accurate for values of the suction parameter, $(v_i/u_e)(\mathrm{Re}_x^{1/2})$, less than -2.0. This is verified in both Figs. 1 and 2. However, for plane stagnation-point flow, Figs. 1 and 2 show a considerable discrepancy between actual and asymptotic shear for values of the suction parameter well below -2.0. And, of course, as the suction parameter approaches zero (small condensation rates), the asymptotic expression gives way to the dry-wall value of the shear for the particular geometry involved. This suggests that a simple interpolation formula between the two extremes could be used for the entire range. A formula of the form [4]

$$\tau = [\tau_a{}^n + \tau_d{}^n]^{1/n}$$

was selected, following Acrivos, who suggested a similar relationship for correlating mass-transfer conductances. A value of $n = 1.375$ was found to give good agreement with the numerical results for both the flat-plate and stagnation-point flow over the entire range of suction parameter.

The numerical results of Fig. 2 were obtained by solving the full conservation equations in boundary-layer form for laminar film condensation: the flat-plate results were for pure steam condensing on a vertical flat plate using a full finite-difference analogue in the vapor phase (as in, e.g., [7]) and the stagnation-point results for both steam–air and steam–methanol mixtures flowing vertically down upon a horizontal cylinder where the vapor-phase solution was determined using a similarity transformation [5]. In view of the good agreement, it is suggested that equation (1) be used for all surface geometries.

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Thermal Conductivity of Two-Phase Systems

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Nomenclature

f = volume fraction of solid phase, dimensionless
K_c = thermal conductivity of liquid phase, Btu/hr-ft-deg F
K_d = thermal conductivity of solid phase, Btu/hr-ft-deg F
K_e = effective thermal conductivity of two-phase system, Btu/hr-ft-deg F
P_1 = one-dimensional porosity, dimensionless
P_2 = two-dimensional porosity, dimensionless
r, t = parameters of beta distribution
$\bar{r}, \bar{t}$ = maximum likelihood estimates of r and t
x = random variable defined as the one-dimensional porosity, P_1
Pr = probability

Introduction

A review of the past work on conductance of heterogeneous systems reveals that there exist a relatively large number of approximate relationships for prediction of the effective thermal conductivity of two-phase systems. Most of these relationships utilize two parameters, e.g., thermal conductivity of the pure phases and the volume fraction of each phase, in describing the thermal conductivity of two-phase systems. Recently some investigators [1, 2][3] have attempted to relate the thermal conductivity of two-phase systems to additional parameters which describe the spatial distribution of the two phases. The object of this brief is to present a model, which accounts for the spatial distribution of solid particles, for prediction of the effective thermal conductivity of solid–liquid two-phase systems.

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Theoretical Development

In a theoretical paper Tsao [1] presented a model for prediction of the two-phase thermal conductivity. He considered a cubical liquid–solid system of unit dimensions and proposed the following equation for prediction of the effective thermal conductivity:

$$K_e = \frac{1}{\displaystyle\int_0^1 \frac{dP_1}{K_c + (K_d - K_c)P_2}} \qquad (1)$$

where P_1 is one-dimensional porosity defined as the fraction of the linear space occupied by solids and P_2 is two-dimensional porosity defined as the fraction of the area occupied by solids.

To solve equation (1), a relation between P_1 and P_2 is required. Based on a stochastic model, Tsao proposed the following equation for relating P_1 and P_2:

$$P_2 = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{P_1}^1 \exp\left\{-\frac{1}{2}[(P_1 - \mu)/\sigma]^2\right\} dP_1 \qquad (2)$$

where μ and σ are the mean and the standard deviations of P_1 respectively.

Tsao suggested the normal distribution, equation (2), as an approximation to the point binomial distribution. Since this expression involves a density function which does not integrate to 1 over its sample space ($0 \le P_1 \le 1$), Baxley [3] suggested the beta distribution, which is the limiting case of the point binomial distribution, for relating P_1 and P_2.

$$P_2 = \frac{\Gamma(r+t)}{\Gamma(r)\Gamma(t)} \int_{P_1}^1 x^{r-1}(1-x)^{t-1}dx \qquad (3)$$

where r and t are the parameters of the beta distribution.

Before the beta distribution can be used for prediction of the effective thermal conductivity, the parameters r and t must be estimated. Leek et al. [4] conducted a study to determine the spatial distribution of particles in a solid–liquid two-phase system. The solid phase consisted of uranium-impregnated Pyrex glass cylinders. A mixture composed of 85 percent glycerol and 15 percent benzyl alcohol was used as the liquid phase. A special spectrophotometer was used for measuring P_1. One-dimensional porosity data on two different-sized glass

Table 1 Thermal conductivity of various two-phase systems

System	K_d/K_c	f	K_e experimental Btu/hr-ft-°F	K_e Deshpande Btu/hr-ft-°F
Zinc sulfate–lard	$\frac{0.354}{0.114} = 3.1$	0.1924	0.1425 [12]	0.1387
		0.2273	0.1475	0.1443
		0.2834	0.1550	0.1543
		0.4245	0.1790	0.1679
		0.4560	0.1850	0.1756
		0.4787	0.190	0.1770
Marble–Vaseline	$\frac{1.72}{0.107} = 16.1$	0.1924	0.1375 [12]	0.1640
		0.2273	0.1450	0.1804
		0.2834	0.1640	0.2134
		0.4245	0.250	0.2168
		0.4560	0.270	0.2441
		0.4787	0.297	0.2374
Selenium–polypropylene glycol	$\frac{3.0}{0.081} = 37$	0.1924	0.1250 [2]	0.1376
		0.2273	0.1380	0.1555
		0.2834	0.1630	0.1936
		0.4245	0.240	0.1816
		0.4560	0.260	0.2117
		0.4787	0.278	0.2004
Aluminum oxide–polypropylene glycol	$\frac{22.5}{0.081} = 278$	0.1924	0.220 [2]	0.1728
		0.2273	0.280	0.2077
		0.2834	0.375	0.2915
		0.4245	0.760	0.2167
		0.4560	0.88	0.2711
		0.4787	0.970	0.2423

cylinders (5 mm diameter by 6 mm long and 3 mm diameter by 4 mm long) were obtained. From the data the estimates $\bar{r}$ and $\bar{t}$ were determined using the maximum-likelihood method [5].

To solve equation (1) for the effective thermal conductivity, the two-dimensional porosity P_2 must be known. There are three methods available for determination of P_2 from equation (3).

Determination of P_2 by Integration of Density Function. By expressing the term $(1 - x)^{t-1}$ in binomial series, equation (3) can be readily evaluated to yield

$$P_2 = 1 - \frac{P_1^{\bar{r}}\,\Gamma(\bar{r} + \bar{t})}{\Gamma(1 - \bar{t})\Gamma(\bar{r})\Gamma(\bar{t})} \sum_{i=0}^{\infty} \frac{\Gamma(1 - \bar{t} + i)}{(i + \bar{r})i!} P_1^{\,i} \qquad (4)$$

In equation (4) the maximum-likelihood estimates $\bar{r}$ and $\bar{t}$ have been substituted for r and t.

Determination of P_2 by Paulson Method. Paulson [6] has shown that if a random variable u is distributed according to the F distribution, the probability that its value is less than or equal to F is given by

$$\Pr(u \leq F) \simeq \frac{1}{2}\,[1 + \operatorname{erf}(v/\sqrt{2})] \qquad (5)$$

where

$$v = \frac{(1 - 1/9\bar{t})F^{1/3} - (1 - 1/9\bar{r})}{[(1/9\bar{t})F^{2/3} + 1/9\bar{r}]^{1/2}} \qquad (6)$$

If a random variable s, on the other hand, has the beta distribution, then the probability that its value is less than or equal to P_1 can be expressed as

$$\begin{aligned}\Pr(S \leq P_1) &= \Pr\left[\frac{\bar{t}}{\bar{r}}\left(\frac{s}{1 - s}\right) \leq \frac{\bar{t}}{\bar{r}}\left(\frac{P_1}{1 - P_1}\right)\right] \\ &= \Pr(u \leq F)\end{aligned} \qquad (7)$$

where

$$F = \frac{\bar{t}}{\bar{r}}\left(\frac{P_1}{1 - P_1}\right)$$

Equation (7) is valid since the relation between s and u is $s/(1 - s) = \bar{r}u/\bar{t}$ (Dunn [7]).

Table 2 Statistical analysis of the results

	Average % error	Error variance	Average bias
Deshpande	17.96	0.495	−5.56
Tsao [1]	17.79	0.423	6.88
Baxley [2]	26.16	0.80	27.17
Jefferson [9]	22.42	0.88	18.34
Maxwell [10]	26.26	0.93	20.04
Rayleigh [11]	26.23	0.93	20.08

The two-dimensional porosity can then be calculated as

$$\begin{aligned}P_2 &= 1 - \Pr(s \leq P_1) \\ &= 1 - \frac{1}{2}\,[1 + \operatorname{erf}(v/\sqrt{2})]\end{aligned} \qquad (8)$$

The approximation is valid for $\bar{t} \geq 3/2$ and $P_1 > \bar{r}/\bar{r} + \bar{t}$.

Determination of P_2 from Pearson's Tables. Pearson [8] developed extensive tables for the determination of the beta probabilities. These tables can be used to obtain P_2 from P_1.

Determination of the Effective Thermal Conductivity. After evaluating P_2, equation (1) can be solved numerically for K_e, the effective thermal conductivity of two-phase systems.

Presentation of Results

All calculations were made on the IBM 7040 computer. The two-dimensional porosity P_2 was evaluated for the beta-distribution model by the three methods discussed in this paper.

Thermal conductivity of 10 two-phase systems was determined by equation (1) using data from the large-cylinder model as well as the small-cylinder model. As an illustration, the results for four of these systems, obtained from the large-cylinder model, are presented in Table 1.

A statistical analysis of the results from the large-cylinder model was made. The performance of the beta-distribution model was compared with that of the models of Tsao [1], Baxley [2], Jefferson [9], Maxwell [10], and Rayleigh [11]. These results are presented in Table 2.

Discussion of Results

The maximum-likelihood estimates $\bar{r}$ and $\bar{t}$ were calculated for the six large-cylinder samples and four small-cylinder samples using the experimental P_1 data of Leek et al. [4].

For some samples computational difficulties were encountered in the determination of P_2 by the direct-integration method and the Dunn–Paulson method. Therefore it was decided to resort to the tables of the incomplete beta function [8] for evaluating P_2.

For all the 10 two-phase systems, the authors' large-cylinder model gave good results when the solid volume fraction f was less than 0.3. For higher-volume-fraction solids, the model was unsatisfactory (see Table 1).

One possible reason for this behavior may be the effect of particle interaction on the thermal conductivity of two-phase systems. These effects may be small in a two-phase system with low-volume-fraction solids so that the spatial-distribution model is adequate for predicting the effective thermal conductivity. The particle interaction effects may become significant, however, as the solid volume fraction is increased, so that the two-phase system can no longer be described by the spatial distribution of the solid particles alone. Consequently, the statistical model fails for high-volume-fraction solids. It may also be noted that the deviation of the model-predicted thermal conductivity from the experimental value becomes more pronounced as the ratio K_d/K_c is increased.

With small cylinders the authors' model fails completely, indicating a strong dependence of particle size on the thermal conductivity of two-phase systems.

From Table 2 it can be seen that the authors' model along with Tsao's model gives, for $f \leq 0.30$, better results than any other model tested.

Conclusions

From the results of this study it was concluded that:

1 With large cylinders, the beta-distribution model predicted the two-phase thermal conductivity very well for low-volume-fraction solids ($f \leq 0.30$). The model failed completely for $f > 0.30$.

2 The thermal conductivity of suspensions was dependent on the size of the suspended particles.

3 Although theoretically unsound, the predictability of Tsao's model (large cylinders, $f \leq 0.30$) was as good as that of the authors' model. His model was also inadequate for high-volume-fraction solids.

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On the Freezing of Tissue

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Nomenclature

c_b = blood specific heat
k, k_f = thermal conductivity of unfrozen and frozen phases, respectively
K_0, K_1 = modified Bessel functions
m_b = blood mass flow rate
r, x = position in field
r_0, x_0 = probe radius or half-thickness
R, L = location of frozen–unfrozen interface
S_b, S_m = heat generation due to blood flow and metabolism, respectively
T, T_f = temperatures in unfrozen and frozen phases, respectively
T_b = systemic arterial blood temperature
T_0 = tissue temperature far from probe
T_{pc} = phase-change temperature
T_s = probe surface temperature
θ = nondimensional unfrozen-tissue temperature, $[(T - T_0)/(T_s - T_0)]$
θ_f = nondimensional frozen-tissue temperature, $[(T_f - T_0)/(T_s - T_0)]$
θ_{pc} = nondimensional phase-change temperature, $[(T_{pc} - T_0)/(T_s - T_0)]$
Φ = nondimensional probe surface temperature, $(-k_f/k) \times [(T_{pc} - T_s)/(T_{pc} - T_0)]$
β = blood flow parameter, $m_b c_b r_0^2/k$ or $m_b c_b x_0^2/k$
R, X = nondimensional position, r/r_0 or x/x_0
r^*, x^* = nondimensional ice-front location, R/r_0 or L/x_0

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Cryosurgery, the technique of surgically destroying tissue by use of extreme cold, is a relatively new procedure which accomplishes tissue destruction by dropping the temperature of the target region below the freezing point. The frozen region, the lesion, is created by use of a cryogenic cannula (cryoprobe) which may be of various geometrical configurations. This technique has been used in a host of surgical applications [1][3] which generally require the creation of one of two basic types of lesions, namely, those formed by an external application of the probe or those formed by inserting the probe deep into the tissue. The latter type of lesion formation has been used extensively in brain surgery as a means of destroying cancerous tumors and also as a treatment for Parkinson's disease.

For the most part, cryosurgery, as presently practiced, is highly empirical [2]. Our objective is to show how the analytical methods of heat transfer can be used to predict the steady-state, or maximum, lesion size which may be formed using standard cylindrical or spherical cryoprobes. A third probe configuration, the planar case, is included mainly as a reference datum to depict geometrical effects.

Bio-Heat Transfer Equation

The steady-state energy equation governing the developed temperature field in *in-vivo* tissue takes the form

$$k\nabla^2 T + S_m + S_b = 0 \tag{1}$$

where S_m represents the effect of metabolic heat generation, S_b accounts for the addition or removal of heat by the local blood flow, and k represents the thermal conductivity of the unfrozen tissue. When equation (1) is applied to the frozen tissue the terms S_m and S_b vanish and there results $\nabla^2 T_f = 0$.

The quantity S_m is a function of the oxygen consumption rate of the tissue and, in a simplistic manner, the quantity S_b can be related to the perfusion rate, blood heat capacity, and the difference between the local tissue temperature and the blood temperature as follows:

$$S_b = m_b c_b (T_b - T) \tag{2}$$

[3] Numbers in brackets designate References at end of technical brief.