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A Novel Triangle-based Method for Scattered Data Interpolation

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Abstract

Local numerical methods for scattered data interpolation often require a smart subdivision of the domain in geometrical polyhedral structures. In particular triangulations in the plane (2D) and tetrahedrizations in the space (3D) are widely used to define interpolation models. In this paper we give a short survey on the main methods for the scattered data problem and we recall preliminaries on triangulations and their related properties. Finally, combining two well-known ideas we present a new triangle-based interpolation method and show its application to a case study.

Mathematics Subject Classification: 41A05, 65D05, 68U05

Keywords: Scattered data, interpolation, triangulation

1 Introduction

Many applications in science and engineering need to construct a multivariate function S_f for which only some samples are known, i.e, the function values

z_i are given on a finite number of uniform or nonuniform points P_i in $\mathbb{R}^s$. Here $\mathbb{R}^s$ denotes the euclidean space $\mathbb{E}_s$, with its affine, metric and topological structures. This problem, commonly referred to as *scattered data interpolation*, is a main issue in several application areas whose partial list includes, for example, medical imaging, learning theory, data mining, numerical solution of partial differential equations and surface reconstruction in computer graphics and data visualization [14]. The scattered data interpolation problem can be formalized as follows [4].

Problem 1.1. *Given n data $\{(P_i, z_i)\}_{i=1,\dots,n}$, with $P_i \in \mathbb{R}^s$, $z_i \in \mathbb{R}$, find a function S_f such that*

$$S_f(P_i) = z_i \qquad i = 1, \dots, n \tag{1}$$

The notation S_f highlights that the values satisfy $z_i = f(P_i)$ for some unknown function f. One dimensional interpolation ($s = 1$) is a classical problem, widely explored and solved. In higher dimensions ($s \geq 2$) things become more difficult. Indeed, in order to achieve a well-posed problem, some a priori assumptions on the P_i-collocation or on the kind of interpolant S_f have to be made. In this work, starting from an overview on the mathematical framework of the problem, we propose a novel triangle-based interpolation method for the 2D case ($s = 2$). The paper is organized as follows: in Section 2 we discuss the mathematical background of the problem, by presenting a short list of the main (global and local) interpolation methods and by recalling preliminaries on triangulations and their related properties; in Section 3, we present our method, discuss some of its issues and show its application to a case study; finally, in Section 4 we draw conclusions.

2 Mathematical preliminaries

Numerical algorithms for solving the scattered data problem are usually classified in two main categories: global methods, where the value of the interpolant at a point P depends on all data points, and local methods, where the value only depends on "nearby" points. Two well-known techniques of the first class are the *Shepard's Method* [5] and the *radial basis functions* [4],[11],[15]:

- in the Shepard's Method the function that solves the problem is a weighted average of the data values at the points P_i. More precisely the interpolant S_f is expressed as:

$$S_f(P) = \sum_{i=1}^{n} \frac{w_i(P)}{\sum_{i=1}^{n} w_i(P)} z_i, \qquad \forall P \in \mathbb{R}^s$$

  where the weights $w_i(P) = d_i^{-\mu}$ (with $\mu = 2$ or $\mu = 4$) depend on the distance $d_i = \|P - P_i\|_2$;

- in a radial basis functions (*RBF*) method the interpolant S_f is a linear combination of radial basis functions $B_i(P) = \phi(\|P - P_i\|_2)$, i.e:

$$S_f(P) = \sum_{i=1}^{n} c_i B_i(P), \qquad \forall P \in \mathbb{R}^s,$$

where $\phi : [0, \infty[\to \mathbb{R}$ is a suitable function. The coefficients $c_1, \dots, c_n$ are obtained by imposing the interpolation conditions (1) and then by solving the linear system $A\mathbf{c} = \mathbf{z}$, where $A_{ji} = B_i(P_j)$, $i, j = 1, \dots, n$, $\mathbf{c} = [c_1, \dots, c_n]^T$ and $\mathbf{z} = [z_1, \dots, z_n]^T$. As exhaustively explained in [11] the well-posedness of such interpolation problem depends on the choice of the function ϕ. Some of the most common functions used in computer graphics applications are: the Gaussian, $\phi(r) = e^{r^2/c^2}$; the thin plate spline, $\phi(r) = r^2 log(r)$; the Hardy's multiquadratic, $\phi(r) = \sqrt{r^2 + c^2}, c \neq 0$.

The use of global methods is not always the best choice (for very large n) since they often involve, as for several RBF functions, the solution of a full linear system of n equations. Alternatively, many local methods have been developed over the last decades. A short list (not exhaustive of all methods) includes:

- *Modified Shepard's Method* [12], an improvement of Shepard's Method in which the weights are defined as[1]:

$$w_i(P) = \left[\frac{(R - d_i)_+}{R d_i}\right]^2 \qquad (\text{with} \;\; d_i = \|P - P_i\|_2, \;\; i = 1, \dots, n)$$

where R is the radius of influence of a node, i.e. Shepard's weights become zero outside the disk of radius R and center P_i.

- *linear triangular* (tetrahedral in 3D) *interpolation method*; given a *triangulation* $\mathcal{T}$ (see below for details), the restriction of the interpolant S_f to a triangle $P_1P_2P_3$ is defined as a linear combination of the values at its vertexes P_1, P_2, P_3, i.e.:

$$S_f|_{P_1P_2P_3}(P) = \lambda_1(P)z_1 + \lambda_2(P)z_2 + \lambda_3(P)z_3, \qquad \forall P \in P_1P_2P_3, \quad (2)$$

where $\lambda_i = \lambda_i(P)$, for $i = 1, 2, 3$, are the barycentric coordinates of P with respect to P_i. It holds that $\sum_{i=1}^{3} \lambda_i = 1$ and $\lambda_1, \lambda_2, \lambda_3 \in [0, 1]$, $\forall P \in P_1P_2P_3$;

- cubic triangular interpolation methods, as the *Clough-Tocher method*; here, in order to obtain C^1-continuity of the interpolant in the *convex hull* (see below for details) of data, Bézier patches are used in three sub-triangles of each triangle of the given triangulation. See [1, 7] for a detailed explanation;

[1] x_+^n denotes the *truncated power function* with exponent n, i.e.: $x_+^n = \begin{cases} x^n & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$

- the *triangle-based blending technique*; in this case, the restriction of the interpolant S_f, to a triangle $P_1P_2P_3$, is defined as in [1, 7]

$$S_f|_{P_1P_2P_3}(P) = w_1 f_1(P) + w_2 f_2(P) + w_3 f_3(P), \qquad \forall P \in P_1P_2P_3, \quad (3)$$

  where $f_i(P)$ is the value at P of the quadratic polynomial which interpolates at P_i and its five nearest neighbors, and $w_i = d_i^k/(d_1^k + d_2^k + d_3^k)$ with d_i distance of P from the triangle edge opposite to P_i. The distances d_i are scaled so that $d_i = 1$ at the vertex P_i and the exponent $k = 3$ is chosen in order to obtain C^1- and C^2-continuity of the interpolation model on edges.

Notice that several local methods, as the last three presented, need a triangulation of the set of points. A complete triangle-based interpolation algorithm for the scattered interpolation problem requires to compute the interpolant function at P and also the following preliminary steps.

(i) **Meshing (and re-meshing)**. Given the set of points $X = \{P_i\}_{i=1,\dots,n}$ compute a triangulation of X (if possible, compute an *optimal* triangulation, such as the *Delaunay* triangulation). There exist several different algorithms for constructing (or adding a new point to) a triangulation: for an overview, we refer to [2, 9, 13].

(ii) **Localization**. Given a point P in the convex-hull of X, find in which triangle (or edge) P is located. Basic algorithms are mentioned in [8].

In the next section we propose a novel triangle-based method for scattered interpolation, assuming that procedures for steps (i) and (ii) are given. First let us provide a rigorous definition of a triangulation and its main properties.

Definition 2.1. *Let be $X \subseteq \mathbb{R}^s$, the* **convex hull** *of X, denoted by $ch(X)$, is the smallest convex set that contains X. If $X = \{P_i\}_{i=0,\dots,k}$ ($k \leq s$) is a set of affinely independent points, the* k**-simplex** *of vertexes $P_0, \dots, P_k$, denoted by $\Delta^k(P_0, \dots, P_k)$, is the convex hull of X. Let be $h < k \leq s$, a h-simplex obtained from a k-simplex Δ^k by deleting $k - h$ vertexes is called a* **face** *of Δ^k.*

In more detail, each point P of a simplex $\Delta^k(P_0, \dots, P_k)$ can be expressed uniquely as a linear combination of its vertexes $P_0, \dots, P_k$:

$$P = \sum_{i=0}^{k} \lambda_i P_i \qquad \text{where} \quad \lambda_0, \dots, \lambda_k \geq 0 \qquad \text{and} \qquad \sum_{i=0}^{k} \lambda_i = 1.$$

The coefficients λ_i are called *barycentric coordinates* of P with respect to P_i. If P belongs to a face of Δ^k, obtained by deleting a vertex P_j, then $\lambda_j = 0$. In particular, if P is a vertex P_i, then $\lambda_i = 1$ and $\lambda_j = 0$ ($j \neq i$). Moreover, if P is an interior point of Δ^k, the barycentric coordinates can be expressed as a ratio of volumes of simplexes:

$$\lambda_i = \frac{vol\big(\Delta^k(P_0, \dots, P_{i-1}, P, P_{i+1}, \dots, P_k)\big)}{vol\big(\Delta^k(P_0, \dots, P_k)\big)} \qquad (i = 0, \dots, k).$$

Here, we are interested in k-simplexes with $k \leq 2$. In particular, note that: a 0-simplex $\Delta^0(P_i) \equiv P_i$ is a point; a 1-simplex $\Delta^1(P_i, P_j) \equiv P_iP_j$ is the segment joining P_i and P_j; a 2-simplex $\Delta^2(P_i, P_j, P_k) \equiv P_iP_jP_k$ is a triangle with vertexes P_i, P_j, P_k. Some finite collections of 0-, 1-, 2-simplexes are called *triangulations*.

Definition 2.2. *Let be $X = \{P_i\}_{i=1,\dots,n} \subset \mathbb{R}^2$. A triangulation $\mathcal{T}$ of the convex hull of X is a subdivision of $ch(X)$ into simplexes such that: if a simplex Δ^h is in $\mathcal{T}$, then each face of Δ^h is in $\mathcal{T}$; if two simplexes are in $\mathcal{T}$ their intersection is either empty or is a common face in $\mathcal{T}$.*

A measure of the density of a triangulation $\mathcal{T}$ is the *mesh* $\mu(\mathcal{T})$, defined as the maximum diameter of the simplexes of $\mathcal{T}$. It can be easily proved that the diameter of a simplex Δ^k is the maximum distance between its vertexes. Generally, given X, there are several admissible triangulations of $ch(X)$. Although all possible triangulations are equivalent from a geometric point of view, usually, for numerical purposes, it is advisable to avoid, as far as possible, triangles with *elongated* or *skinny* shape. This can be done by taking the *Delaunay* triangulation $\mathcal{DT}$: this triangulation is optimal in the sense that maximizes the minimum internal angle among all its triangles; moreover such triangulation has the property that circles circumscribing any of its triangles do not contain other points of X. It can be proved that $\mathcal{DT}$ is unique if no four points of X are co-circular. In Figure 1 we show two different triangulations of a set X of 25 Halton points [4] in the square $[0, 1]^2$.

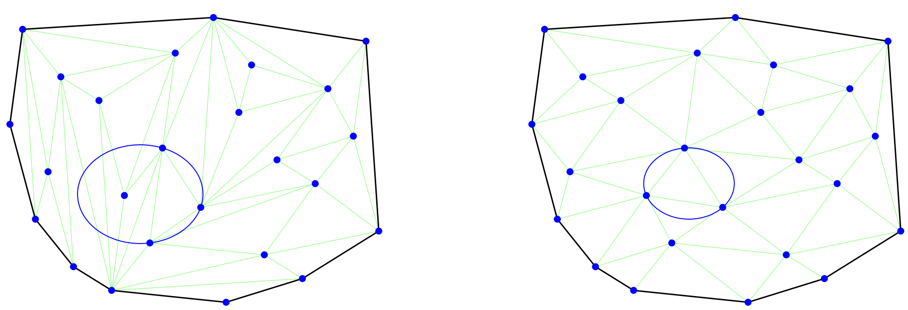

Figure 1: Left: a triangulation of X. Right: the Delaunay triangulation of X.

3 A novel triangle-based method

Let be $X = \{P_i\}_{i=1,\dots,n} \subset \mathbb{R}^2$ a set of points and let be $\mathcal{T}$ a given *point set triangulation* of X (a triangulation of $ch(X)$). All the triangle-based methods shown in the previous section define the interpolant in each triangle and they offer regularity conditions by matching their values on edges. Here, we are interested to obtain the same regularity result by using a technique similar to Shepard's method. Unlike Shepard's our idea is that the closeness of a point P to the vertexes P_i is measured in terms of barycentric coordinates and not by distances. Let us define the method.

Definition 3.1. [The Method]
Let be $X = \{P_i\}_{i=1,\dots,n} \subset \mathbb{R}^2$ and $\mathcal{T}$ a triangulation of $ch(X)$. Let $m \geq 0$ be an integer and let $\omega : [0,1] \to \mathbb{R}$ be a weight function with the properties:
(i) $\omega(\lambda) \geq 0$, $\forall \lambda \in [0,1]$; *(ii) ω is strictly increasing on $[0,1]$;*
(iii) $\omega \in C^m([0,1])$ and $\omega^{(j)}(0) = 0$, for $j = 0, \dots, m$.
The restriction of the interpolant S_f, to a triangle $P_iP_jP_k$ of $\mathcal{T}$, is defined as:

$$S_f|_{P_iP_jP_k}(P) = \frac{\omega(\lambda_i)z_i + \omega(\lambda_j)z_j + \omega(\lambda_k)z_k}{\omega(\lambda_i) + \omega(\lambda_j) + \omega(\lambda_k)}, \qquad \forall P \in P_iP_jP_k, \tag{4}$$

where $\lambda_i, \lambda_j, \lambda_k$ are the barycentric coordinates of P with respect to P_i, P_j, P_k.

Condition (i) on the weight function entails not negativity of the weights $\omega(\lambda_i)$ in (4). Condition (ii) guarantees that a weight $\omega(\lambda_i)$ increases as P approaches the vertex P_i. Finally, condition (iii) ensures that the interpolant is in $C^m\big(ch(X)\big)$. The proof of the continuity ($m = 0$) is immediate: for instance, if two triangles $P_1P_2P_3$ and $P_1P_2P_4$ share a common edge, P_1P_2, then the barycentric coordinates λ_1 and λ_2 of a point $P \in P_1P_2$ are the same in both triangles, while λ_3 and λ_4 are zero; then $\omega(\lambda_3) = \omega(\lambda_4) = 0$ and the values $S_f|_{P_1P_2P_3}(P)$ and $S_f|_{P_1P_2P_4}(P)$ coincide. The proof for the case $m > 0$ may be obtained observing that the partial derivatives of the interpolant S_f are proportional to the derivatives of ω. Then the matching of the interpolant derivatives on edges is due to condition (iii).

Notice that a suitable choice of the weight function makes our model very similar to the triangle-based blending technique ($\omega(\lambda) = \lambda^3$) or to the linear triangular method in (2) (it can be considered a special case of our method with $\omega(\lambda) = \lambda$). In spite of its general formulation, the proposed method preserves the ability of the linear triangular method of well approximating Lipschitz continuous functions, as proved below.

Theorem 3.2. *Let be $X = \{P_i\}_{i=1,\dots,n} \subset \mathbb{R}^2$, $\mathcal{T}$ a triangulation of $ch(X)$ with mesh $\mu(\mathcal{T})$ and $f : ch(X) \to \mathbb{R}$ a Lipschitz continuous function with Lipschitz constant $L > 0$. Let the interpolant S_f be as in Definition 3.1, with $z_i = f(P_i)$, then it holds that:*

$$|S_f(P) - f(P)| \leq L \cdot \mu(\mathcal{T}), \qquad \forall P \in ch(X).$$

Proof. Without loss of generality, let us assume that $P_1P_2P_3$ is a triangle of $\mathcal{T}$ and $P \in P_1P_2P_3$. The statement is proved by these straight inequalities:

$$|S_f(P)-f(P)|=\left|\sum_{i=1}^{3}\frac{\omega(\lambda_i)}{\sum_{i=1}^{3}\omega(\lambda_i)}f(P_i)-f(P)\right|\leq\sum_{i=1}^{3}\frac{\omega(\lambda_i)}{\sum_{i=1}^{3}\omega(\lambda_i)}|f(P_i)-f(P)|$$
$$\leq\sum_{i=1}^{3}\frac{\omega(\lambda_i)}{\sum_{i=1}^{3}\omega(\lambda_i)}L\cdot\|P_i-P\|_2\leq\sum_{i=1}^{3}\frac{\omega(\lambda_i)}{\sum_{i=1}^{3}\omega(\lambda_i)}L\cdot\mu(\mathfrak{T})=L\cdot\mu(\mathfrak{T}).$$

□

The choice of the weight function offers the possibility of tailoring the method by adapting it to specific problems. Examples of weight functions are:
(i) $\omega(\lambda)=\lambda^\alpha$ with $\alpha\in\mathbb{R}^+$; (ii) $\omega(\lambda)=\log(1+k\lambda^\alpha)$ with $k,\alpha\in\mathbb{R}^+$;
(iii) $\omega(\lambda)=\lambda^\alpha e^{-\alpha\lambda}$ with $\alpha\in\mathbb{R}^+$.
In Figure 2 it is shown the behaviour of three different functions S_f which interpolate the Egg-Holder function [16] in 25 Halton points in the square $[0,1]^2$. In this test we used $\omega(\lambda)=\lambda$ (left, the linear triangular method), $\omega(\lambda)=\log(1+1000\lambda^3)$ (center) and $\omega(\lambda)=\lambda^3e^{-3\lambda}$ (right).

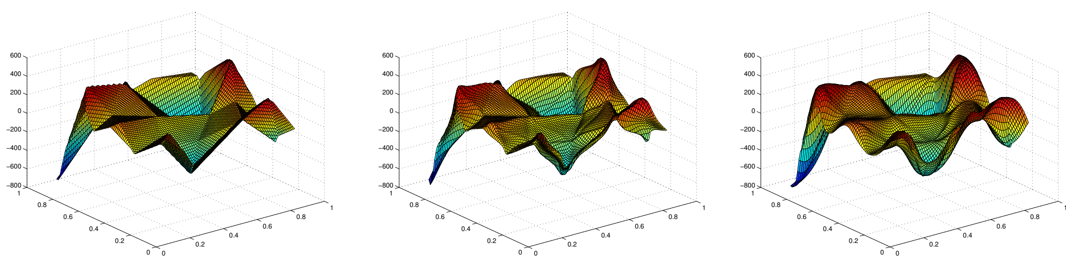

Figure 2: Left: $\omega(\lambda)=\lambda$. Center: $\omega(\lambda)=\log(1+1000\lambda^3)$. Right: $\omega(\lambda)=\lambda^3e^{-3\lambda}$.

4 Conclusions

In this work we have dealt with the scattered data interpolation problem. Firstly we have introduced the mathematical background of the problem, by presenting a short list of the main (global and local) interpolation methods and by recalling preliminaries on triangulations and their related properties. Then we have presented a novel interpolation method similar to the Shepard's method: the main idea is to use the barycentric coordinates as a measure of the closeness between the points. Finally, by means a case study we have shown that out method is capable of maintaining regularity properties despite it is piecewise defined.

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