 a counterexample which is easy to understand and shows that — and why — general multidimensional quadratic BSDEs do not have solutions. This gives a mathematical flavor for the absence of an equilibrium in the later-presented financial model, because we establish a relation between existence of equilibria and solutions to such a BSDE. Section 3 contains the arguments and results explained above on the (non-)existence of an equilibrium. In Section 4, we come back to the BSDE counterexample and discuss its mathematical scope. Finally, the Appendix contains some proofs and auxiliary results.

2 An illustrative counterexample

After some preparation, we give a counterexample to the existence of solutions of multidimensional quadratic BSDEs. Throughout the paper, we fix $T > 0$ and $d, n \in \mathbb{N}$ and work on a canonical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a d-dimensional Brownian motion $W = (W^1, \dots, W^d)$ restricted to the time interval $[0, T]$. We denote by $\mathcal{F} = (\mathcal{F}_t)_{0 \le t \le T}$ its augmented natural filtration and assume $\mathcal{F} = \mathcal{F}_T$. For an equivalent probability measure $\mathbb{Q}$, we define:

- the space $\mathcal{S}^\infty$ of bounded predictable processes;
- the space $\mathcal{H}^2_{n,d}(\mathbb{Q})$ of $(n\times d)$-dimensional predictable processes $(Z_t)_{0\le t\le T}$ normed by $\|Z\|_{\mathcal{H}^2_{n,d}(\mathbb{Q})} := \mathbb{E}_\mathbb{Q}\big[\int_0^T \operatorname{trace}(Z_t Z_t^\top)\,\mathrm{d}t\big]^{1/2}$;
- the space $BMO(\mathbb{Q})$ of square-integrable martingales M with $M_0 = 0$ and satisfying

$$\|M\|^2_{BMO(\mathbb{Q})} := \sup_\tau \big\| \mathbb{E}_\mathbb{Q}[\langle M\rangle_T - \langle M\rangle_\tau | \mathcal{F}_\tau]\big\|_{L^\infty} < \infty,$$

  where the supremum is taken over all stopping times τ valued in $[0, T]$.

In the case $\mathbb{Q} = \mathbb{P}$, we usually omit the symbol $\mathbb{P}$. A *solution* of a BSDE

$$\mathrm{d}Y_t = f(t, Y_t, Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \quad 0 \le t \le T, \quad Y_T = \xi, \tag{2.1}$$

with given n-dimensional random variable ξ and generator function f is a pair (Y, Z) satisfying (2.1) with a semimartingale Y and $Z \in \mathcal{H}^2_{n,d}$.

The counterexample, for which we take $d = 1$ (dimension of W), consists of the two-dimensional ($n = 2$) BSDE

$$\mathrm{d}Y^1_t = Z^1_t\,\mathrm{d}W_t, \qquad 0 \le t \le T, \quad Y^1_T = \xi, \tag{2.2}$$

$$\mathrm{d}Y^2_t = -\Big(|Z^1_t|^2 + \frac{1}{2}|Z^2_t|^2\Big)\,\mathrm{d}t + Z^2_t\,\mathrm{d}W_t, \quad 0 \le t \le T, \quad Y^2_T = 0, \tag{2.3}$$

where the terminal condition $\xi \in L^\infty$ is given. There is an explicit solution for the first component, which does not depend on the second. The generator of the second component depends quadratically on the control variables of both the first and the second dimension of the BSDE. For some choices of the terminal condition, the second component explodes, leading to insolvability.

Theorem 2.1. *For some $\xi \in L^\infty$, the BSDE (2.2), (2.3) has no solution.*

Proof. From (2.2), it follows that Y^1 is explicitly given by $Y_t^1 = \mathbb{E}[\xi|\mathcal{F}_t]$ and Z^1 is uniquely defined via Itô's representation theorem through

$$\xi = \mathbb{E}[\xi] + \int_0^T Z_t^1 \,\mathrm{d}W_t, \qquad \mathbb{E}\left[\int_0^T |Z_t^1|^2 \,\mathrm{d}t\right] < \infty.$$

We now use Z^1 in (2.3), which implies

$$\mathbb{E}\left[\exp\left(\int_0^T |Z_t^1|^2 \,\mathrm{d}t\right)\right] = \exp\big(Y_0^2\big)\mathbb{E}\left[\mathcal{E}\left(\int Z^2 \,\mathrm{d}W\right)_T\right] \le \exp\big(Y_0^2\big)$$

since the stochastic exponential $\mathcal{E}(\int Z^2 \,\mathrm{d}W)$ is a positive supermartingale. This gives $Y_0^2 = \infty$ if $\mathbb{E}\big[\exp\big(\int_0^T |Z_t^1|^2 \,\mathrm{d}t\big)\big] = \infty$, and the result follows by setting $\xi = \int_0^T \zeta_t \,\mathrm{d}W_t \in L^\infty$ for ζ given in Lemma A.1 in the Appendix. □

The underlying mathematical reason presented in Lemma A.1 is that there exists a bounded martingale whose quadratic variation has an infinite exponential moment. Since the generator in (2.3) depends quadratically on both Z^1 and Z^2, this leads to explosion. In Section 4, we will discuss some mathematical aspects of this phenomenon in more detail.

3 Maximizing the relative performance

After we have seen that multidimensional quadratic BSDEs need not have solutions, we study a financial problem, its link to existence issues for such BSDEs and how altering the problem can lead to solvability. We start by introducing the problem formulation and then group the results based on different types of trading restrictions for the agents.

3.1 Model setup and preliminaries

The financial market we consider consists of a risk-free bank account yielding zero interest and m traded risky assets $S = (S^j)_{j=1,\dots,m}$ with dynamics

$$\mathrm{d}S_t^j = S_t^j \mu_t^j \,\mathrm{d}t + \sum_{k=1}^d S_t^j \sigma_t^{jk} \,\mathrm{d}W_t^k, \quad 0 \le t \le T,\ S_0^j > 0, \quad j = 1, \dots, m;$$

the drift vector $\mu = (\mu^j)_{j=1,\dots,m}$ as well as the lines of the volatility matrix $\sigma = (\sigma^{jk})_{\substack{j=1,\dots,m,\\ k=1,\dots,d}}$ are predictable and uniformly bounded. We assume that σ has full rank and that there exists a constant C such that

$$C|\beta|^2 \geq \beta^\top \sigma\sigma^\top \beta \geq \frac{1}{C}|\beta|^2 \quad \text{a.e. on } \Omega \times [0,T] \text{ for all } \beta \in \mathbb{R}^m.$$

The market price of risk $\theta := \sigma^\top\big(\sigma\sigma^\top\big)^{-1}\mu$ is then also uniformly bounded and $\hat{W} := W + \int \theta \,\mathrm{d}t$ is a Brownian motion under the probability measure $\hat{\mathbb{P}}$ given by $\frac{\mathrm{d}\hat{\mathbb{P}}}{\mathrm{d}\mathbb{P}} := \mathcal{E}\big(-\int \theta \,\mathrm{d}W\big)_T$.

We consider n agents. Any agent i can trade in S subject to some personal restrictions and has to pay (or is endowed with) a claim $F_i \in L^\infty$ at time T. This means that agent i uses some self-financing trading strategy $\pi^i = (\pi^{i1}, \dots, \pi^{im})$ valued in A_i, where A_i is a closed and convex subset of $\mathbb{R}^m$. We denote by P^i_t the projection onto $A_i\sigma_t$, i.e., $P^i_t(x) := \operatorname*{argmin}_{z \in A_i\sigma_t} |x - z|$ for $x \in \mathbb{R}^d$. If agent i starts with zero initial capital, her wealth at time t related to a strategy π^i is given by

$$X^{\pi^i}_t := \int_0^t \sum_{j=1}^m \frac{\pi^{ij}_s}{S^j_s} \,\mathrm{d}S^j_s = \int_0^t \pi^i_s \sigma_s \,\mathrm{d}\hat{W}_s.$$

Any agent i measures her preferences by an exponential utility function $U_i(x) = -\exp(-\eta_i x)$, $x \in \mathbb{R}$, for a fixed $\eta_i > 0$. Instead of maximizing the classical expected utility $\mathbb{E}[U_i(X^{\pi^i}_T - F_i)]$, agent i takes also the relative performance into consideration and maximizes over π^i the value

$$\begin{aligned} V^\pi_i &:= \mathbb{E}\left[U_i\left((1-\lambda_i)X^{\pi^i}_T + \lambda_i\Big(X^{\pi^i}_T - \frac{1}{n-1}\sum_{j\neq i} X^{\pi^j}_T\Big) - F_i\right)\right] \\ &= \mathbb{E}\left[U_i\left(X^{\pi^i}_T - \frac{\lambda_i}{n-1}\sum_{j\neq i} X^{\pi^j}_T - F_i\right)\right] \end{aligned} \tag{3.1}$$

for a fixed $\lambda_i \in [0,1]$ and given the other agents $j \neq i$ use strategies π^j. The set $\mathcal{A}_i$ of admissible strategies for agent i is given by

$$\mathcal{A}_i := \Big\{\pi^i \ \mathbb{R}^m\text{-valued, predict.} \,\Big|\, \pi^i \in A_i \text{ a.e. on } \Omega \times [0,T],\ X^{\pi^i} \in BMO\big(\hat{\mathbb{P}}\big)\Big\}.$$

We set $\mathcal{A} := \mathcal{A}_1 \times \dots \times \mathcal{A}_n$. Because we assume that each agent maximizes her expected utility without cooperating with the other agents, we are interested in Nash equilibria.

Definition 3.1. *In this setting, a strategy $\hat{\pi} \in \mathcal{A}$ is a* Nash equilibrium *if for every i, $V^{\hat{\pi}}_i \geq V^{\pi^i, \hat{\pi}^{j\neq i}}_i$ for all $\pi^i \in \mathcal{A}_i$.*

The classical problem of maximizing $\mathbb{E}[U_i(X^{\pi^i} - F_i)]$ has been studied by Hu et al. [10] in the same setting, but with not necessarily convex A_i. They give in Theorem 7 a BSDE characterization for the optimal strategy and the maximal expected utility. Although their definition of admissibility slightly differs from ours (class (D)- instead of BMO-condition), their Theorem 7 still holds under our definition in the case $\lambda_i = 0$ for all i, which can be seen from its proof and which we later use several times. Our choice of admissibility allows for both regaining the assertion of Hu et al. [10] in the case $\lambda_i = 0$ for all i and deriving in Lemma 3.2 a BSDE characterization for general λ_i. By Theorem 3.6 of Kazamaki [11], the condition $X^{\pi^i} \in BMO\big(\hat{\mathbb{P}}\big)$ is equivalent to $\int \pi^i \sigma \, \mathrm{d}W \in BMO(\mathbb{P})$ because θ is bounded.

In contrast to optimizing $\mathbb{E}[U_i(X_T^{\pi^i} - F_i)]$, we maximize $\mathbb{E}[U_i(X_T^{\pi^i} - \tilde{F}_i)]$ with $\tilde{F}_i := \frac{\lambda_i}{n-1} \sum_{j \neq i} X_T^{\pi^j} + F_i$. Since $\tilde{F}_i$ is unbounded and depends on the other agents' strategies, the study is more involved. This problem of agents concerning the relative performance has also been considered in the PhD thesis of Espinosa [6]. In a simpler setting where σ and θ are deterministic and without claims F_i, he proved the existence of a Nash equilibrium and gave a characterization in his Theorem 4.40. In our stochastic model, a counterexample in Section 3.3 will show that there need not exist a Nash equilibrium and only a notion weaker than a Nash equilibrium might be satisfied.

The following result, which relates a Nash equilibrium to a BSDE, is an analogue to Theorem 7 of Hu et al. [10]. However, one has here no uniqueness and existence result for the BSDE. In fact, the counterexample in Section 3.3 shows that existence does not hold in general. To formulate the result, we recall the reverse Hölder inequality $R_p(\mathbb{Q})$. For $p > 1$, an equivalent probability measure $\mathbb{Q}$ and an adapted positive process M, we say

$$M \text{ satisfies } R_p(\mathbb{Q}) \iff \exists\, C \text{ s.t. } \operatorname*{ess\,sup}_{\tau \text{ stop. time}} \mathbb{E}_{\mathbb{Q}}[(M_T/M_\tau)^p | \mathcal{F}_\tau] \leq C. \tag{3.2}$$

Lemma 3.2. *There is a one-to-one correspondence between the following:*

(i) a Nash equilibrium $\hat{\pi} \in \mathcal{A}$ such that for any i, there exists $p > 1$ with

$$\mathbb{E}\Big[U_i\Big(X_T^{\hat{\pi}^i} - \frac{\lambda_i}{n-1} \sum_{j \neq i} X_T^{\hat{\pi}^j}\Big)\Big|\mathcal{F}_\cdot\Big] \text{ satisfies } R_p(\mathbb{P}); \tag{3.3}$$

(ii) a solution (Y, Z) with $\int Z \, \mathrm{d}W \in BMO$ of the multidimensional BSDE

$$\begin{aligned} \mathrm{d}Y_t^i &= \left(\frac{|\theta_t|^2}{2\eta_i} - \frac{\eta_i}{2}\Big|Z_t^i + \frac{1}{\eta_i}\theta_t - P_t^i\Big(Z_t^i + \frac{1}{\eta_i}\theta_t\Big)\Big|^2\right) \mathrm{d}t + Z_t^i \, \mathrm{d}\hat{W}_t, \\ Y_T^i &= \frac{\lambda_i}{n-1} \sum_{j \neq i} \int_0^T P_t^j\Big(Z_t^j + \frac{1}{\eta_j}\theta_t\Big) \mathrm{d}\hat{W}_t + F_i, \quad i = 1, \dots, n \end{aligned} \tag{3.4}$$

The relation is given by $\hat{\pi}^i \sigma = P^i\big(Z^i + \frac{1}{\eta_i}\theta\big)$ and $V_i^{\hat{\pi}} = -\exp(\eta_i Y_0^i)$.

Proof. Assume (i) holds and fix i. One can show by dynamic programming similarly to Lemma 4.24 of Espinosa [6] that for any $\pi^i \in \mathcal{A}_i$, M^{π^i} given by

$$M_t^{\pi^i} := \mathrm{e}^{-\eta_i X_t^{\pi^i}} \operatorname*{ess\,sup}_{\kappa \in \mathcal{A}_i} \mathbb{E}\left[U_i\left(X_T^\kappa - X_t^\kappa - \frac{\lambda_i}{n-1} \sum_{j \neq i} X_T^{\hat{\pi}^j} - F_i \right) \middle| \mathcal{F}_t \right] \tag{3.5}$$

has a continuous version which is a supermartingale and a martingale for $\pi^i = \hat{\pi}^i$. This uses that for any $\pi^i, \tilde{\pi}^i \in \mathcal{A}$ and stopping time τ, we have $\pi^i \mathbb{1}_{]\!]0,\tau]\!]} + \tilde{\pi}^i \mathbb{1}_{]\!]\tau,T]\!]} \in \mathcal{A}_i$. A variant of Itô's representation theorem implies

$$M^{\hat{\pi}^i} = M_0^{\hat{\pi}^i} \mathcal{E}\left(\int \tilde{Z}^i \,\mathrm{d}W \right) \quad \text{for } \tilde{Z}^i \text{ with } \int_0^T \left| \tilde{Z}_t^i \right|^2 \mathrm{d}t < \infty \text{ a.s. and } M_0^{\hat{\pi}^i} < 0.$$

Theorem 3.3 of Kazamaki [11] yields $\int \tilde{Z}^i \,\mathrm{d}W \in BMO$ because of (3.3) and the boundedness of F_i. We set $Z^i := \frac{1}{\eta} \tilde{Z}^i + \hat{\pi}^i \sigma$, which again satisfies $\int Z^i \,\mathrm{d}W \in BMO$ because $\hat{\pi}^i \in \mathcal{A}_i$. For any $\pi^i \in \mathcal{A}_i$, we obtain

$$\begin{aligned} M^{\pi^i} &= \exp\bigl(\eta_i X^{\hat{\pi}^i} - \eta_i X^{\pi^i}\bigr) M^{\hat{\pi}^i} = M_0^{\hat{\pi}^i} N^{\pi^i} B^{\pi^i}, \text{ where} \\ N^{\pi^i} &:= \mathcal{E}\left(\eta_i \int (Z^i - \pi^i \sigma) \,\mathrm{d}W \right), \\ B^{\pi^i} &:= \exp\left(\frac{\eta_i^2}{2} \int \left(\left| Z^i + \frac{1}{\eta_i}\theta - \pi^i \sigma \right|^2 - \left| Z^i + \frac{1}{\eta_i}\theta - \hat{\pi}^i \sigma \right|^2 \right) \mathrm{d}t \right). \end{aligned}$$

The $\mathbb{P}$-supermartingale property of M^{π^i} implies that $M^{\pi^i}/N^{\pi^i} = M_0^{\hat{\pi}^i} B^{\pi^i}$ is a $\mathbb{Q}^{\pi^i}$-supermartingale where $\frac{\mathrm{d}\mathbb{Q}^{\pi^i}}{\mathrm{d}\mathbb{P}} := N_T^{\pi^i}$, using that N^{π^i} is a $\mathbb{P}$-martingale by Theorem 2.3 of Kazamaki [11]. Because B^{π^i} is a continuous $\mathbb{Q}^{\pi^i}$-submartingale and of finite variation, it is nondecreasing, i.e., for any $\pi^i \in \mathcal{A}_i$ $|Z^i - \frac{1}{\eta_i}\theta - \pi^i \sigma| \geq |Z^i + \frac{1}{\eta_i}\theta - \hat{\pi}^i \sigma|$ a.e. Hence, we get $\hat{\pi}^i \sigma = P^i\bigl(Z^i + \frac{1}{\eta_i}\theta\bigr)$, using that a strategy $\tilde{\pi}^i$ satisfying $\tilde{\pi}^i \sigma = P^i\bigl(Z^i + \frac{1}{\eta_i}\theta\bigr)$ can be chosen predictable by Lemma 11 of Hu et al. [10]. We set $Y^i := \frac{1}{\eta_i} \log\bigl(-M^{\hat{\pi}^i} \exp(\eta_i X^{\hat{\pi}^i})\bigr)$ and obtain for $\mathrm{d}Y_t^i$ the expression in (3.4) after a straightforward calculation. Moreover, (3.5) implies $Y_T^i = \frac{\lambda_i}{n-1} \sum_{j \neq i} \int_0^T \hat{\pi}_t^j \sigma_t \,\mathrm{d}\hat{W}_t + F_i$. Since this holds for any i, we have $\hat{\pi}^i \sigma = P^i\bigl(Z^i + \frac{1}{\eta_i}\theta\bigr)$ for all i and (3.4) follows.

Suppose (ii) holds, define $\hat{\pi}$ by $\hat{\pi}^j \sigma = P^j\bigl(Z^j + \frac{1}{\eta_j}\theta\bigr)$ for all j and fix i. Like in Lemma 12 of Hu et al. [10], we obtain $\int P^i\bigl(Z^i + \frac{1}{\eta_i}\theta\bigr) \,\mathrm{d}W \in BMO$ so that $\hat{\pi}^i \in \mathcal{A}_i$. For $\pi^i \in \mathcal{A}_i$, we set $R^{i,\pi^i} := -\exp\bigl(-\eta_i(X^{\pi^i} - Y^i)\bigr)$, which satisfies

$$\begin{aligned} R^{i,\pi^i} &= -\exp(\eta_i Y_0^i)\, \mathcal{E}\left(\eta_i \int (Z^i - \pi^i \sigma) \,\mathrm{d}W \right) \\ &\qquad \times \exp\left(\frac{\eta_i^2}{2} \int \left| Z^i + \frac{1}{\eta_i}\theta - \pi^i \sigma \right|^2 - \left| Z^i + \frac{1}{\eta_i}\theta - P^i\Bigl(Z^i + \frac{1}{\eta_i}\theta \Bigr) \right|^2 \mathrm{d}t \right). \end{aligned}$$

We deduce that $R^{i,\hat{\pi}^i}$ is a martingale and $V_i^{\hat{\pi}} = -\exp(\eta_i Y_0^i)$. For any $\pi^i \in \mathcal{A}_i$, R^{i,π^i} is a supermartingale and we have $V_i^{\hat{\pi}} = R_0^{i,\pi^i} \geq \mathbb{E}\big[R_T^{i,\pi^i}\big] = V_i^{\pi^i,\hat{\pi}^{j\neq i}}$. $\square$

In the specific case where all $F_i = 0$ and μ as well as σ are deterministic, one can construct a solution to the BSDE (3.4) by choosing a deterministic Z with $Z^i = \frac{\lambda_i}{n-1}\sum_{j\neq i} P^j\Big(Z^j + \frac{1}{\eta_j}\theta\Big)$ for all i if $\Pi_{j=1}^n \lambda_j < 1$. This is possible because then the mapping φ defined by

$$z \mapsto \varphi_t^i(z) := z^i - \frac{\lambda_i}{n-1}\sum_{j\neq i} P_t^j(z^j) \tag{3.6}$$

is invertible by Lemma 4.41 of Espinosa [6], who also shows that φ^{-1} is Lipschitz-continuous uniformly in t. Since $\int Z\,\mathrm{d}W$ is in BMO for this deterministic Z, the strategy $\hat{\pi}$ satisfying $\hat{\pi}^i\sigma = P^i\big(Z^i + \frac{1}{\eta_i}\theta\big)$ is a Nash equilibrium by Lemma 3.2. Hence, we regain the form of a Nash equilibrium stated in Theorem 4.40 of Espinosa [6] for this deterministic case. In the following, we give a brief alternative derivation which does not use BSDEs.

Remark. In this remark, we fix i and assume $F_i = 0$ and that μ and σ are deterministic. Supposing $\pi^j \in \mathcal{A}_j$ for $j \neq i$ are deterministic, we obtain from (3.1) for any (possibly stochastic) $\pi^i \in \mathcal{A}_i$ that

$$\begin{aligned}
-V_i^{\pi} &= \mathbb{E}_{\hat{\mathbb{P}}}\bigg[\exp\bigg(\int_0^T \Big(\frac{\eta_i\lambda_i}{n-1}\sum_{j\neq i}\pi_t^j\sigma_t - \eta_i\pi_t^i\sigma_t + \theta_t\Big)\,\mathrm{d}\hat{W}_t\bigg)\bigg]\mathrm{e}^{-\frac{1}{2}\int_0^T|\theta_t|^2\,\mathrm{d}t}\\
&= \mathbb{E}_{\hat{\mathbb{P}}}\bigg[\mathcal{E}\bigg(\int\Big(\frac{\eta_i\lambda_i}{n-1}\sum_{j\neq i}\pi^j\sigma - \eta_i\pi^i\sigma + \theta\Big)\,\mathrm{d}\hat{W}\bigg)_T\\
&\qquad\times\exp\bigg(\frac{1}{2}\int_0^T\Big|\frac{\eta_i\lambda_i}{n-1}\sum_{j\neq i}\pi_t^j\sigma_t - \eta_i\pi_t^i\sigma_t + \theta_t\Big|^2\,\mathrm{d}t\bigg)\bigg]\mathrm{e}^{-\frac{1}{2}\int_0^T|\theta_t|^2\,\mathrm{d}t}\\
&\geq \mathbb{E}_{\hat{\mathbb{P}}}\bigg[\mathcal{E}\bigg(\int\Big(\frac{\eta_i\lambda_i}{n-1}\sum_{j\neq i}\pi^j\sigma - \eta_i\pi^i\sigma + \theta\Big)\,\mathrm{d}\hat{W}\bigg)_T\bigg]\\
&\qquad\times\exp\bigg(\frac{\eta_i^2}{2}\int_0^T\Big|\frac{\lambda_i}{n-1}\sum_{j\neq i}\pi_t^j\sigma_t + \frac{1}{\eta_i}\theta_t - \hat{\pi}_t^i\sigma_t\Big|^2\,\mathrm{d}t - \frac{1}{2}\int_0^T|\theta_t|^2\,\mathrm{d}t\bigg),
\end{aligned}$$

where $\hat{\pi}^i\sigma = P^i\big(\frac{\lambda_i}{n-1}\sum_{j\neq i}\pi^j\sigma + \frac{1}{\eta_i}\theta\big)$. Thus we have

$$\sup_{\pi^i\in\mathcal{A}_i} V_i^{\pi} = \exp\bigg(\frac{\eta_i^2}{2}\int_0^T\Big|\frac{\lambda_i}{n-1}\sum_{j\neq i}\pi_t^j\sigma_t + \frac{1}{\eta_i}\theta_t - \hat{\pi}_t^i\sigma_t\Big|^2\,\mathrm{d}t - \frac{1}{2}\int_0^T|\theta_t|^2\,\mathrm{d}t\bigg).$$

This shows the existence of a Nash equilibrium $\hat{\pi} \in \mathcal{A}$ given by

$$\begin{aligned}
\hat{\pi}^i \sigma &:= P^i\Big(\varphi^{-1,i}\Big(\frac{1}{\eta_1}\theta, \dots, \frac{1}{\eta_n}\theta\Big)\Big) \\
&= P^i\bigg(\frac{\lambda_i}{n-1}\sum_{j\neq i} P^j\Big(\varphi^{-1,j}\Big(\frac{1}{\eta_1}\theta, \dots, \frac{1}{\eta_n}\theta\Big)\Big) + \frac{1}{\eta_i}\theta\bigg) \\
&= P^i\bigg(\frac{\lambda_i}{n-1}\sum_{j\neq i} \hat{\pi}^j \sigma + \frac{1}{\eta_i}\theta\bigg), \qquad i = 1, \dots, n,
\end{aligned}$$

where $\varphi^{-1,i}$ denotes the i-th component of the inverse of φ given in (3.6). $\diamondsuit$

While we do not need the BSDE formulation in the presence of deterministic parameters, it is helpful in the general case. The multidimensional BSDE (3.4) is coupled via its terminal condition. By using the mapping φ defined in (3.6), we can rewrite (3.4) as

$$\begin{aligned}
\mathrm{d}\Gamma_t^i &= -\frac{\eta_i}{2}\big|\varphi_t^{-1,i}(\zeta_t) - P_t^i\big(\varphi_t^{-1,i}(\zeta_t)\big)\big|^2 \,\mathrm{d}t + \zeta_t^i \,\mathrm{d}\hat{W}_t, \quad 0 \le t \le T, \\
\Gamma_T^i &= F_i + \frac{1}{\eta_i}\int_0^T \theta_s \,\mathrm{d}\hat{W}_s - \frac{1}{2\eta_i}\int_0^T |\theta_s|^2 \,\mathrm{d}s, \qquad i = 1, \dots, n, \qquad (3.7)
\end{aligned}$$

where $\zeta^i := \varphi^i\big(Z^1 + \frac{1}{\eta_1}\theta, \dots, Z^n + \frac{1}{\eta_n}\theta\big)$ and

$$\Gamma_t^i := Y_t^i - \frac{\lambda_i}{n-1}\sum_{j\neq i}\int_0^t P_s^j\Big(Z_s^j + \frac{1}{\eta_j}\theta_s\Big)\,\mathrm{d}\hat{W}_s + \frac{1}{\eta_i}\int_0^t \theta_s \,\mathrm{d}\hat{W}_s - \frac{1}{2\eta_i}\int_0^t |\theta_s|^2 \,\mathrm{d}s. \qquad (3.8)$$

Because φ^{-1} is Lipschitz-continuous, (3.7) shows that we are dealing with a multidimensional quadratic BSDE.

In the following remark, we briefly mention two articles related to other financial applications of multidimensional quadratic BSDEs.

Remark. El Karoui and Hamadène [4] consider certain games with two players. In a Markovian framework, they give a characterization for an equilibrium in terms of a solution of a multidimensional quadratic BSDE. For their setting, the coupling of that BSDE is weak, namely it is assumed that the i-th entry of the driver f is dominated by $C(1 + |z^i|^2)$ for some positive constant C. However, no existence result for such a BSDE is provided.

Cheridito et al. [2] follow in the footsteps of Horst et al. [9] to solve a problem of valuing a derivative in an incomplete market by equilibrium considerations. In Horst et al. [9], the problem can be solved in a one-dimensional framework, since the derivative is assumed to complete the market. Cheridito et al. [2] do not impose this condition, which makes the analysis much more involved. The authors solve the problem in a discrete framework, but close their work with considerations on the continuous case. The latter leads to a fully coupled multidimensional quadratic BSDE, whose solvability is unkown. $\lozenge$

3.2 Agents having the same constraints

In a situation where all agents are faced to the same constraints given by a linear subspace of $\mathbb{R}^d$, there exists a unique Nash equilibrium $\hat{\pi}$ and we can give a BSDE characterization for $\hat{\pi}$ similarly to Hu et al. [10].

Proposition 3.3. *Assume that $A_i = A$ are the same linear subspace for all $i = 1, \ldots, n$ and $\Pi_{i=1}^n \lambda_i < 1$, and set $P = P^i$. Then for $i = 1, \ldots, n$, the decoupled BSDEs*

$$\mathrm{d}\Gamma_t^i = \left(\frac{|\theta_t|^2}{2\eta_i} - \frac{\eta_i}{2} \left| \zeta_t^i + \frac{1}{\eta_i}\theta_t - P_t\Big(\zeta_t^i + \frac{1}{\eta_i}\theta_t\Big) \right|^2 \right) \mathrm{d}t + \zeta_t^i \, \mathrm{d}\hat{W}_t, \quad \Gamma_T^i = F_i \tag{3.9}$$

have a unique solution $(\Gamma, \zeta) \in \mathcal{S}^\infty \times \mathcal{H}^2_{1,d}$. There is a unique Nash equilibrium $\hat{\pi} \in \mathcal{A}$. It is given by $\psi^i(\hat{\pi}_t)\sigma_t = P_t\big(\zeta_t^i + \frac{1}{\eta_i}\theta_t\big)$, where the linear mapping ψ is defined by $z \mapsto \psi^i(z) := z^i - \frac{\lambda_i}{n-1}\sum_{j\neq i} z^j$. Moreover, $V_i^{\hat{\pi}} = -\exp(\eta_i \Gamma_0^i)$.

Proposition 3.3 shows that the agents' maximal expected utility is the same as in the case without interaction. However, the optimal strategies are different. Since all agents have the same constraints, an agent can completely hedge against the others agents' behavior. This implies that the optimal strategy accounts for the others agents' behavior, while the maximal expected utility is unaltered compared to the situation without interdependencies.

Proof of Proposition 3.3. Because $A_i = A$ is a linear space and ψ is invertible due to $\Pi_{i=1}^n \lambda_i < 1$, we have $\pi \in \mathcal{A} \iff \psi(\pi) \in \mathcal{A}$. This implies $\sup_{\pi^i \in \mathcal{A}_i} V_i^{\pi^i, \hat{\pi}^{j\neq i}} = \sup_{p \in \mathcal{A}_i} \mathbb{E}\big[U_i\big(X_T^p - F_i\big)\big]$ for all $\hat{\pi}^j \in \mathcal{A}_j$. Applying Theorem 7 of Hu et al. [10] to the latter optimization problem yields the result. $\square$

Remarks. 1) The proof shows as well that there exists no $\pi \in \mathcal{A}$ with

$$V_i^\pi > V_i^{\hat{\pi}} \text{ for some } i \text{ and } V_j^\pi \geq V_j^{\hat{\pi}} \text{ for all } j,$$

which means that $\hat{\pi}$ from Proposition 3.3 is also a Pareto optimum.

2) The BSDEs (3.9) correspond to (3.4). Indeed, define (Y, Z) by

$$\Big(\zeta_t^1 + \frac{1}{\eta_1}\theta_t, \ldots, \zeta_t^n + \frac{1}{\eta_n}\theta_t\Big) = \varphi_t\Big(Z_t^1 + \frac{1}{\eta_1}\theta_t, \ldots, Z_t^n + \frac{1}{\eta_n}\theta_t\Big),$$

$$Y = \Gamma + \frac{\lambda_i}{n-1}\sum_{j\neq i}\int P\Big(Z^j + \frac{1}{\eta_j}\theta\Big)\,\mathrm{d}\hat{W},$$

with the invertible linear mapping φ_t given by (3.6). Because of $A_i = A$ for all i, $\frac{1}{n-1}\sum_{j\neq i} P_t\big(Z_t^j + \frac{1}{\eta_j}\theta_t\big)$ is in $\sigma_t A$, and we obtain

$$\begin{aligned}\Big|\zeta_t^i + \frac{1}{\eta_i}\theta_t - P_t\Big(\zeta_t^i + \frac{1}{\eta_i}\theta_t\Big)\Big| &= \Big|\zeta_t^i + \frac{1}{\eta_i}\theta_t + \frac{\lambda_i}{n-1}\sum_{j\neq i} P_t\Big(Z_t^j + \frac{1}{\eta_j}\theta_t\Big) \\ &\quad - P_t\Big(\zeta_t^i + \frac{1}{\eta_i}\theta_t + \frac{\lambda_i}{n-1}\sum_{j\neq i} P_t\Big(Z_t^j + \frac{1}{\eta_j}\theta_t\Big)\Big)\Big| \\ &= \Big|Z_t^i + \frac{1}{\eta_i}\theta_t - P_t\Big(Z_t^i + \frac{1}{\eta_i}\theta_t\Big)\Big|.\end{aligned}$$

Therefore, the BSDEs (3.9) are equivalent to (3.4). $\Diamond$

We now give an easy counterexample for the case $\lambda_i = 1$ where the BSDE (3.4) has no solution. We take $n = 2$ (number of agents), $d = 1$ (dimension of W), $\sigma = 1$, $\theta = 1$, $A_1 = A_2 = \mathbb{R}$ (no constraints), $\eta_1 = \eta_2 = 1$ and $\lambda_1 = \lambda_2 = 1$ (only the relative performance matters). The BSDE (3.4) equals

$$\begin{aligned}\mathrm{d}Y_t^1 &= \frac{1}{2}\,\mathrm{d}t + Z_t^1\,\mathrm{d}\hat{W}_t, \qquad Y_T^1 = \int_0^T (Z_s^2 + 1)\,\mathrm{d}\hat{W}_s, \\ \mathrm{d}Y_t^2 &= \frac{1}{2}\,\mathrm{d}t + Z_t^2\,\mathrm{d}\hat{W}_t, \qquad Y_T^2 = \int_0^T (Z_s^1 + 1)\,\mathrm{d}\hat{W}_s.\end{aligned}$$

By combining these equations, we obtain

$$\begin{aligned}\frac{T}{2} + \int_0^T Z_t^1\,\mathrm{d}\hat{W}_t &= \int_0^T (Z_t^2 + 1)\,\mathrm{d}\hat{W}_t - Y_0^1 \\ &= \hat{W}_T - Y_0^1 - \frac{T}{2} + \int_0^T (Z_t^1 + 1)\,\mathrm{d}\hat{W}_t - Y_0^2.\end{aligned}$$

This implies

$$2\hat{W}_T = T + Y_0^1 + Y_0^2,$$

which is a contradiction, because the right-hand side is stochastic while the left-hand side is deterministic. One can interpret this example as follows: Both agents care only about the relative wealth. Since the market price of risk θ is nonzero, there is some risk inherent in the model and each agent wants to hedge against this risk. For any given strategy $\pi^2 \in \mathcal{A}_2$ of agent 2, the optimal strategy of agent 1 is $\hat{\pi}^1 = \pi^2 + \theta = \pi^2 + 1 \in \mathcal{A}_1$. Analogously, $\hat{\pi}^2 = \pi^1 + 1 \in \mathcal{A}_2$ is the best response of agent 2 to any given strategy $\pi^1 \in \mathcal{A}_1$ of agent 1. By trying to hedge, the first agent transfers the risk to the second agent, who then transfers it back to the first. Because of $\lambda_i = 1$, no agent reduces the risk, but instead each agent iteratively passes the buck to the other. In the end, both agents break down so that there is no Nash equilibrium. This counterexample can also be interpreted in the context of copycat hedge funds, which try to imitate the strategy of a successful hedge fund. If a hedge fund copies the strategy of another fund which itself mimics the former fund, then no equilibrium can exist because the interdependence mutually amplifies the strategies.

3.3 Agents with ordered constraints

Throughout this section, we assume $\Pi_{i=1}^n \lambda_i < 1$ and that A_i are linear subspaces of $\mathbb{R}^d$ satisfying

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n.$$

We start with a counterexample for the case where two agents have different constraints. The first agent copes with a bounded claim F_1 by choosing a suitable hedging strategy. However, the second agent is affected by the first agent's hedging strategy, which makes the second agent break down.

Theorem 3.4. *There exists a counterexample with $n = 2$, linear spaces $A_1 \supsetneq A_2$ and $\lambda_1 \lambda_2 < 1$ where there is no Nash equilibrium.*

Proof. We take $d = 2$ (dimension of W), $\sigma = (2 \times 2)$-identity matrix, $\theta = 0$, $A_1 = \{(x, x)|x \in \mathbb{R}\}$, $A_2 = \{(0, 0)\}$, $\eta_1 = 2/(\pi^2 + 1)$ (this choice will later simplify computations; π denotes here the number 3.141... and not a strategy), $\eta_2 = 1$, $F_2 = 0$, F_1 to be chosen later and $\lambda_1 = \lambda_2 = 1/2$. We obtain for the corresponding BSDE (3.4) that

$$\mathrm{d}Y_t^1 = -\frac{1}{2(\pi^2+1)}|Z_t^{1,1} - Z_t^{1,2}|^2 \,\mathrm{d}t + Z_t^1 \,\mathrm{d}W_t, \quad Y_T^1 = F_1, \tag{3.10}$$

$$\mathrm{d}Y_t^2 = -\frac{1}{2}|Z_t^2|^2 \,\mathrm{d}t + Z_t^2 \,\mathrm{d}W_t, \quad Y_T^2 = \frac{1}{4}\int_0^T (Z_s^{1,1} + Z_s^{1,2}) \,\mathrm{d}(W_s^1 + W_s^2). \tag{3.11}$$

The first component (3.10) does not depend on the second, and has for any bounded F_1 a unique solution (Y^1, Z^1) with $\int Z^1 \,\mathrm{d}W \in BMO$. This solution is plugged in (3.11) to solve for the second component. Similarly to the counterexample presented in Section 2, we construct an F_1 such that Y^2 explodes. The difference to the first counterexample is that (3.10) has a quadratic generator and (3.11) depends on Z^1 via a $\mathrm{d}W$- and not a $\mathrm{d}t$-integral.

We set $F_1 := (\pi^2 + 1) \log \mathcal{E}\left(\int \zeta \,\mathrm{d}W^1\right)_T$ for ζ from Lemma A.2 with

$$\log \mathcal{E}\left(\int \zeta \,\mathrm{d}W^1\right) \in \mathcal{S}^\infty \quad \text{and} \quad \mathbb{E}\left[\exp\left(\frac{\pi^2+1}{4}\int_0^T \zeta_t \,\mathrm{d}W_t^1\right)\right] = \infty. \tag{3.12}$$

The BSDE (3.10) has the explicit solution

$$Y^1 = (\pi^2+1)\log\mathcal{E}\Big(\int \zeta\,\mathrm{d}W^1\Big), \quad Z^{1,1} = (\pi^2+1)\zeta, \quad Z^{1,2}=0.$$

From (3.11) and (3.12), it follows that

$$\mathrm{e}^{Y_0^2} = \mathbb{E}\Big[\exp\Big(\frac{1}{4}\int_0^T Z_t^{1,1}\,\mathrm{d}(W_t^1+W_t^2)\Big)\Big] \geq \mathbb{E}\Big[\exp\Big(\frac{\pi^2+1}{4}\int_0^T \zeta_t\,\mathrm{d}W_t^1\Big)\Big] = \infty \tag{3.13}$$

by conditioning on the σ-field generated by W^1 and using Jensen's inequality. Therefore, the coupled BSDE (3.10), (3.11) has no solution and there is no Nash equilibrium satisfying (3.3) by Lemma 3.2. To see that there exists no Nash equilibrium at all (even without (3.3)), we note that a candidate Nash equilibrium $\hat{\pi} \in \mathcal{A}$ must satisfy $\hat{\pi}^1 = \frac{Z^{1,1}+Z^{1,2}}{2}(1,1)$ (optimality for agent 1) and $\hat{\pi}^0 = 0$ (trading restrictions of agent 2). But this gives $V_2^{\hat{\pi}} = \mathbb{E}\big[U_2\big(-\lambda_2\int_0^T \frac{1}{2}(Z_t^{1,1}+Z_t^{1,2})\,\mathrm{d}(W_t^1+W_t^2)\big)\big] = -\infty$ by (3.13). □

The trading constraints in the counterexample might look restrictive, but it is possible to generalize the counterexample to higher-dimensionsional W, while giving the agents more trading possibilities. For $d > 2$, one can deduce an analogous counterexample with $A_1 = \{(x,x,y_1,\dots,y_{d-3}) | x, y_i \in \mathbb{R}\}$, $A_2 = \{(0,0,y_1,\dots,y_{d-3}) | y_i \in \mathbb{R}\}$; in that case, Y^2 satisfies

$$\begin{aligned} \mathrm{d}Y_t^2 &= -\frac{1}{2}\big(|Z_t^{2,1}|^2 + |Z_t^{2,2}|^2\big)\,\mathrm{d}t + Z_t^2\,\mathrm{d}W_t, \\ \mathrm{e}^{Y_0^2} &= \mathbb{E}\Big[\exp\Big(Y_T^2 - \sum_{i=3}^d \int_0^T Z_t^{2,i}\,\mathrm{d}W_t^i\Big)\Big] \geq \mathbb{E}\big[\exp\big(\mathbb{E}[Y_T^2|\mathcal{G}]\big)\big], \end{aligned}$$

where $\mathcal{G}$ denotes the σ-field generated by W^1 and W^2.

Theorem 3.4 shows that having A_i as ordered linear spaces is not enough to guarantee the existence of a Nash equilibrium. Even if the first agent does not concern the relative performance, her choice of a hedging strategy for F_1 may bankrupt the other agents. While the first agent can hedge against all other strategies, her strategy may negatively influence the other agents and ruin them. Assuming that the first agent wants to avoid the ruin of the other agents, she might be willing to reduce her wealth a little bit. Continuing this idea for the other agents, we come to the following relaxation of a Nash equilibrium.

Definition 3.5. *We say that there exists an* additively approximated equilibrium *if for every $\epsilon > 0$, there is $(\hat{\pi}^{\epsilon,1},\dots,\hat{\pi}^{\epsilon,n}) \in \mathcal{A}$ such that for any i,*

$$V_i^{\hat{\pi}^\epsilon} + \epsilon \geq V_i^{\pi^i,\hat{\pi}^{\epsilon,j\neq i}} \quad \textit{for all } \pi^i \in \mathcal{A}_i. \tag{3.14}$$

A multiplicatively approximated equilibrium *exists if for every* $\epsilon > 0$, *there is* $(\hat{\pi}^{\epsilon,1}, \ldots, \hat{\pi}^{\epsilon,n}) \in \mathcal{A}$ *such that for any* i,

$$(1-\epsilon)V_i^{\hat{\pi}^\epsilon} \geq V_i^{\pi^i, \hat{\pi}^{\epsilon, j\neq i}} \quad \text{for all } \pi^i \in \mathcal{A}_i. \tag{3.15}$$

Note that we use $(1-\epsilon)$ and not $(1+\epsilon)$ in (3.15), because V_i is negative. In the literature, there exists the notion of ϵ-equilibrium, which corresponds to the situation where (3.14) holds for a fixed $\epsilon > 0$, instead of all $\epsilon > 0$. Given the existence of a Nash equilibrium, calculating such a fixed ϵ-approximation instead of the true Nash equilibrium can be more efficient and easier to implement; see for example Hémon et al. [8]. In our setting, we can relate the notion of ϵ-equilibrium to the aspect of solidarity. If the more powerful agents are willing to deviate little from the expected utility associated to the best response, then the other agents do not break down and they can even find themselves "almost" optimal strategies in the sense of Definition 3.5.

Theorem 3.6. *There exists an additively as well as multiplicatively approximated equilibrium.*

Because of its length, we present the constructive proof of Theorem 3.6 in the Appendix, but give here a brief outline. Its idea is that for agent i, only the strategies of agents $1, \ldots, i-1$ really matter because she can hedge the other strategies. Therefore, one starts to consider the first agent's optimization problem when the strategies of all other agents are zero, and constructs an auxiliary strategy which leads to a deviation of at most $\epsilon > 0$ from the optimum and whose wealth process is bounded. Then one builds an auxiliary strategy for the second agent taking into account the first agent's strategy. To keep "almost" optimality for the first agent, her strategy has to be updated. One iteratively continues with the third until the n-th agent. One could slightly adapt the proof to show the existence of an approximated equilibrium such that additionally the strategy for agent n is optimal, i.e., (3.14) and (3.15) hold for $i = n$ also with $\epsilon = 0$. The underlying reason is that agent n cannot negatively affect the other agents because her strategy is hedgeable by the others. The following result says more about convergence of approximated equilibria in the case of two agents.

Corollary 3.7. *Assume* $n = 2$ *and let* $(\epsilon_k)_{k\in\mathbb{N}}$ *be a strictly positive sequence with* $\lim_{k\to\infty} \epsilon_k = 0$, *and let for each* k, $\hat{\pi}^{\epsilon_k} \in \mathcal{A}$ *be an approximated equilibrium constructed as in the Appendix (proof of Theorem 3.6 with* ϵ *replaced by* ϵ_k*). Suppose that there exists a Nash equilibrium* $\hat{\pi}^\star \in \mathcal{A}$ *with* $\left\| X^{\hat{\pi}^{\star,1}} \right\|_{BMO_1(\hat{\mathbb{P}})} < \frac{1}{4\eta_2\lambda_2}$, *where*

$$\left\| X^{\hat{\pi}^{\star,1}} \right\|_{BMO_1(\hat{\mathbb{P}})} := \sup_\tau \left\| \mathbb{E}_{\hat{\mathbb{P}}}\left[\left| X_T^{\hat{\pi}^{\star,1}} - X_\tau^{\hat{\pi}^{\star,1}} \right| \middle| \mathcal{F}_\tau \right] \right\|_{L^\infty}$$

with the supremum taken over all stopping times τ valued in $[0,T]$. Then we have $\lim_{k\to\infty} V_i^{\hat{\pi}^{\epsilon_k}} = V_i^{\hat{\pi}^\star}$ *for* $i = 1, 2$.

The proof of Corollary 3.7, which is based on the convergence of the BSDEs related to $V_i^{\hat{\pi}^{\epsilon_k}}$, is contained in the Appendix.

3.4 A glimpse of general constraints

We conclude the financial considerations by some results for general closed, convex sets A_i. We do not impose any restrictions on the relations of the A_i.

3.4.1 Sequentially delayed equilibria

We first introduce a further relaxation of a Nash equilibrium.

Definition 3.8. *We say that there exists a* sequentially delayed equilibrium *if for any strictly positive sequence* $(\epsilon_k)_{k\in\mathbb{N}}$ *with* $\lim_{k\to\infty} \epsilon_k = 0$*, there is* $(\hat{\pi}^k)_{k\in\mathbb{N}} \subset \mathcal{A}$ *such that for any* $k \in \mathbb{N}$ *and* $i = 1, \ldots, n$*,*

$$V_i^{\hat{\pi}^{k,i}, \hat{\pi}^{k-1, j\neq i}} + \epsilon_k \geq V_i^{\pi^i, \hat{\pi}^{k-1,j\neq i}} \quad \text{for all } \pi^i \in \mathcal{A}_i, \tag{3.16}$$

where we set $\hat{\pi}^0 = 0$.

Roughly speaking, (3.16) says that $\hat{\pi}^{k,i}$ is "almost" optimal (up to ϵ_k) for agent i when the other agents use the delayed strategies $\hat{\pi}^{k-1,j\neq i}$. Definition 3.5 would correspond to (3.16) if $\hat{\pi}^{k-1,j\neq i}$ were replaced by $\hat{\pi}^{k,j\neq i}$. In a way, the concept of sequentially delayed equilibria is opposed to that of trembling-hand perfect equilibria. That notion, which has been introduced by Selten [15], is a refinement of a Nash equilibrium. Roughly speaking, a trembling-hand perfect equilibrium is robust against small deviations ("trembling hand"). In contrast, a sequentially delayed equilibrium is a weaker notion than that of a Nash equilibrium and gives a way of approaching a status which can be acceptable for all agents. The idea behind Definition 3.8 is that the delay makes the problem easier to handle and in the limit $k \to \infty$, it does not matter whether one has $\hat{\pi}^{k-1,j\neq i}$ or $\hat{\pi}^{k,j\neq i}$ in (3.16). Before making this statement precise in Corollary 3.10, we give an existence result.

Proposition 3.9. *For any family* $(A_i)_{i=1,\ldots,n}$ *of closed sets, there exists a sequentially delayed equilibrium.*

Proof. Let $(\epsilon_k)_{k\in\mathbb{N}}$ be a strictly positive sequence with $\lim_{k\to\infty} \epsilon_k = 0$. We construct iteratively a sequence $(\hat{\pi}^k)_{k\in\mathbb{N}} \subset \mathcal{A}$ satisfying (3.16). Fix $k \in \mathbb{N}$ and

$i \in \{1, \dots, n\}$, set $\hat{\pi}^0 = 0$ and assume that for any $j \in \{1, \dots, n\}$, $X_T^{\hat{\pi}^{k-1,j}}$ is bounded. By Theorem 7 of Hu et al. [10], there exists $\hat{p} \in \mathcal{A}_i$ such that

$$\sup_{\pi^i \in \mathcal{A}_i} V_i^{\pi^i, \hat{\pi}^{k-1, j\neq i}} = V_i^{\hat{p}, \hat{\pi}^{k-1,j\neq i}}.$$

We define a sequence of stopping times by

$$\tau_\ell := \inf \left\{ t \in [0,T] \text{ such that } \left| X_t^{\hat{p}} \right| \geq \ell \right\} \wedge T, \quad \ell \in \mathbb{N}$$

and set $p^{(\ell)} := \hat{p} \mathbb{1}_{]\!]0,\tau_\ell]\!]} \in \mathcal{A}_i$ such that $X_T^{p^{(\ell)}} = X_{\tau_\ell}^{\hat{p}}$. Using that the random variable $F_i + \frac{\lambda_i}{n-1} \sum_{j \neq i} X_T^{\hat{\pi}^{k-1,j}}$ is bounded, the a.s.-converging sequence $\big(U_i\big(X_{\tau_j}^{\hat{p}} - F_i - \frac{\lambda_i}{n-1} \sum_{j\neq i} X_T^{\hat{\pi}^{k-1,j}}\big)\big)_{j \in \mathbb{N}}$ is uniformly integrable by the same argument as above (A.2). Therefore, we have

$$\lim_{\ell \to \infty} V_i^{p^{(\ell)}, \hat{\pi}^{k-1,j\neq i}} = \lim_{\ell \to \infty} \mathbb{E}\left[U_i \left(X_T^{p^{(\ell)}} - F_i - \frac{\lambda_i}{n-1} \sum_{j \neq i} X_T^{\hat{\pi}^{k-1,j}} \right) \right] = V_i^{\hat{p}, \hat{\pi}^{k-1,j\neq i}}.$$

Choose $L \in \mathbb{N}$ such that

$$V_i^{p^{(L)}, \hat{\pi}^{k-1,j\neq i}} \geq V_i^{\hat{p}, \hat{\pi}^{k-1,j\neq i}} - \epsilon_k,$$

and set $\hat{\pi}^{k,i} := p^{(L)}$. By construction, (3.16) is satisfied and $X_T^{\hat{\pi}^{k,i}}$ is bounded. The proof follows from iteratively using the above procedure. □

Corollary 3.10. *Let $(\hat{\pi}^k)_{k\in\mathbb{N}} \subset \mathcal{A}$ satisfy (3.16). Fix i and assume that there exists $\hat{\pi}^\infty \in \mathcal{A}$ with $\int_0^T \hat{\pi}_t^{k,i} \, \mathrm{d}\hat{W}_t \to \int_0^T \hat{\pi}_t^{\infty,i} \, \mathrm{d}\hat{W}_t$ a.s., and that both $U_i\big(X_T^{\hat{\pi}^{k+1,i}} - \frac{\lambda_i}{n-1} \sum_{j\neq i} X_T^{\hat{\pi}^{k,j}}\big)$ and $U_i\big(-\frac{\lambda_i}{n-1} \sum_{j \neq i} X_T^{\hat{\pi}^{k,j}}\big)$, $k \in \mathbb{N}$, are uniformly integrable. Then $V_i^{\hat{\pi}^\infty} \geq V_i^{\pi^i, \hat{\pi}^{\infty,j\neq i}}$ for all $\pi^i \in \mathcal{A}_i$ with bounded $X_T^{\pi^i}$.*

Proof. Fix $\pi^i \in \mathcal{A}_i$ with bounded $X_T^{\pi^i}$. Using the uniform integrability, we obtain both $\lim_{k\to\infty} V_i^{\pi^i, \hat{\pi}^{k,j\neq i}} = V_i^{\pi^i, \hat{\pi}^{\infty,j\neq i}}$ and $\lim_{k\to\infty} V_i^{\hat{\pi}^{k+1,i} \hat{\pi}^{k,j\neq i}} = V_i^{\hat{\pi}^\infty}$. The assertion follows from (3.16). □

3.4.2 Models close to the martingale case

In the martingale case, where S is a $\mathbb{P}$-martingale, θ is zero. Then the strategy $\hat{\pi} = 0$ is a Nash equilibrium by Jensen's inequality if $F_i = 0$ for all i. The idea behind the following result is that we still can find a Nash equilibrium if we are not exactly in the martingale case, but in some sense, close to it.

Proposition 3.11. *Assume $\Pi_{i=1}^n \lambda_i < 1$, that every A_i contains zero and that for any i there exists a constant c_i such that*

$$\left\| \left(F_i + \frac{1}{\eta_i} \int_0^T \theta_t \, \mathrm{d}\hat{W}_t - \frac{1}{2\eta_i} \int_0^T |\theta_t|^2 \, \mathrm{d}t \right) - c_i \right\|_{L^\infty} \le \epsilon_i \tag{3.17}$$

for some sufficiently small $\epsilon_i > 0$ depending on $(\eta_j)_{j=1,\dots,n}$, $(\lambda_j)_{j=1,\dots,n}$ and n. Then the BSDE (3.7) has a unique solution (Γ, ζ) with sufficiently small $\sup_t \|\Gamma_t - c\|_{L^\infty}$ and $\left\| \int \zeta \, \mathrm{d}\hat{W} \right\|_{BMO(\hat{\mathbb{P}})}$. Defining $\hat{\pi}_t^i \sigma_t = P_t^i\big(\varphi_t^{-1,i}(\zeta_t)\big)$ for φ given in (3.6), $\hat{\pi}$ is a Nash equilibrium.

Proof. We first show existence and uniqueness of a solution of (3.7) by applying Proposition 1 of Tevzadze [16]. To this end, we verify that the generator is purely quadratic, i.e., there exists C such that

$$\Big| \big|\varphi_t^{-1,i}(a) - P_t^i\big(\varphi_t^{-1,i}(a)\big)\big|^2 - \big|\varphi_t^{-1,i}(b) - P_t^i\big(\varphi_t^{-1,i}(b)\big)\big|^2 \Big| \le C\big(|a| + |b|\big)|a - b| \tag{3.18}$$

for all $a, b \in \mathbb{R}^{n \times d}$. Setting $\tilde{a} = \varphi_t^{-1,i}(a)$ and $\tilde{b} = \varphi_t^{-1,i}(b)$ and using $0 \in A_i$, we have that

$$\begin{aligned}
&\Big| |\tilde{a} - P_t^i(\tilde{a})|^2 - \big|\tilde{b} - P_t^i\big(\tilde{b}\big)\big|^2 \Big| \\
&= \Big(|\tilde{a} - P_t^i(\tilde{a})| + \big|\tilde{b} - P_t^i\big(\tilde{b}\big)\big| \Big) \Big| |\tilde{a} - P_t^i(\tilde{a})| - \big|\tilde{b} - P_t^i\big(\tilde{b}\big)\big| \Big| \\
&\le \big(|\tilde{a}| + \big|\tilde{b}\big|\big) \big|\tilde{a} - \tilde{b} + P_t^i(\tilde{a}) - P_t^i\big(\tilde{b}\big)\big| \\
&\le 2\big(|\tilde{a}| + \big|\tilde{b}\big|\big)\big|\tilde{a} - \tilde{b}\big|.
\end{aligned}$$

By Lemma 4.41 of Espinosa [6], φ is invertible and φ^{-1} is Lipschitz-continuous with a constant L depending on $(\lambda_j)_{j=1,\dots,n}$ and n. Therefore, we obtain $\big|\tilde{a} - \tilde{b}\big| \le L|a - b|$ as well as $|\tilde{a}| \le L|a|$ and $\big|\tilde{b}\big| \le L|b|$ using $\varphi^{-1}(0) = 0$. This yields (3.18) with $C := 2L^2$. Proposition 1 of Tevzadze [16] now gives existence and uniqueness of a solution (Γ, ζ) of (3.7) under the assumption (3.17). Setting $Z^i = \varphi^{-1,i}(\zeta) - \frac{1}{\eta_i}\theta$ and defining Y via (3.8), the pair (Y, Z) solves the BSDE (3.4). Since $\int \zeta \, \mathrm{d}\hat{W} \in BMO\big(\hat{\mathbb{P}}\big)$ and θ is bounded, $\int \zeta \, \mathrm{d}W$ is in $BMO(\mathbb{P})$ and so is $\int Z \, \mathrm{d}W$ because φ^{-1} is Lipschitz-continuous. Hence, the assertion follows from Lemma 3.2. □

4 About the scope of the counterexample

We now put the counterexample of Section 2 into a broader context by presenting and discussing some related results.

4.1 Limitations of the counterexample

In this section, we show that the counterexample works only if the dependence of (2.3) is quadratic in both Z^1 and Z^2. If it is subquadratic in Z^1 or Z^2, one can always construct a non-exploding solution, even when the dimension of Y is higher than two, with Y^j for $j \geq 3$ satisfying equations analogous to (2.3). This is a consequence of the following two results.

Proposition 4.1. *Let $\xi \in L^\infty$, β be a predictable process with $\int \beta \, \mathrm{d}W$ in BMO, and $\epsilon \in (0, 2]$. Then the one-dimensional BSDE*

$$\mathrm{d}Y_t = -\Big(|\beta_t|^{2-\epsilon} + \frac{1}{2}|Z_t|^2\Big)\,\mathrm{d}t + Z_t \,\mathrm{d}W_t, \quad 0 \leq t \leq T, \quad Y_T = \xi \tag{4.1}$$

has a unique solution (Y, Z) with $\big(\int_0^T |Z_t|^2 \,\mathrm{d}t\big)^{1/2}$, $\exp\big(\sup_{t\in[0,T]} |Y_t|\big) \in L^p$ for all $p \in [1, \infty)$. The martingale $\int Z \,\mathrm{d}W$ is in BMO and Y is bounded.

Theorem 2.1 shows that the assertion is not true for $\epsilon = 0$. However, the assertion still holds if $\frac{1}{2}|Z_t|^2$ in (4.1) is replaced by $c|Z_t|^2$ for some constant c. This can be seen by multiplying the new BSDE with $2c$ and setting $\tilde{Z} = 2cZ$ and $\tilde{Y} = 2cY$.

Proof of Proposition 4.1. Since $\int \beta \,\mathrm{d}W \in BMO$, there is $k > 0$ such that

$$K := \sup_\tau \left\| \mathbb{E}\left[\exp\left(k \int_\tau^T |\beta_t|^2 \,\mathrm{d}t\right) \middle| \mathcal{F}_\tau\right]\right\|_{L^\infty} < \infty.$$

by the John-Nirenberg inequality (Theorem 2.2 of Kazamaki [11]). We recall that this inequality says

$$\mathbb{E}\left[\exp\left(k \int_\tau^T |\beta_t|^2 \,\mathrm{d}t\right)\middle|\mathcal{F}_\tau\right] \leq \frac{1}{1 - k^2 \| \int \beta \,\mathrm{d}W\|^2_{BMO}} \quad \text{a.s.} \tag{4.2}$$

for $0 < k < \frac{1}{\| \int \beta \,\mathrm{d}W\|_{BMO}}$ and any stopping time τ. Let $p \geq 1$ and use

$$p|x|^{2-\epsilon} \leq p^{2/\epsilon}/k^{2/\epsilon - 1} + k|x|^2 \quad \text{for } x \in \mathbb{R}$$

to obtain

$$\mathbb{E}\left[\exp\left(p \int_\tau^T |\beta_t|^{2-\epsilon} \,\mathrm{d}t\right)\middle|\mathcal{F}_\tau\right] \leq K \exp\left(\frac{p^{2/\epsilon}}{k^{2/\epsilon-1}}\right) < \infty. \tag{4.3}$$

This implies by Corollary 6 of Briand and Hu [1] the existence of a unique solution (Y, Z) of (4.1) with $\big(\int_0^T |Z_t|^2 \,\mathrm{d}t\big)^{1/2}$, $\exp\big(\sup_{t\in[0,T]} |Y_t|\big) \in L^p$ for all $p \in [1, \infty)$. The solution satisfies $Y_t = \log \mathbb{E}\big[\exp\big(\int_t^T |\beta_s|^{2-\epsilon} \,\mathrm{d}s + \xi\big)\big|\mathcal{F}_t\big]$ for $t \in [0, T]$, which is bounded by (4.3). The BMO-property of $\int Z \,\mathrm{d}W$ now follows from considering (4.1) over an interval $[\tau, T]$ and taking $\mathcal{F}_\tau$-conditional expectations. □

We next give a completely analogous result to Proposition 4.1 when (4.1) is subquadratic in β_t instead of Z_t. However, its proof is different.

Proposition 4.2. *Let $\xi \in L^\infty$, β be a predictable process with $\int \beta \,\mathrm{d}W$ in BMO, and $\epsilon \in (0,2]$. Then the one-dimensional BSDE*

$$\mathrm{d}Y_t = -\Big(|\beta_t|^2 + \frac{1}{2}|Z_t|^{2-\epsilon}\Big)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \quad 0 \le t \le T, \quad Y_T = \xi \qquad (4.4)$$

has a unique solution (Y,Z) with $\big(\int_0^T |Z_t|^2\,\mathrm{d}t\big)^{1/2}$, $\exp\big(\sup_{t\in[0,T]}|Y_t|\big) \in L^p$ for all $p \in [1,\infty)$. The martingale $\int Z\,\mathrm{d}W$ is in BMO and Y is bounded.

Proof. Set $c_1 := \big(4\|\int \beta\,\mathrm{d}W\|^2_{BMO}\big)^{-1/\epsilon}$. By defining $\widetilde{Y} := c_1 Y$ and $\widetilde{Z} := c_1 Z$, the BSDE (4.4) is seen to be equivalent to

$$\mathrm{d}\widetilde{Y}_t = -\Big(c_1|\beta_t|^2 + \frac{1}{2c_1^{1-\epsilon}}\big|\widetilde{Z}_t\big|^{2-\epsilon}\Big)\,\mathrm{d}t + \widetilde{Z}_t\,\mathrm{d}W_t, \quad 0 \le t \le T, \quad \widetilde{Y}_T = c_1\xi. \quad (4.5)$$

By the John-Nirenberg inequality (4.2), $c_1\int_0^T |\beta_t|^2\,\mathrm{d}t$ has an exponential moment of order $2c_1^{\epsilon-1}$. Therefore, Theorems 2.1 and 3.3 of Delbaen et al. [3] imply the existence and uniqueness of a solution $\big(\widetilde{Y},\widetilde{Z}\big)$ to (4.5) in a suitable class, and there exist constants c_2 and c_3 such that

$$c_2 \le \widetilde{Y}_t \le c_3 + c_1^{1-\epsilon}\log\mathbb{E}\Big[\exp\Big(c_1^\epsilon\int_t^T |\beta_s|^2\,\mathrm{d}s\Big)\Big|\mathcal{F}_t\Big], \quad t \in [0,T].$$

This shows by (4.2) that $\widetilde{Y}$ is bounded, and so is Y for the solution (Y,Z) of (4.4). To prove the BMO-property, we may assume that $\int Z\,\mathrm{d}W$ is a true martingale (otherwise we use a localization argument). From Itô's formula, it follows that, for any stopping time τ,

$$\mathrm{e}^{2\xi} - \mathrm{e}^{2Y_\tau} = \int_\tau^T 2\mathrm{e}^{2Y_t}Z_t\,\mathrm{d}W_t + \int_\tau^T 2\mathrm{e}^{2Y_t}\Big(-|\beta_t|^2 - \frac{1}{2}|Z_t|^{2-\epsilon} + |Z_t|^2\Big)\,\mathrm{d}t.$$

Since $|Z_t|^{2-\epsilon} \le |Z_t|^2 + 1$ and Y is bounded, we obtain that $\mathbb{E}\big[\int_\tau^T |Z_t|^2\,\mathrm{d}t\big|\mathcal{F}_\tau\big]$ is bounded, which concludes the proof. □

4.2 A second counterexample: resonance

In the counterexample of Section 2, explosion happens because the generator of the second component depends quadratically on the control variables of the first and second dimension of the BSDE. In view of Section 4.1, the explosion can happen only if it is quadratic in both components. We next present another counterexample, where the reason for the explosion is different. In this case, the components of the BSDE are mutually dependent, which leads to an explosion and causes insolvability.

We take again $\dim W = d = 1$ and consider the three-dimensional BSDE

$$\mathrm{d}Y^1_t = Z^1_t\,\mathrm{d}W_t, \qquad Y^1_T = \xi, \tag{4.6}$$

$$\mathrm{d}Y^2_t = -\big(|Z^1_t|^2 + |Z^3_t|^2\big)\,\mathrm{d}t + Z^2_t\,\mathrm{d}W_t, \qquad Y^2_T = 0, \tag{4.7}$$

$$\mathrm{d}Y^3_t = -4|Z^2_t|^2\,\mathrm{d}t + Z^3_t\,\mathrm{d}W_t, \qquad Y^3_T = 0, \tag{4.8}$$

where the terminal condition $\xi \in L^\infty$ is given. In contrast to the first counterexample, the generator of (4.7) has no (quadratic) dependence on Z^2. We will see that the explosion here is related to the mutual dependence of (4.7) and (4.8). For a certain choice of $\xi \in L^\infty$, Z^1 is big in some sense, and the size is amplified by the interdependence of (4.7) and (4.8), which finally leads to a collapse of the whole BSDE system (4.6)–(4.8). We call this phenomenon resonance due to its analogy in physics.

Theorem 4.3. *For some $\xi \in L^\infty$, the BSDE (4.6)–(4.8) has no solution.*

Proof. As in Theorem 2.1, we take $\xi = \int_0^T \zeta_t\,\mathrm{d}W_t$ with ζ given in Lemma A.1 and satisfying $\xi \in L^\infty$ and $\mathbb{E}\big[\exp\big(\int_0^T |\zeta_t|^2\,\mathrm{d}t\big)\big] = \infty$. We have $Y^1_t = \mathbb{E}[\xi|\mathcal{F}_t]$ and $Z^1 = \zeta$ by (4.6). From (4.7) and (4.8), it follows that

$$\mathrm{e}^{Y^2_0+Y^3_0}\mathcal{E}\bigg(\int (Z^2+Z^3)\,\mathrm{d}W\bigg)_T = \exp\bigg(\int_0^T \Big(|Z^1_t|^2+\frac{7}{2}|Z^2_t|^2+\frac{1}{2}|Z^3_t|^2-Z^2_tZ^3_t\Big)\,\mathrm{d}t\bigg).$$

Using $-Z^2_tZ^3_t \geq -\frac{1}{2}|Z^2_t|^2 - \frac{1}{2}|Z^3_t|^2$ and that $\mathcal{E}\big(\int (Z^2 + Z^3)\,\mathrm{d}W\big)$ is a positive supermartingale, we obtain by taking expectations that

$$\mathrm{e}^{Y^2_0+Y^3_0} \geq \mathbb{E}\bigg[\exp\bigg(\int_0^T \big(|Z^1_t|^2 + 3|Z^2_t|^2\big)\,\mathrm{d}t\bigg)\bigg] = \infty.$$

This shows $Y^2_0+Y^3_0 = \infty$ and that there cannot exist a solution to (4.6)–(4.8) for $\xi = \int_0^T \zeta_t\,\mathrm{d}W_t \in L^\infty$ with ζ given in Lemma A.1. □

Remarks. 1) The reason for the explosion is that the possible solutions to (4.7), (4.8) amplify each other. Assume that (4.7), (4.8) has a solution for the Z^1 given above. From (4.7), it follows that

$$\mathbb{E}\bigg[\exp\bigg(\int_0^T Z^2_t\,\mathrm{d}W_t\bigg)\bigg] \geq \mathrm{e}^{-Y^2_0}\mathbb{E}\bigg[\exp\bigg(\int_0^T |Z^1_t|^2\,\mathrm{d}t\bigg)\bigg] = \infty. \tag{4.9}$$

For any local martingale M, we have

$$\mathbb{E}\big[\exp(M_T)\big] = \mathbb{E}\big[\mathcal{E}(2M)_T^{1/2}\exp\big(\langle M\rangle_T\big)\big] \le \mathbb{E}\big[\exp\big(2\langle M\rangle_T\big)\big]^{1/2}$$

by Hölder's inequality, using that $\mathcal{E}(2M)$ is a positive supermartingale. Applying this to (4.9) yields $\mathbb{E}\big[\exp\big(2\int_0^T |Z_t^2|^2\,\mathrm{d}t\big)\big] = \infty$. The combination with (4.8) gives

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T Z_t^3\,\mathrm{d}W_t\right)\right] = \mathrm{e}^{-Y_0^3/2}\mathbb{E}\left[\exp\left(2\int_0^T |Z_t^2|^2\,\mathrm{d}t\right)\right] = \infty,$$

and thus $\mathbb{E}\big[\exp\big(\frac{1}{2}\int_0^T |Z_t^3|^2\,\mathrm{d}t\big)\big] = \infty$. Analogously to (4.9), we obtain that $\mathbb{E}\big[\exp\big(\frac{1}{2}\int_0^T Z_t^2\,\mathrm{d}W_t\big)\big] = \infty$. Proceeding iteratively like above, we deduce $\mathbb{E}\big[\exp\big(\frac{1}{2^k}\int_0^T Z_t^2\,\mathrm{d}W_t\big)\big] = \infty$ for every $k \in \mathbb{N}$, which implies that $\int Z^2\,\mathrm{d}W$ is *not* in BMO by the variant of the John-Nirenberg inequality stated in Theorem 2.1 of Kazamaki [11]. As this procedure only shows that there exists no solution (Y,Z) with $\int Z\,\mathrm{d}W \in BMO$, it is not enough for proving Theorem 4.3, but it gives intuition for the explosion.

2) We can interpret the counterexample in terms of financial economics. Consider the following three agents:

- a bank, which is based in a country A, wants to achieve a value ξ;
- country A, whose economy depends on the bank's portfolio fluctuation;
- country B, whose economy depends on country A, but not on the bank.

If we denote by Y^i the wealth development of agent i, the system (4.6)–(4.8) can be understood as a toy model for this situation. The counterexample tells us that if the bank wants to achieve a highly risky (but still bounded) value ξ, the economies in both countries break down, because the bank's risk is transferred to country A and then leveraged by the dependencies between countries A and B. $\Diamond$

Our counterexamples show that dimensions matter in stochastics. This issue of dimensionality has already been pointed out by Emery [5]. While the stochastic exponential of any bounded continuous martingale is a true martingale, he gave an example of a bounded continuous matrix-valued martingale whose stochastic exponential is not a true martingale. Both Emery [5] and our counterexamples show that integrability properties of stochastic processes may crucially depend on the dimension, although Emery [5] and our counterexamples are in completely different settings.

4.3 An extension of the first counterexample

The counterexample presented in Section 2 falls into the class of multi-dimensional quadratic BSDEs with generators that have linearly dependent components. For their definition, we take an n-dimensional $\xi \in L^\infty$, a vector $a_1 = (a_{11}, \dots, a_{1n}) \in \mathbb{R}^n$ and a function $g : [0,T] \times \mathbb{R}^{n\times d} \to \mathbb{R}$ satisfying, for some positive constant C,

$$|g(t,z)| \le C\big(1 + \mathrm{trace}(zz^\top)\big) \quad \text{for } (t,z) \in [0,T] \times \mathbb{R}^{n\times d}.$$

Define the generator $G : [0,T] \times \mathbb{R}^{n\times d} \to \mathbb{R}^n$ by

$$G(t,z) := a_1 g(t,z) = \big(a_{11}g(t,z), \dots, a_{1n}g(t,z)\big) \quad \text{for } (t,z) \in [0,T] \times \mathbb{R}^{n\times d}. \tag{4.10}$$

With the above elements, we analyze the BSDE

$$\mathrm{d}Y_t = G(t,Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \quad 0 \le t \le T, \quad Y_T = \xi. \tag{4.11}$$

Take now a set of vectors $\{a_i\}_{i=2,\dots,n}$ in $\mathbb{R}^n$ such that $\{a_i\}_{i=1,\dots,n}$ form an orthogonal basis of $\mathbb{R}^n$, i.e.,

$$\mathrm{span}\{a_1, \dots, a_n\} = \mathbb{R}^n \text{ and } a_i \cdot a_j = 0 \text{ for } i \neq j. \tag{4.12}$$

We set $A = (a_{ij})_{i,j=1,\dots,n}$, which defines an $(n \times n)$-dimensional matrix whose inverse A^{-1} exists by construction.

We define a new system of BSDEs by applying A to the BSDE (4.11). We set $(U,V) = (AY, AZ)$ and $h(t,z) = AG(t,z)$. Because of (4.10) and (4.12), we have

$$h_i(t,z) = a_i \cdot G(t,z) = a_i \cdot a_1 g(t,z) = 0 \quad \text{for every } i = 2, \dots, n.$$

Hence, for $i \neq 1$ the BSDEs for (U^i, V^i) are trivial and given by

$$\mathrm{d}U_t^i = V_t^i\,\mathrm{d}W_t, \quad 0 \le t \le T, \quad U_T^i = a_i \cdot \xi.$$

Since $\xi \in L^\infty$, a unique solution (U^i, V^i) of the above equation exists by Itô's representation theorem and it satisfies $\big(U^i, \int V^i\,\mathrm{d}W\big) \in \mathcal{S}^\infty \times BMO$. However, the case $i = 1$ is different. It leads to the BSDE

$$\begin{aligned} \mathrm{d}U_t^1 &= h_1(t,Z_t)\,\mathrm{d}t + V_t^1\,\mathrm{d}W_t = |a_1|^2 g(t, A^{-1}V_t)\,\mathrm{d}t + V_t^1\,\mathrm{d}W_t, \quad 0 \le t \le T, \\ U_T^1 &= a_1 \cdot \xi. \end{aligned} \tag{4.13}$$

At this point, all (U^i, V^i) for $i \neq 1$ are determined. So (4.13) reduces to the usual case of a one-dimensional quadratic BSDE. Its generator satisfies

$$|a_1|^2 |g(t, A^{-1}V_t)| \le C(1 + H_t + |V_t^1|^2),$$

with the positive process H representing a linear combination of quadratic terms of $V^2, \dots, V^n$ such that $\int \sqrt{H}\,\mathrm{d}W \in BMO$. If the one-dimensional BSDE (4.13) can be solved, then so can the multidimensional BSDE (4.11).

The problem of solving (4.13) boils down to the question of how random H can be, and for this, we recall some existing results for one-dimensional BSDEs. In Kobylanski [12] and Morlais [13], existence and uniqueness are obtained under the assumption that H is uniformly bounded. This assumption allows to conclude that the solution is bounded and satisfies a BMO-property. Unfortunately, the restrictions imposed by these results are too strong for our BSDE (4.13).

The only viable options in the literature are Briand and Hu [1] and Delbaen et al. [3]. The latter strengthens the result of the former by proving uniqueness in a bigger class of solutions. Proposition 3 of Briand and Hu [1] yields existence under the assumption $\mathbb{E}\big[\exp\big(2C^2 p \int_0^T H_t\,\mathrm{d}t\big)\big] < \infty$ for some $p > 1$. For our BSDE (4.13) with $\int \sqrt{H}\,\mathrm{d}W \in BMO$, exponential moments of $\int_0^T H_t\,\mathrm{d}t$ exist only under certain conditions and are implied by the John-Nirenberg inequality (4.2). If $\big\|\int \sqrt{H}\,\mathrm{d}W\big\|_{BMO}$ is small enough, we can conclude that a unique solution (Y, Z) exists, Y is bounded and $\int Z\,\mathrm{d}W$ is in BMO. Our counterexample of Section 2 demonstrates that $\int \sqrt{H}\,\mathrm{d}W \in BMO$ is not enough to guarantee existence of a solution, and one really needs a further condition on the exponential moments of $\int_0^T H_t\,\mathrm{d}t$.

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Appendix

A.1 Auxiliary results

Lemma A.1. *There exists $\zeta \in \mathcal{H}^2_{1,1}$ with*

$$\int \zeta\,\mathrm{d}W^1 \in \mathcal{S}^\infty \quad \textit{and} \quad \mathbb{E}\bigg[\exp\bigg(\int_0^T |\zeta_t|^2\,\mathrm{d}t\bigg)\bigg] = \infty.$$

Proof. The following construction is inspired by the proof of Lemma 2.7 of Kazamaki [11]. Define

$$M_t := \int_0^t \frac{1}{\sqrt{T-s}}\,\mathrm{d}W_s^1, \quad t\in[0,T) \tag{A.1}$$

so that $(M_t)_{0\le t\le u}$ is a continuous martingale on $[0,u]$ for every $u<T$. We set $\tau := \inf\{t\ge 0 : |M_t|>1\}$ and $\zeta_t := \frac{\pi}{2\sqrt{2}\sqrt{T-t}}\mathbb{1}_{[\![0,\tau]\!]}(t)$ so that $\left|\int \zeta\,\mathrm{d}W^1\right|$ is bounded by $\frac{\pi}{2\sqrt{2}}$. It remains to show that $\mathbb{E}\big[\exp\big(\int_0^T|\zeta_t|^2\,\mathrm{d}t\big)\big]=\infty$. For this, we define an auxiliary function $h:[0,\infty)\to[0,T)$ by $h(t):=T(1-\mathrm{e}^{-t})$, which fulfills

$$\int_0^{h(t)}\frac{1}{T-s}\,\mathrm{d}s = \log\frac{T}{T-h(t)} = t, \quad t\in[0,\infty).$$

We set $B_t := M_{h(t)}$, $0\le t<\infty$, implying that $(B_t)_{0\le t<\infty}$ is an $(\mathcal{F}_{h(t)})_{0\le t<\infty}$-Brownian motion. The random variable $h^{-1}(\tau)$ is the $(\mathcal{F}_{h(t)})_{0\le t<\infty}$-stopping time when B first leaves $[-1,1]$. From Lemma 1.3 of Kazamaki [11], it follows that $\mathbb{E}\big[\exp\big(\frac{\alpha^2}{2}h^{-1}(\tau)\big)\big]=\frac{1}{\cos(\alpha)}$ for all $\alpha\in[0,\pi/2)$. Therefore, we obtain

$$\begin{aligned}\mathbb{E}\left[\exp\left(\int_0^T|\zeta_t|^2\,\mathrm{d}t\right)\right] &= \mathbb{E}\left[\exp\left(\frac{\pi^2}{8}\int_0^\tau\frac{1}{T-t}\,\mathrm{d}t\right)\right] = \mathbb{E}\left[\exp\left(\frac{\pi^2}{8}h^{-1}(\tau)\right)\right]\\ &= \lim_{\alpha\nearrow\pi/2}\mathbb{E}\left[\exp\left(\frac{\alpha^2}{2}h^{-1}(\tau)\right)\right] = \lim_{\alpha\nearrow\pi/2}\frac{1}{\cos(\alpha)}=\infty.\end{aligned}$$ □

The result is unchanged if one replaces in the definition (A.1) of M the function $s\mapsto\frac{1}{\sqrt{T-s}}$ by another continuous function $g:[0,T)\to\mathbb{R}$ which satisfies $\int_0^T|g(s)|^2\,\mathrm{d}s=\infty$ and $\int_0^t|g(s)|^2\,\mathrm{d}s<\infty$ for every $t\in[0,T)$. For any given $\zeta\in\mathcal{H}^2_{1,d}$ with $\int\zeta\,\mathrm{d}W^1\in\mathcal{S}^\infty$, there exists a constant c such that $\mathbb{E}\big[\exp\big(c\int_0^T|\zeta_t|^2\,\mathrm{d}t\big)\big]<\infty$ by the John-Nirenberg inequality (4.2). Lemma A.1 shows conversely that for any fixed constant c, there exists $\zeta\in\mathcal{H}^2_{1,d}$ with $\int\zeta\,\mathrm{d}W^1\in\mathcal{S}^\infty$ and $\mathbb{E}\big[\exp\big(c\int_0^T|\zeta_t|^2\,\mathrm{d}t\big)\big]=\infty$.

Lemma A.2. *There exists $\zeta\in\mathcal{H}^2_{1,1}$ with*

$$\log\mathcal{E}\left(\int\zeta\,\mathrm{d}W^1\right)\in\mathcal{S}^\infty \quad \textit{and} \quad \mathbb{E}\left[\exp\left(\frac{\pi^2+1}{4}\int_0^T\zeta_t\,\mathrm{d}W_t^1\right)\right]=\infty.$$

Proof. Similarly to (A.1), we define $M_t := \int_0^t\frac{1}{\sqrt{T-s}}\,\mathrm{d}W_s^1$ for $t\in[0,T)$ and set $\tau := \inf\big\{t\ge 0 : \big|M_t-\frac{1}{2}\log\frac{T}{T-t}\big|>1\big\}$ and $\zeta_t := \frac{1}{\sqrt{T-t}}\mathbb{1}_{[\![0,\tau]\!]}(t)$. Because we have $\langle M\rangle_t=\int_0^t\frac{1}{T-s}\,\mathrm{d}s=\log\frac{T}{T-t}$ for $t\in[0,T)$, $\log\mathcal{E}\big(\int\zeta\,\mathrm{d}W^1\big)$ is bounded, and τ is the first time that $\mathcal{E}(M)$ leaves $[1/\mathrm{e}, \mathrm{e}]$. We recall the function $h : [0, \infty) \to [0, T)$ given by $h(t) := T(1 - \mathrm{e}^{-t})$, which is the inverse of $t \mapsto \log \frac{T}{T-t}$. We set $B_t := M_{h(t)}$, $0 \le t < \infty$, so that $(B_t)_{0 \le t < \infty}$ is an $(\mathcal{F}_{h(t)})_{0 \le t < \infty}$-Brownian motion. The random variable $h^{-1}(\tau)$ is the $(\mathcal{F}_{h(t)})_{0 \le t < \infty}$-stopping time when the drifted $(\mathcal{F}_{h(t)})_{0 \le t < \infty}$-Brownian motion $(B_t - t/2)_{0 \le t < \infty}$ first leaves $[-1, 1]$. Lemma 1.3 of Kazamaki [11] implies

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\frac{\alpha^2}{2} h^{-1}(\tau)\right)\right] = \frac{1}{\cos(\alpha)} \quad \text{for all } \alpha \in [0, \pi/2),$$

where $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} := \mathcal{E}\big(\frac{1}{2}B\big)_{h^{-1}(\tau)} = \mathcal{E}\big(\frac{1}{2}M\big)_{\tau}$. For $\beta \ge (\pi^2 + 1)/8$, we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[\exp\left(\beta \int_0^{\tau} \frac{1}{T-t}\,\mathrm{d}t\right)\right] &= \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\mathcal{E}(\frac{1}{2}M)_{\tau}} \exp\big(\beta h^{-1}(\tau)\big)\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\mathcal{E}(M)_{\tau}^{1/2}} \exp\left(\beta h^{-1}(\tau) - \frac{1}{8}\langle M \rangle_{\tau}\right)\right] \\
&\ge \mathrm{e}^{-1/2}\mathbb{E}_{\mathbb{Q}}\left[\exp\left(\left(\beta - \frac{1}{8}\right) h^{-1}(\tau)\right)\right] \\
&= \infty
\end{aligned}$$

and hence

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}[\exp(2\beta M_{\tau})] &= \mathbb{E}_{\mathbb{P}}\left[\mathcal{E}(M)_{\tau}^{2\beta} \exp\left(\beta \int_0^{\tau} \frac{1}{T-t}\,\mathrm{d}t\right)\right] \\
&\ge \mathrm{e}^{-2\beta}\mathbb{E}_{\mathbb{P}}\left[\exp\left(\beta \int_0^{\tau} \frac{1}{T-t}\,\mathrm{d}t\right)\right] \\
&= \infty
\end{aligned}$$

so that $\mathbb{E}_{\mathbb{P}}\big[\exp\big(\frac{\pi^2+1}{4} \int_0^T \zeta_t \,\mathrm{d}W_t^1\big)\big] = \mathbb{E}_{\mathbb{P}}\big[\exp\big(\frac{\pi^2+1}{4} M_{\tau}\big)\big] = \infty$. □

A.2 Proofs of Theorem 3.6 and Corollary 3.7

Proof of Theorem 3.6. Fix $\epsilon > 0$ to show (3.14) and (3.15). We assume $\epsilon < 1$ without loss of generality.

1. Step: Construction of an auxiliary strategy for agent 1.

We start by looking at an auxiliary problem for the first agent. By Theorem 7 of Hu et al. [10], there exists $\hat{p} \in \mathcal{A}_1$ such that

$$\sup_{p \in \mathcal{A}_1} \mathbb{E}\big[U_1\big(X_T^p - F_1\big)\big] = \mathbb{E}\big[U_1\big(X_T^{\hat{p}} - F_1\big)\big].$$

We define a sequence of stopping times by

$$\tau_k := \inf\left\{t \in [0,T] \text{ such that } \left|X_t^{\hat{p}}\right| \geq k\right\} \wedge T, \quad k \in \mathbb{N}$$

and set $p^{(k)} := \hat{p}\mathbb{1}_{]\!]0,\tau_k]\!]} \in \mathcal{A}_1$ such that $X_T^{p^{(k)}} = X_{\tau_k}^{\hat{p}}$. Because F_1 is bounded and $\big(U_1\big(X_t^{\hat{p}}\big)\big)_{0\leq t\leq T}$ can be written as the product of a martingale and a bounded process (see the proof of Theorem 7 of Hu et al. [10]), the process $\big(U_1\big(X_t^{\hat{p}} - F_1\big)\big)_{0\leq t\leq T}$ is of class (D). Hence, the sequence $\big(U_1\big(X_{\tau_k}^{\hat{p}} - F_1\big)\big)_{k\in\mathbb{N}}$ converging almost surely is uniformly integrable and thus, we have

$$\lim_{k\to\infty} \mathbb{E}\Big[U_1\Big(X_T^{p^{(k)}} - F_1\Big)\Big] = \mathbb{E}\big[U_1\big(X_T^{\hat{p}} - F_1\big)\big]. \tag{A.2}$$

Choose $K \in \mathbb{N}$ such that

$$\mathbb{E}\Big[U_1\Big(X_T^{p^{(K)}} - F_1\Big)\Big] \geq \max\Big\{\mathbb{E}\big[U_1\big(X_T^{\hat{p}} - F_1\big)\big] - \epsilon, \frac{1}{1-\epsilon}\mathbb{E}\big[U_1\big(X_T^{\hat{p}} - F_1\big)\big]\Big\}.$$

For notational convenience, we set $\pi^{(1,1)} := p^{(K)}$, where $\pi^{(i,j)}$ stands for the auxiliary strategy of agent i in the j-th iteration.

2. Step: Construction of an auxiliary strategy for agent 2 and adaptation of the first agent's auxiliary strategy.

We now construct an auxiliary strategy $\pi^{(2,1)}$ for agent 2 in a similar way; we simply replace η_1 by η_2, U_1 by U_2, and F_1 by $F_2 + \frac{\lambda_2}{n-1}X_T^{\pi^{(1,1)}}$, which is bounded by construction. Because there is interdependence between agents 1 and 2, we need to adapt the strategies by setting

$$\pi^{(2,2)} := \frac{1}{1-\lambda_1\lambda_2/(n-1)^2}\pi^{(2,1)}, \quad \pi^{(1,2)} := \pi^{(1,1)} + \frac{\lambda_1}{n-1}\pi^{(2,2)}$$

to achieve that

$$\pi^{(2,2)} - \frac{\lambda_2}{n-1}\pi^{(1,2)} = \pi^{(2,1)} - \frac{\lambda_2}{n-1}\pi^{(1,1)}, \quad \pi^{(1,2)} - \frac{\lambda_1}{n-1}\pi^{(2,2)} = \pi^{(1,1)}.$$

Since A_1, A_2 are linear subspaces with $A_1 \supseteq A_2$, we have $\pi^{(2,2)} \in \mathcal{A}_2$ and $\pi^{(1,2)} \in \mathcal{A}_1$.

3. Step: Construction of an auxiliary strategy for agent i and adaptation of the auxiliary strategy of agents $1, \ldots, i-1$.

Like above, we construct an auxiliary strategy $\pi^{(3,1)}$ for the third agent, replacing η_1 by η_3, U_1 by U_3, and F_1 by $F_3 + \frac{\lambda_3}{n-1}\big(X_T^{\pi^{(1,2)}} + X_T^{\pi^{(2,2)}}\big)$. To account for the interdependence, we set $\lambda_{1,2}^n := \frac{\frac{\lambda_1}{n-1}\left(1+\frac{\lambda_2}{n-1}\right)}{1-\frac{\lambda_1\lambda_2}{(n-1)^2}}$ and define

$$\pi^{(3,3)} := \frac{1}{1-(\lambda_{1,2}^n + \lambda_{2,1}^n)\lambda_3/(n-1)}\pi^{(3,1)},$$
$$\pi^{(2,3)} := \pi^{(2,2)} + \lambda_{1,2}^n\pi^{(3,3)}, \qquad \pi^{(1,3)} := \pi^{(1,2)} + \lambda_{2,1}^n\pi^{(3,3)},$$

achieving that

$$\begin{aligned}
\pi^{(3,3)} - \frac{\lambda_3}{n-1}\big(\pi^{(1,3)} + \pi^{(2,3)}\big) &= \pi^{(3,1)} - \frac{\lambda_3}{n-1}\big(\pi^{(1,2)} + \pi^{(2,2)}\big),\\
\pi^{(2,3)} - \frac{\lambda_2}{n-1}\big(\pi^{(1,3)} + \pi^{(3,3)}\big) &= \pi^{(2,2)} - \frac{\lambda_2}{n-1}\pi^{(1,2)},\\
\pi^{(1,3)} - \frac{\lambda_1}{n-1}\big(\pi^{(2,3)} + \pi^{(3,3)}\big) &= \pi^{(1,2)} - \frac{\lambda_1}{n-1}\pi^{(2,2)}.
\end{aligned}$$

Continuing iteratively like this, we finally obtain strategies $\pi^{(1,n)}, \dots, \pi^{(n,n)}$. (The procedure works since we can solve in each step a system of linear equations with non-zero determinant because of the assumption $\Pi_{i=1}^n \lambda_i < 1$.)
4. Step: Definition of $\hat{\pi}^\epsilon$ and verification of (3.14) and (3.15).
We set $\hat{\pi}^{\epsilon,j} := \pi^{(j,n)} \in \mathcal{A}_j$ for all j. For fixed i, we have by construction that

$$V_i^{\hat{\pi}^\epsilon} = \mathbb{E}\bigg[U_i\bigg(X_T^{\pi^{(1,i)}} - \frac{\lambda_i}{n-1}\sum_{j=1}^{i-1} X_T^{\pi^{(j,i-1)}} - F_i\bigg)\bigg] \geq \max\Big\{a_i - \epsilon, \frac{1}{1-\epsilon}a_i\Big\},$$

$$\begin{aligned}
\text{where } a_i &:= \sup_{p\in\mathcal{A}_i} \mathbb{E}\bigg[U_i\bigg(X_T^p - \frac{\lambda_i}{n-1}\sum_{j=1}^{i-1} X_T^{\pi^{(j,i-1)}} - F_i\bigg)\bigg]\\
&= \sup_{p\in\mathcal{A}_i} \mathbb{E}\bigg[U_i\bigg(X_T^p - \frac{\lambda_i}{n-1}\sum_{j\neq i} X_T^{\hat{\pi}^{\epsilon,j}} - F_i\bigg)\bigg] = \sup_{\pi^i\in\mathcal{A}_i} V_i^{\pi^i,\hat{\pi}^{\epsilon,j\neq i}}.
\end{aligned}$$

Therefore, both (3.14) and (3.15) are satisfied by this $(\hat{\pi}^{\epsilon,1}, \dots, \hat{\pi}^{\epsilon,n})$. □

Proof of Corollary 3.7. From (A.2), we obtain $\lim_{k\to\infty} V_1^{\hat{\pi}^{\epsilon_k}} = V_1^{\hat{\pi}^\star}$, where $\hat{\pi}^{\epsilon_k,1} = \hat{\pi}^{\star,1}\mathbb{1}_{]\!]0,\tau^k]\!]}$ for some stopping time τ^k. We study the BSDEs related to $V_2^{\hat{\pi}^{\epsilon_k}}$. By construction and Theorem 7 of Hu et al. [10], we have

$$-\exp\big(\eta_2 Y_0^{(k)}\big) - \epsilon_k \leq V_2^{\hat{\pi}^{\epsilon_k}} \leq -\exp\big(\eta_2 Y_0^{(k)}\big),$$

where $\big(Y^{(k)}, Z^{(k)}\big)$ is the unique solution in $(\mathcal{S}^\infty, \mathcal{H}^2_{1,d})$ of the BSDE

$$\begin{aligned}
\mathrm{d}Y_t^{(k)} &= \bigg(\frac{|\theta_t|^2}{2\eta_2} - \frac{\eta_2}{2}\Big|Z_t^{(k)} + \frac{1}{\eta_2}\theta_t - P_t^2\Big(Z_t^{(k)} + \frac{1}{\eta_2}\theta_t\Big)\Big|^2\bigg)\,\mathrm{d}t + Z_t^{(k)}\,\mathrm{d}\hat{W}_t,\\
Y_T^{(k)} &= \lambda_2 X_T^{\hat{\pi}^{\epsilon_k,1}} + F_2 = \lambda_2 X_{\tau_k}^{\hat{\pi}^{\star,1}} + F_2.
\end{aligned}$$

Because F_2 and θ are bounded, there exist constants c_1 and c_2 (not depending on k) such that for any stopping time ν, $Y_\nu^{(k)} \geq \lambda_2 \mathbb{E}_{\hat{\mathbb{P}}}\big[X_{\tau_k}^{\hat{\pi}^{\star,1}}\big|\mathcal{F}_\nu\big] + c_1$ and

$$\frac{\mathcal{E}\big(\int\big(\eta_2 Z^{(k)} + \theta\big)\,\mathrm{d}\hat{W}\big)_T}{\mathcal{E}\big(\int\big(\eta_2 Z^{(k)} + \theta\big)\,\mathrm{d}\hat{W}\big)_\nu} \leq \exp\bigg(\int_\nu^T \theta_t\,\mathrm{d}\hat{W}_t + \eta_2\lambda_2\big(X_{\tau_k}^{\hat{\pi}^{\star,1}} - \mathbb{E}_{\hat{\mathbb{P}}}\big[X_{\tau_k}^{\hat{\pi}^{\star,1}}\big|\mathcal{F}_\nu\big]\big) + c_2\bigg)$$

so that Hölder's inequality implies for any $p,q>1$ and some $c_3>0$ (depending on p and q but not on k or ν)

$$\begin{aligned}&\mathbb{E}_{\hat{\mathbb{P}}}\left[\frac{\mathcal{E}\big(\int\big(\eta_2 Z^{(k)}+\theta\big)\,\mathrm{d}\hat{W}\big)_T^p}{\mathcal{E}\big(\int\big(\eta_2 Z^{(k)}+\theta\big)\,\mathrm{d}\hat{W}\big)_\nu^p}\middle|\mathcal{F}_\nu\right]\\ &\le c_3\mathbb{E}_{\hat{\mathbb{P}}}\left[\exp\left(qp\eta_2\lambda_2\big(X^{\hat{\pi}^\star,1}_{\tau_k}-\mathbb{E}_{\hat{\mathbb{P}}}\big[X^{\hat{\pi}^\star,1}_{\tau_k}\big|\mathcal{F}_\nu\big]\big)\right)\middle|\mathcal{F}_\nu\right]^{1/q}.\end{aligned}$$

The assumption $\big\|X^{\hat{\pi}^\star,1}\big\|_{BMO_1(\hat{\mathbb{P}})}<\frac{1}{4\eta_2\lambda_2}$ enables us to choose $p,q>1$ with $qp\big\|X^{\hat{\pi}^\star,1}\big\|_{BMO_1(\hat{\mathbb{P}})}<\frac{1}{4\eta_2\lambda_2}$. Using $\big\|X^{\hat{\pi}^\star,1}_{\tau_k\wedge\cdot}\big\|_{BMO_1(\hat{\mathbb{P}})}\le\big\|X^{\hat{\pi}^\star,1}\big\|_{BMO_1(\hat{\mathbb{P}})}$, we obtain from the variant of the John-Nirenberg inequality stated in Theorem 2.1 of Kazamaki [11] that

$$\mathbb{E}_{\hat{\mathbb{P}}}\left[\exp\left(qp\eta_2\lambda_2\big(X^{\hat{\pi}^\star,1}_{\tau_k}-\mathbb{E}_{\hat{\mathbb{P}}}\big[X^{\hat{\pi}^\star,1}_{\tau_k}\big|\mathcal{F}_\nu\big]\big)\right)\middle|\mathcal{F}_\nu\right]\le\frac{1}{1-4qp\eta_2\lambda_2\big\|X^{\hat{\pi}^\star,1}\big\|_{BMO_1(\hat{\mathbb{P}})}},$$

which shows that there exists $p>1$ such that $\mathcal{E}\big(\int(\eta_2 Z^{(k)}+\theta)\,\mathrm{d}\hat{W}\big)$ satisfies the reverse Hölder inequality $R_p\big(\hat{\mathbb{P}}\big)$ uniformly in k; compare (3.2). This implies by Theorem 3.3 of Kazamaki [11] that the $BMO\big(\hat{\mathbb{P}}\big)$-norm of $\int Z^{(k)}\,\mathrm{d}\hat{W}$ is bounded uniformly in k. One can now show similarly to the proof of Theorem 2.1 of Frei [7] that $\lim_{k\to\infty}Y_0^{(k)}=Y_0^{(\infty)}$, where $\big(Y^{(\infty)},Z^{(\infty)}\big)$ is the solution of the BSDE related to $V_2^{\hat{\pi}^\star}$. Therefore, we obtain

$$\lim_{k\to\infty}V_2^{\hat{\pi}^{\epsilon_k}}=-\lim_{k\to\infty}\exp\big(\eta_2 Y_0^{(k)}\big)=-\exp\big(\eta_2 Y_0^{(\infty)}\big)=V_2^{\hat{\pi}^\star},$$

which concludes the proof. □

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