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The controlled estimation method in the multiobjective linear fractional problem

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Abstract

This paper introduces a new method to estimate the weakly efficient set for the Multiobjective Linear Fractional Programming problem. The main idea is based on the procedure proposed by Tzeng and Hsu (In: G.H. Tzeng, H.F. Wang, U.P. Wen, L. Yu (Eds.), Multiple Criteria Decision Making, Springer, New York, 1994, pp. 459–470), called CONNISE. However, as we will explain in this paper, the CONNISE method is not always convergent for problems with more than two objectives. For this reason, we have developed a new method, called "The Controlled Estimation Method", based on the same concept as CONNISE regarding the decision-maker being able to control distances between points from the estimation set he/she wants to find, while ensuring the method is convergent with problems with more than two objectives. Thus, we propose an algorithm able to calculate a discrete estimation of the weakly efficient set that verifies this property of the CONNISE method, but further, improves it thanks to its convergence and the fact that it satisfies the three good properties suggested by Sayin (Math. Programming 87(3) (2000) 543): Coverage, Uniformity, and Cardinality.

Keywords: Multiobjective programming; Fractional programming

1. Introduction

The term *Fractional Programming* is used to denote a type of optimization problems where the objective function is a quotient, $f(x)/g(x)$, subject to certain constraints. *Linear Fractional Programming* refers to the same kind of optimization problems, but where the numerator and the denominator are affine functions, and the feasible set is a convex polyhedron. Fractional Programming

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has been widely reviewed by many authors (Schaible [1], Stancu-Minasian [2], and [3]), and there are entire books and chapters devoted to this subject (Craven [4], Stancu-Minasian [5], Horst and Pardalos [6], and Horst et al. [7]). Stancu-Minasian has also provided over the years an updated bibliography on the subject (Stancu-Minasian [8–12]).

This paper will focus on the *Multiobjective Linear Fractional Programming* problem (MOLFP), i.e., we will be dealing with the following problem:

$$\begin{aligned} \max \quad & \left\{ \varphi_1(x) = \frac{c_1^t x + \alpha_1}{d_1^t x + \beta_1}, \ldots, \varphi_p(x) = \frac{c_p^t x + \alpha_p}{d_p^t x + \beta_p} \right\} \\ \text{s.t} \quad & Ax \leqslant b, \\ & x \geqslant 0, \end{aligned} \qquad \text{(MOLFP)}$$

where $c, d \in \mathbb{R}^n, \alpha_i, \beta_i \in \mathbb{R}; A \in M_{mxn}(\mathbb{R})$ and $b \in \mathbb{R}^m$. Let X be the feasible set for this problem; that is, $X = \{x \in \mathbb{R}^n / Ax \leqslant b, x \geqslant 0\}$; and $f_i(x) = c_i^t x + \alpha_i$, $g_i(x) = d_i^t x + \beta_i$.

In this problem the definitions of efficient point and weakly efficient point used in Multiobjective Programming are extended in a natural way. From now on, we will denote as E the efficient set of the problem, and E^{w} the weakly efficient set.

The method introduced in this paper is inspired by a procedure suggested by Tzeng and Hsu [14], CONNISE, which is a combination of the Constraint Method and the NISE Method, Cohon et al. [15], for solving multiobjective linear programming problems.

The CONNISE method has been specially developed for Multiobjective Linear Fractional Programming. The CONNISE method is based on finding an estimation of the weakly efficient frontier ($\varphi(E^{\mathrm{w}})$) containing points from all regions of the set, such that the distances between these points do not exceed a given quantity previously established by the decision-maker. However, although the idea is good, our experience of this method for solving problems with more than two objectives reveals that convergence is not always attained. The next section will be devoted to justifying this assertion in detail.

Given that the idea of the decision-maker controlling the distance between the points of the estimation obtained is highly interesting, we decided to develop a method using the same concept, which calculates a discrete estimation of the weakly efficient set of the MOLFP problem using the Constraint Method as a basis, but whose convergence is ensured. Besides controlling the distances, the method we introduce satisfies the three properties proposed by Sayin [13] for discrete estimations of the efficient and weakly efficient sets of multiobjective problems: Coverage (i.e., the estimation must have points from all the regions of the set), Uniformity (the estimation should not include redundant information), and Cardinality (the estimation should have a reasonable number of points).

The rest of the paper is as follows.

The next section will be devoted to a brief description of the CONNISE method and the reasons for asserting that this method is not convergent, which is the main motivation for developing a new method. The Controlled Estimation Method is introduced in Section 3, with a description of its theoretical framework as well as a proof of its convergence and the fact that it satisfies the required property of control of the distances between the estimation points, and the properties suggested by Sayin. Then, we introduce the algorithm step by step in Section 4, and illustrate it with an example. Finally, Section 5 presents the conclusions derived from our work.

2. The CONNISE method

We begin this section by briefly reviewing the CONNISE method, proposed by Tzeng and Hsu [14], for the MOLFP problem. Following the description of the procedure used in this method, we demonstrate the lack of convergence. To this end, a linear fractional problem with three objectives is used as an example. As we will see, the problem lies in the fact that setting the bounds between certain values in the Constraint Method does not necessarily mean that the functions evaluated in the optimum solution will also be between those values.

In the CONNISE method, after calculating the Euclidean distance between the ideal and the anti-ideal vector of the problem, the decision-maker is asked to give the percentage of this distance he/she is willing to set as the maximum distance between the points obtained by the method (always in the objective space). The value resulting from applying the chosen percentage to the distance between the ideal and the anti-ideal is denoted by c^*. From this point onwards, the iterative process of the method is laid out. Each step aims at solving a constraint problem, and so the function chosen as the one to be optimized is maintained throughout the whole process. This method is based on choosing the bounds in the constraint problems which are different for each iteration.

In the first iteration, bounds are set between the ideal value (denoted as D^*) and the anti-ideal value (D_-) previously obtained (the mean point of this value is usually taken). Given that the resulting solution to the constraint problem—in the objective space—is denoted by $D^k = \{\varphi_1^k, \varphi_2^k, \ldots, \varphi_p^k\}$, then the CONNISE method divided the objective space into 2^p subspaces, each of them with different extreme values (called "ideal" and "anti-ideal" values of the subspace), where $D_1^* = D^*, D_{1-} = D^k; D_{2p}^* = D^k, D_{2p-} = D_-$, and the remaining subspaces bounded by different component-wise combinations between these three vectors.

For each subspace established in the previous step (except for the first and the last one, since they will not give weakly efficient points), the "maximum error possible" (c_i) is calculated as the normalized distance between the ideal and anti-ideal solutions of this ith subspace. This distance c_i will be compared to the maximum distance allowed, c^*.

The iterative process is thus established. Taking into account that we are in the ith subspace, a new constraint problem is solved by taking the bounds between the extreme values of this subspace, that is, between D_i^* and D_{i-}. With the solution obtained, the subspace is again split into $2^p - 2$ subspaces, and the distances, c_i, are measured again, and so on. The algorithm ends when all the maximum errors obtained in the different subspaces that have been generated are lower than the allowed maximum distance c^*. It is necessary to obtain a list of solutions D^k such that $c_i < c^*$ for each i in *all* the subspaces created.

Thus, this method assumes that the errors calculated, c_i, get smaller and thus there will come a time when all the errors will be smaller than the one chosen by the decision-maker, c^*. For the errors to grow smaller (and since the errors are calculated from the distance between the extreme values of the subspace) the solution point in the given subspace should lie between the corresponding ideal and anti-ideal of such a subspace. However, this does not always have to be the case.

Let us assume that we are not in the first iteration of the problem, and we are in a given subspace "s" such that the associated error is $c_s \geqslant c^*$, which means that we must stay in this subspace. Then, a constraint problem is solved where the bounds are taken from between the vectors D_s^* and D_{s-} and we obtain the corresponding solution D_s^k. The fact that the bounds were taken from between the ideal and the anti-ideal values of the subspace (since this is not the first iteration and that these values have been created in an artificial way without representing any *actual* ideal or anti-ideal value) it does not necessarily mean that the solution D_s^k is found among the vectors D_s^* and D_{s-} component-wise. In other words, D_s^k can be such that $dist(D_s^k, D_s^*) > dist(D_{s-}, D_s^*)$ or $dist(D_s^k, D_{s-}) > dist(D_{s-}, D_s^*)$, and in such a case, the error associated with the subspaces arising from the current s will not be strictly smaller than c_s; on the contrary, it will be greater. Therefore, the method will never stop iterating in this subspace.

Let us demonstrate this by way of an example. We will use the example Kornbluth and Steuer used [16] to illustrate their method to locate all the vertices of the weakly efficient set of the MOLFP problem. The problem is the following:

Example 1. Let the problem be

$$\begin{aligned} \max \quad & \left\{ -x_1 + x_2, \frac{x_1 - 4}{-x_2 + 3}, \frac{-x_1 + 4}{x_2 + 1} \right\} \\ \text{s.t} \quad & -x_1 + 4x_2 \leqslant 0, \\ & x_1 - 0.5x_2 \leqslant 4, \\ & x_i \geqslant 0, \quad i = 1, 2. \end{aligned} \tag{1}$$

We apply the CONNISE algorithm to it. The calculation of the ideal and anti-ideal vectors for this problem yields the following results:

$D^* = (0, 0.307692, 4)$, $\quad D_- = (-3.42857, -1.333333, -0.266667)$.

In order to apply the Constraint Method, the second objective function is chosen to be optimized. Once the decision-maker has provided a percentage regarding the distance between these values as the maximum distance between the points that will be obtained, the problem to solve is as follows in the first iteration:

$$\begin{aligned} \max \quad & \frac{x_1 - 4}{-x_2 + 3} \\ \text{s.t} \quad & -x_1 + 4x_2 \leqslant 0, \\ & x_1 - 0.5x_2 \leqslant 4, \\ & -x_1 + x_2 \geqslant -1.714285, \\ & \frac{-x_1 + 4}{x_2 + 1} \geqslant 1.86667, \\ & x_i \geqslant 0, \quad i = 1, 2. \end{aligned}$$

The solution to this constraint problem, for the objective space, is the following:

$D^1 = (-1.714285, -0.749709, 1.86667)$

So, when establishing the splitting subspaces of the objective space, we have as extreme values as in Table 1:

Table 1
Values of the ideal and anti-ideal solutions in the subspaces created

Subspace	Ideal solution	Anti-ideal solution
1	$D_1^* = (0, 0.307692, 4)$	$D_{1-} = (-1.714285, -0.749709, 1.86667)$
2	$D_2^* = (-1.714285, 0.307692, 4)$	$D_{2-} = (-3.42857, -0.749709, 1.86667)$
3	$D_3^* = (0, -0.749709, 4)$	$D_{3-} = (-1.714285, -1.33333, 1.86667)$
4	$D_4^* = (0, 0.307692, 1.86667)$	$D_{4-} = (-1.714285, -0.749709, -0.26667)$
...	...	...

In order to find the counterexample, let us assume that we are iterating in the fourth subspace; that is, we are in a subspace of the objective space where the extreme values are the following:

$D_4^* = (0, 0.307692, 1.86667)$,

$D_{4-} = (-1.71429, -0.749709, -0.266667)$.

Using again the Constraint Method by setting as bounds the mean points between these ideal and anti-ideal values for the remaining objective functions, we obtain the following problem:

$$
\begin{aligned}
\max \quad & \frac{x_1 - 4}{-x_2 + 3} \\
\text{s.t} \quad & -x_1 + 4x_2 \leqslant 0, \\
& x_1 - 0.5x_2 \leqslant 4, \\
& -x_1 + x_2 \geqslant -0.857145, \\
& \frac{-x_1 + 4}{x_2 + 1} \geqslant 0.8, \\
& x_i \geqslant 0, \quad i = 1, 2.
\end{aligned}
$$

and the result is

$D_4^k = (-0.857145, -1.04762, 3.14286)$.

Note that the components of this point are not among the ideal and anti-ideal components of such a subspace:

$D_4^* = (0, 0.307692, 1.86667)$,

$D_{4-} = (-1.71429, -0.749709, -0.266667)$.

Thus, for the second objective function, the solution D_4^k takes a value *smaller* than the corresponding anti-ideal, and for the third function, it is *greater* than the ideal of that subspace. This means that the errors resulting from possible subspaces obtained from these three solutions *will not be increasingly smaller*, and so the iteration will never end via this subspace. In other words, the CONNISE method will not be convergent for this problem.

Although initially we thought that the solution to the problem would lie in choosing other bounds between the ideal and the anti-ideal vectors (and not always the mean values), further analysis of the results obtained revealed that this was not the case. Even with a more rational selection of bounds, there always comes a moment when the errors arising from a given subspace do not decrease, and therefore, the algorithm is not convergent.

3. The controlled estimation method

We now introduce the method suggested in this paper. As outlined earlier, our objective was to obtain a method with the same good property of distance control as the CONNISE method, but to eliminate its disadvantage and ensure convergence. However, given that we are dealing with a discrete estimation of the weakly efficient set of a multiobjective problem, we have created a method that satisfies the three properties which according to Sayin [13], any estimation should satisfy: Coverage, Uniformity, and Cardinality.

Besides fulfilling these three properties, the proposed method gives the decision-maker the possibility of choosing whether the estimation is established in relation to the set E^{w}, or in relation to its image $\varphi(E^{\mathrm{w}})$.

Let us now view the procedure followed by the controlled estimation method.

We begin by calculating the pay-off matrix of the problem in order to obtain the ideal and the anti-ideal points for each objective function, x_i^*, and their respective values in the objectives space, φ_i^*. We also calculate the anti-ideal values φ_{i-}, $i=1,\dots,p$, and then we calculate the distance between the ideal and the anti-ideal points in $\mathbb{R}^p$ (using the norm L^∞), that is,

$$\mathscr{E} = \max_{i=1,\dots,p} |\varphi_i^* - \varphi_{i-}|.$$

In this way, $\mathscr{E}$ is a bound for the maximum distance that can be between two points of the weakly efficient frontier.

If the decision-maker was interested in estimating the weakly efficient set E^{w}—in the decision space instead of in the objective space—we will also need to make a previous estimation of the maximum distance possible between two points of this set. To achieve this, let us maximize and minimize the ideal points of each objective function component-wise. That is, if $x_i^* = (x_{1i}^*, x_{2i}^*, \dots, x_{ni}^*)$, let us build the points $x\_\max = (\max_{i=1,\dots,p} x_{1i}^*, \max_{i=1,\dots,p} x_{2i}^*, \dots, \max_{i=1,\dots,p} x_{ni}^*)$ and $x\_\min$ in a similar way by minimizing component-wise. Thus, the distance L^∞ between this two points of $\mathbb{R}^n$, denoted by $\mathscr{D}$, is considered to be the reference of the maximum error that can be between two points of the estimation of the weakly efficient set. In other words, it is $\mathscr{D} = \|x\_\max - x\_\min\|_\infty$.

As with the CONNISE method, we then ask the decision-maker to provide a percentage of this maximum error, which in both cases will be the maximum distance between the solution points. Then, we use the *Parametric Constraint Method* to obtain a first estimation of the set sought. This method is a parametric version of the Constraint Method, where we progressively take one objective function from all the ones available, and the bounds for the rest of them are taken by varying between the corresponding ideal and anti-ideal in a uniform way. This version of the Constraint Method permits us to estimate the set E^{w}, in such a way that no region from the set is uncovered. We briefly explain this.

The idea is to take a number of divisions of the anti-ideal–ideal interval (m) of the problem, where the bounds k_i will be calculated. Each objective function is consecutively chosen as the function to be optimized and all possible bound vectors $k=(k_i)$, $i=1,\dots,p$, $i \neq j$, fulfilling the relationship

$$k_i = \varphi_{i-} + (s_i/m)\cdot(\varphi_i^* - \varphi_{i-}), \quad \text{with } s_i \in \{0,1,2,3,\dots,m\}$$

are generated for each function.

We know that the Constraint Method produces Pareto weak optimum solutions without the need to fulfill any other condition. Thus, if the feasible set is convex and all the objective functions are quasi-concave or concave, then the reciprocal case is true. Such a result was proved by Ruiz-Canales and Rufián-Lizana [17]. In the Linear Fractional case, these conditions are verified and thus we can be sure that by using the Constraint Method we obtain weakly efficient solutions; and also *all the weakly efficient solutions can be obtained by solving a constraint problem.* Therefore, the great advantage of using the parametric version of the Constraint Method is the fact that, thanks to this result, the solution set generated contains points from all the regions of $\varphi(E^{\mathrm{w}})$ since the objective function has been varying among the one available, and the bounds used when applying the Constraint Method have also been changing. The disadvantage of the Parametric Constraint Method is that it provides the decision-maker with a large amount of useless information, because many of the points obtained are either the same or very close to others already obtained, and so the same information is repeated over and over again. The Controlled Estimation Method will solve this problem.

In the Controlled Estimation Method, the percentage the decision-maker has initially established is used—after being divided by 100—to decide the number of interval divisions used to apply the Parametric Constraint Method. If d is the percentage established by the decision-maker, the integer part of $1/d$ is the number of interval divisions that are used when applying the Parametric Constraint Method. After applying the Parametric Constraint Method as stated, we obtain a series of solution points (all of them weakly efficient, since they are the solutions to a constraint problem). If we are interested in the estimation of $\varphi(E^{\mathrm{w}})$, we will use their images.

As stated, we can be sure that the solution set generated so far contains points from all regions of E^{w}, or $\varphi(E^{\mathrm{w}})$. In other words, the Coverage of the first estimation is ensured. Obviously, the coverage level is given by the number of interval divisions used, since the greater the number, the more constraint problems are solved, and as a consequence the coverage of this first estimation is greater. Next we deal with the Cardinality and Uniformity of the estimation.

In order to solve this issue, we will filter the points obtained in the previous step and the decision-maker can choose whether the filtering will be done in the objective space or in the decision space. To this end, let us consider a filtering range $\varepsilon = \alpha d$, where $\alpha \in (0, 1)$ (usually $\alpha = 0.8$).

Let us assume first that the decision-maker is interested in the objective space. Given the filtering range, ε, we measure the distances of the values for each objective function in the points obtained, and these distances are compared to $\varepsilon/|\varphi_i^* - \varphi_{i\_}|$. If, for every $i = 1, \ldots, p$, this distance is less than the value compared to, one of the two points being compared is eliminated from the point list. This is done with all the points in the list, that is, with all the points that were obtained by applying the Parametric Constraint Method.

Assuming that the decision-maker was interested in the decision space, the distance between the points would be measured and compared, component-wise, with $\varepsilon/|x\_\mathrm{max}_j - x\_\mathrm{min}_j|$ for $j = 1, \ldots, n$.

Once the filtering is concluded, we can be sure that we are left only with those points that—either in the objective space or in the decision space—have a distance between each other, at least in one component, that is less than or equal to $\varepsilon/|\varphi_i^* - \varphi_{i\_}|$ or $\varepsilon/|x\_\mathrm{max}_j - x\_\mathrm{min}_j|$, respectively. In this way, Uniformity is ensured in the sense that the estimation does not include any redundancy. Cardinality is also be ensured since both properties are closely related, i.e., if the estimation does not contain redundancies, the number of points included has been necessarily reduced.

However, when creating the Controlled Estimation Method we also aimed at guaranteeing the decision-maker that the points obtained estimate either E^{w}, or $\varphi(E^{\mathrm{w}})$, with a maximum error equal to a percentage d of the maximum distance, $\mathscr{D}$ or $\mathscr{E}$. To achieve this, the following step is to calculate the distances between the points obtained after filtering, and so we hold those whose distances to their closest point is higher than the permitted value $d\mathscr{D}$ or $d\mathscr{E}$. These are called "loose points". If such points do not exist, the method ends by displaying the points obtained after the filtering and ensuring that we have a non-redundant estimation of the set sought and that the distances between points are controlled by the decision-maker.

On the other hand, if we obtain loose points, the algorithm has to carry on iterating. For each loose point found, we consider its closest point in the estimation, take the values of the objective functions in these two points, and use them as the new ideal and anti-ideal vectors to once again apply the Parametric Constraint Method, where the number of interval divisions increases by the unit, and the bounds vary, in this case, within the values of the functions in the loose point and its closest point. In this way, we are sure to obtain a set of new weakly efficient points whose images are found in the region of the E^{w} set less covered initially. We will probably find weakly efficient points from other regions of the set—this is well-illustrated in the example given in the next section—since changing the bounds within certain values in order to apply the Constraint Method does not mean that all the resulting points will necessarily be found among these values. However, in contrast to the CONNISE method, this fact is no hindrance in the Controlled Estimation Method, since a new filtering is carried out taking into account both the new points as well as those obtained in the previous step.

So, as we have just said, in order not to give the decision-maker recurrent information, the next step is to again filter the new points obtained while also including those obtained in the previous step. However, to make sure that not too many points are discarded and a loop is not created, the filtering range is increasingly reduced. Every time an iteration is carried out during the filtering, we take as new ε the value resulting from multiplying α—recall that α was smaller than the unit—by the earlier ε. In this way, as the filtering range decreases, the possibility of falling into a loop is eliminated.

Once this is done for all the loose points in the first round, we are left with a much more refined estimation of the set, whether, E^{w} or $\varphi(E^{w})$, where the uniformity is now given by the new smallest ε that has been considered. This process will have to be repeated until, for all the points obtained, the distances to their closest point in the estimation are smaller than the value established by the decision-maker ($d\mathscr{D}$ or $d\mathscr{E}$). In other words, the algorithm ends when we do not find any other loose point among the points we have in the estimation. In this way, the control of distance between the points in the estimation is ensured.

Next, we will show the convergence of the Controlled Estimation Method, before presenting a step-by-step procedure of the method in the next section.

Our method can ensure convergence mainly because, in the Linear Fractional Multiobjective problem, *all the weakly efficient points can be obtained through the constraint method.* As shown, the Controlled Estimation Method makes use of a parametric version of the Constraint Method combining all the objective functions available and taking the bounds in a uniform way with an increasingly finer sweep of the bounds in the regions which have not been so well covered by previous iterations. The application of the Parametric Constraint Method in an iterative and increasingly more refined way—and given that all the points of the estimated set have to be necessarily obtained by solving a constraint problem—guarantees that we obtain new weakly efficient points close to the loose points during the pre-filtering estimations.

This fact, together with the gradual reduction of the filtering range—i.e., the same points will not always be discarded and so there is no danger of falling into a loop—means that at some stage there will be no more loose points. As a consequence, the convergence of the Controlled Estimation Method is proved.

However, the more the iterations needed to avoid loose points, the worse the uniformity of the estimation will be, since the filtering range has been reduced.

4. The algorithm

We will now present and develop the proposed algorithm in detail assuming that the decision-maker is interested in points from the objective space. The algorithm would be analogous for the decision space.

Algorithm.

Step 1. Calculate $\mathscr{E} = \max_{i=1,\dots,p} |\varphi_i^* - \varphi_{i\text{-}}|$. Ask the decision-maker for the percentage d wanted. Let $\varepsilon = d/100$.

Let the solution lists be, $S = \emptyset$, $A = \emptyset$. (S: *Points of current estimation*, A: *Loose points*.)

Step 2. Let $m = E(1/d)$. Iterate using the Parametric Constraint Method with this number of interval divisions.

The solutions obtained will be added to the list S.

Step 3. Let $\varepsilon = \alpha\varepsilon$, with $\alpha \in (0, 1)$. Let us assume that $S = \{x^1, \dots, x^N\}$.

For each x^j from S, take x^k of S for every $k = 1, \dots, N$; $k \neq j$.

If $|\varphi_i(x^j) - \varphi_i(x^k)| < \varepsilon / |\varphi_i^* - \varphi_{i\text{-}}|$ for every $i = 1, \dots, p$, then eliminate point x^k from the list S. Let $N = N - 1$.

Carry out this procedure until there are no more points to compare in S.

Step 4. Given that $S = \{x^1, \dots, x^N\}$, for each x^j, $j = 1, \dots, N$ find the closest point x^k, $k = 1, \dots, N$; $k \neq j$ (in the objective space) with the distance d_∞, to x^j. Let us assume that the closest point is x^K.

If $\|\varphi(x^j) - \varphi(x^K)\|_\infty > \mathrm{d}\mathscr{E}$, then x^j is a loose point. Introduce x^j in the list A.

Repeat this step for every x^j of S. Go to Step 5.

Step 5. If $A = \emptyset$, **END**. The solution points are those included in the list S.

If $A \neq \emptyset$, go to Step 6.

Step 6. Let $m = m + 1$.

Assuming that $A = \{x^1, \dots, x^M\}$, for each x^j of A, take the point x^k of S closer to x^j, calculated in Step 4.

Take $\varphi(x^j)$ and $\varphi(x^k)$, and make an iteration using the Parametric Constraint Method where the bounds are taken from these values, and with the number of interval divisions m. Add the solutions obtained to the list S.

Once this step is carried out for every point of A, go to Step 7.

Step 7. Make $A = \emptyset$ and go back to Step 3.

Let us conclude the development of the method suggested by illustrating its use with a specific problem.

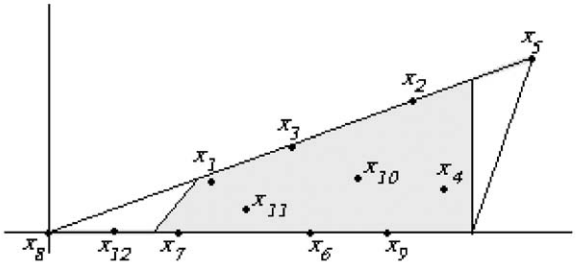

Fig. 1. Problem set E^w and the points obtained in the estimation.

Example 2. Let us apply the Controlled Estimation Method to problem (1) used to demonstrate that the CONNISE method did not converge

$$\begin{aligned} &\max \quad \left\{-x_1+x_2, \frac{x_1-4}{-x_2+3}, \frac{-x_1+4}{x_2+1}\right\} \\ &\text{s.t} \quad -x_1+4x_2 \leqslant 0, \\ &\qquad x_1-0.5x_2 \leqslant 4, \\ &\qquad x_i \geqslant 0, \quad i=1,2. \end{aligned} \tag{2}$$

Let us assume that we are interested in estimating the set E^w (in this way, we have the possibility to depict the estimation graphically). The distance between the ideal and the anti-ideal vectors of this problem is calculated, $\mathscr{D} = 4.57143$, then we assume the decision-maker wants a 25% error on this distance. This means that the first number of interval divisions to be applied is 4. After applying the Parametric Constraint Method with this number of interval divisions, we obtain 49 weakly efficient points covering the whole set, but not all of them are different from each other. A filtering process is applied to these points with a range of $\varepsilon = 0.8 \times 0.25 = 0.2 (\alpha = 0.8)$. After this procedure, 8 points are left ($x_1, \ldots, x_8$ in Fig. 1) distributed throughout the set E^w, and with no redundancy in the information provided.

When checking the distance between these points, we see that point $x_8=(0,0)$ is a loose point, since the distance to its closest point in the decision space, $x_7 = (1.23077, 0)$, is 1.23077, which is greater than the distance chosen by the decision-maker (in this case, $0.25 \times 4.57143 = 1.142857$). Therefore, setting the bounds by moving among the values the objective functions take between points x_7 and x_8, the Parametric Constraint Method is applied again, but the new number of interval divisions is now 5, and then we filter the points obtained using a new filtering range, $\varepsilon = 0.8 \times 0.2 = 0.16$.

After this step, two new points are obtained for the estimation—x_9 and x_{10} in Fig. 1—but, when calculating the distances of these two points plus those previously obtained, $x_8 = (0, 0)$ is again a loose point, since the closest point to it in this new estimation is still $x_7 = (1.23077, 0)$.

Thus, we have to repeat the process, although now the points obtained after applying the Parametric Constraint Method are filtered with a range of $\varepsilon = 0.8 \times 0.16 = 0.128$. Then, the estimation increases in two new points, $x_{11} = (2.46154, 0)$ and $x_{12} = (0.61539, 0)$, the latter being located between x_8 and x_7, and thus when checking for loose points no more are found and so the algorithm ends.

The final estimation of the set E^w, $\{x_1, x_2, \ldots, x_{12}\}$ has the required properties: a reasonable number of elements to be offered to the decision-maker; it has no redundancies, and it also covers the totality of the set E^w ensuring that in the estimation the maximum distance from a point to its closest one is smaller than or the same as the distance requested by the decision-maker. The points obtained are shown in Fig. 1.

If another error percentage had been chosen, the results would have been analogous. As expected, the smaller the error allowed, the greater are the number of points per estimation. For instance, in our example, using a 10% error on the distance between the ideal and anti-ideal vectors, we make a first estimation resulting in 30 points with two loose points and subsequently, carry out a second iteration that yield 37 points. These 37 points are distributed throughout the weakly efficient set, and none of them are a loose point. The filtering range, which to a certain degree measures uniformity, has been reduced in this instance to a value of $\varepsilon = 0.064$. On the other hand, if the decision-maker allows a 50% error over the maximum distance, our example would not yield any loose points in the first iteration, and the final iteration would yield only 4 points, presenting a filtering range of $\varepsilon = 0.4$. Therefore, we think it is important to be able to chose a greater or smaller percentage according to how fine or coarse is the estimation required.

The Controlled Estimation Method has been implemented in a software package for Windows, called *PFLMO*, together with other solution search methods for the MOLFP problem. The *PFLMO* software is one of the few applications including all the main methodologies available for solving MOLFP problems.

We have tested our method with this software in order to test its efficiency and its compliance with the properties proposed by Sayin [13]. The testing has been carried out by solving a large number of test problems randomly generated and with different dimensions. The results were positive in all cases.

5. Conclusions

We have developed a new method to solve Multiobjective Linear Fractional problems which uses generative techniques and is called the Controlled Estimation Method. As demonstrated in this work, this new method is able to establish a discrete estimation of the problem's weakly efficient set such that the distances between the values of the objective functions are less than the value set by the decision-maker. This concept is behind the creation of the new method. We think that it is important to offer the decision-maker the possibility to obtain an estimation of the efficient set with some control over the errors between the estimation points, although this should be done without including redundant information.

The method we have just described offers an estimation of the chosen set, E^{w} or $\varphi(E^{\mathrm{w}})$, that is given by those points included in the list S once the algorithm concludes. The points of this set are all weakly efficient points of the multiobjective linear fractional problem, and the estimation of the set (or its image), is such that the following properties are verified:

- It contains points from all the regions in the set (Coverage).
- It is uniform in the sense of not offering recurrent information. The points in this estimation have a distance between each other, component-wise, of at least ε, which has been normalized by using the maximum distance of the set (Uniformity).
- Thanks to this uniformity, the estimation has a good degree of cardinality in the sense that the decision-maker is not given a large number of points of the set (Cardinality).

- Finally, the errors between the estimation points do not exceed the maximum established by the decision-maker by a percentage of the maximum distance between two points of the set (the CONNISE method tried to fulfill this property).

The method has been illustrated by using an example that clearly shows these properties.

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