Journal of Mathematical Inequalities
Volume 7, Number 2 (2013), 227–237
doi:10.7153/jmi-07-21

NON–SYMMETRIC STOLARSKY MEANS

SAAD IHSAN BUTT, JOSIP PEČARIĆ AND ATIQ UR REHMAN

(*Communicated by A. Vukelić*)

Abstract. In this paper we construct n-exponentially convex functions and exponentially convex functions using the functional defined as the difference of the right parts of the Hermite-Hadamard inequality, for different classes of functions. Applying these results on some starshaped functions, we derive non-symmetric means of Stolarsky type.

1. Introduction

Let us consider the sets of continuous, convex, starshaped and superadditive functions on $[a,b]$ respectively, given by

$$\begin{aligned}
C[a,b] &= \{f : [a,b] \to \mathbb{R} \text{ continuous}\},\\
K[a,b] &= \{f \in C[a,b]; f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y),\\
&\qquad\qquad\qquad\qquad \forall\, x,y \in [a,b], \forall\, \lambda \in [0,1]\},\\
St[a,b] &= \{f \in C[a,b]; (f(x)-f(a))/(x-a) \text{ is an increasing function} \;\forall\, x > a\},\\
S[a,b] &= \{f \in C[a,b]; f(x)+f(y) \leqslant f(x+y-a)+f(a),\ \ \forall\, x,y,x+y-a \in [a,b]\}.
\end{aligned}$$

Let us denote integral arithmetic mean of f on $[a,b]$ and arithmetic mean of a and b as follows:

$$A(f;a,b) = \frac{1}{b-a}\int_a^b f(x)dx \quad \text{and} \quad A(a,b) = \frac{a+b}{2}. \tag{1}$$

The inequalities of Hermite-Hadamard, valid for every function f from $K[a,b]$ are:

$$f\left(A(a,b)\right) \leqslant A(f;a,b) \leqslant A(f(a),f(b)). \tag{2}$$

For $a=0$ we denote by $C(b)$, $K(b)$, $St(b)$ and $S(b)$ the corresponding set of functions, submitted also to the condition $f(0)=0$.

A. M. Bruckner and E. Ostrow have proved in [1] the strict inclusions:

$$K(b) \subset St(b) \subset S(b).$$

The following results are given in Gh. Toader [7]. For positive identity function, $p(x) = x$, the following inequality

$$\int_0^x p(t)\left[\frac{f(x)}{x} - \frac{f(t)}{t}\right] dt \geqslant 0, \ \ \forall x \in [0,b] \tag{3}$$

holds for every $f \in S(b)$.

Mathematics subject classification (2010): 26D15.
Keywords and phrases: Convex function, log-convex functions, power sums, mean value theorems.

Paper JMI-07-21

REMARK 1.1. Of course, for $f \in St(b)$ the inequality (3) is valid for all positive p.

LEMMA 1.2. *For every $f \in S(b)$, the following inequality holds*

$$\int_0^x f(t)dt \leqslant \frac{xf(x)}{2}, \ \forall x \in [0,b]. \tag{4}$$

We can write (4) as:

$$\frac{1}{x}\int_0^x f(t)dt \leqslant \frac{f(x)+f(0)}{2}, \quad \text{where} \quad x \neq 0, \tag{5}$$

which is one of Hermite-Hadamard's inequalities. In (5) we see that the inequality

$$A(f;a,b) \leqslant A\left(f(a),f(b)\right),$$

holds for all $f \in S(b)$. Since $St(b) \subset S(b)$, the above inequality also holds for all $f \in St(b)$.

Gh. Toader [7] considered the above means and proved the following result.

THEOREM 1.3. *Let $f \in St[a,b]$. Then the following inequality is valid:*

$$A(f;a,b) \leqslant A\left(f(a),f(b)\right), \tag{6}$$

where $A(f;a,b)$ and $A(a,b)$ are defined in (1).

REMARK 1.4. If $(f(x)-f(a))/(x-a)$ is strictly increasing for $x \in [a,b]$, then strict inequality holds in (6).

In the next section we prove two mean-value theorems. In Section 3 we construct n-exponentially convex functions and exponentially convex functions by using the functional defined as the difference of the right and the left side of inequality (6), for different classes of functions. In the last section, we define non-symmetric Stolarsky means using the same functional for some star-shaped functions, and the mean-value theorem proved in Section 2.

2. Mean value theorems

To prove related mean value theorems of Lagrange and Cauchy type, we consider the functions ϕ_1 and ϕ_2 defined in a following lemma.

LEMMA 2.1. *Let $f \in C^1[a,b]$ and denote*

$$G_f(x) = \frac{f'(x)(x-a)-f(x)+f(a)}{(x-a)^2}, \quad \textit{where} \quad x \neq a. \tag{7}$$

Let $m, M \in \mathbb{R}$ be such that

$$m \leqslant G_f(x) \leqslant M \quad \textit{for all} \quad x \in [a,b]. \tag{8}$$

Let the functions ϕ_1 and ϕ_2 be defined by

$$\phi_1(x) = M(x-a)^2 - f(x) + f(a) \tag{9}$$

and

$$\phi_2(x) = f(x) - f(a) - m(x-a)^2. \tag{10}$$

Then $\phi_1, \phi_2 \in St[a,b]$.

Proof. Now

$$\left(\frac{\phi_1(x)-\phi_1(a)}{x-a}\right)' = \left(M(x-a) - \frac{f(x)-f(a)}{x-a}\right)' = M - G_f(x) \geqslant 0,$$

and

$$\left(\frac{\phi_2(x)-\phi_2(a)}{x-a}\right)' = \left(\frac{f(x)-f(a)}{x-a} - m(x-a)\right)' = G_f(x) - m \geqslant 0.$$

This gives that $\phi_1, \phi_2 \in St[a,b]$. $\square$

THEOREM 2.2. *Let $f \in C^1[a,b]$, $G_f \in C[a,b]$ as defined in Lemma 2.1, $A(f;a,b)$ the integral arithmetic mean of f on $[a,b]$ and $A(f(a),f(b))$ the arithmetic mean of $f(a)$ and $f(b)$. Then there exists $\xi \in [a,b]$ such that*

$$A(f(a),f(b)) - A(f;a,b) = \frac{(b-a)^2}{6} G_f(\xi).$$

Proof. Since G_f is continuous on a compact set, it attains its maximum and minimum value on $[a,b]$. Let us consider

$$m = \min\{G_f(x)\}$$

and

$$M = \max\{G_f(x)\}.$$

Applying Theorem 1.3 on functions ϕ_1 and ϕ_2 defined in Lemma 2.1, we get the following inequalities:

$$A(f(a),f(b)) - A(f;a,b) \leqslant M\frac{(b-a)^2}{6}$$

$$A(f(a),f(b)) - A(f;a,b) \geqslant m\frac{(b-a)^2}{6}.$$

Combining both inequalities, we get

$$m\frac{(b-a)^2}{6} \leqslant A(f(a),f(b)) - A(f;a,b) \leqslant M\frac{(b-a)^2}{6}.$$

Since G_f is continuous on $[a,b]$, there exists $\xi \in [a,b]$ such that

$$A(f(a),f(b)) - A(f;a,b) = \frac{(b-a)^2}{6}G_f(\xi)$$

holds and the proof is completed. $\square$

THEOREM 2.3. *Let $f,g \in C^1[a,b]$, $G_f, G_g \in C[a,b]$ as defined in Lemma 2.1, $A(f;a,b)$ the integral arithmetic mean of f on $[a,b]$ and $A(f(a),f(b))$ the arithmetic mean of $f(a)$ and $f(b)$. Then there exists $\xi \in [a,b]$ such that the following equality is true:*

$$\frac{A(f(a),f(b)) - A(f;a,b)}{A(g(a),g(b)) - A(g;a,b)} = \frac{f'(\xi)(\xi-a) - f(\xi) + f(a)}{g'(\xi)(\xi-a) - g(\xi) + g(a)},$$

provided that the denominators are nonzero.

Proof. Consider

$$k = c_1 f - c_2 g,$$

where c_1 and c_2 are defined as

$$c_1 = A(g(a),g(b)) - A(g;a,b),$$

$$c_2 = A(f(a),f(b)) - A(f;a,b).$$

Then $k \in C^1[a,b]$ and $G_k = c_1 G_f - c_2 G_g \in C[a,b]$, so using Theorem 2.2 with $f = k$, we have

$$0 = \left(c_1 G_f(\xi) - c_2 G_g(\xi)\right)\left[\frac{(b-a)^2}{6}\right]. \tag{11}$$

Since

$$\left[\frac{(b-a)^2}{6}\right] \neq 0,$$

(11) gives

$$\frac{c_2}{c_1} = \frac{f'(\xi)(\xi-a) - f(\xi) + f(a)}{g'(\xi)(\xi-a) - g(\xi) + g(a)}. \tag{12}$$

$\square$

3. n-exponential convexity and exponential convexity

We start this section by giving some definitions and notions which are used frequently in the results. Throughout this section I is an interval in $\mathbb{R}$. The following results for n-exponentially convex functions have been cited from [4].

DEFINITION 1. A function $f : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I, if

$$\sum_{i,j=1}^{n} \xi_i \xi_j f\left(\frac{x_i + x_j}{2}\right) \geqslant 0$$

holds for all choices $\xi_i \in \mathbb{R}$ and every $x_i \in I$, $i = 1,\ldots,n$.

A function $f : I \longrightarrow \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

REMARK 3.1. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leqslant n$.

By using some linear algebra and definition of positive semi-definite matrices, we have the following proposition.

PROPOSITION 3.2. *If f is an n-exponentially convex function in the Jensen sense then the matrix*

$$\left[f\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k}$$

is a positive semi-definite matrix for all $k \in \mathbb{N}$, $k \leqslant n$. In particular,

$$\det\left[f\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k} \geqslant 0$$

for all $k \in \mathbb{N}$, $k \leqslant n$.

DEFINITION 2. A function $f : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $f : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous on I.

LEMMA 3.3. *It is known (and easy to show) that $f : I \to \mathbb{R}^{+}$ is $\log$-convex in the Jensen sense if and only if*

$$l^2 f(t) + 2lm f\left(\frac{t+r}{2}\right) + m^2 f(r) \geqslant 0$$

holds for each $l,m \in \mathbb{R}$ and $r,t \in I$.

It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense. Also, using basic convexity theory it follows that a positive function is log-convex if and only if it is 2-exponentially convex.

The following lemma is equivalent to the definition of convex function [5, page 2].

LEMMA 3.4. *If $x_1, x_2, x_3 \in I$ are such that $x_1 < x_2 < x_3$, then the function $f : I \to \mathbb{R}$ is convex if and only if inequality*

$$(x_3 - x_2) f(x_1) + (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3) \geqslant 0$$

holds.

LEMMA 3.5. *If Φ is a convex function on an interval I and if $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{\Phi(x_2) - \Phi(x_1)}{x_2 - x_1} \leqslant \frac{\Phi(y_2) - \Phi(y_1)}{y_2 - y_1}. \tag{13}$$

If the function Φ is concave then the inequality reverses.

Divided differences are found to be very handy and interesting when we have to operate with different functions having different degree of smoothness. Let $f : I \to \mathbb{R}$ be a function. Then for distinct points $u_i \in I$, $i = 0, 1$, the divided difference of first order is defined as follows:

$$[u_i; f] = f(u_i) \quad (i = 0, 1),$$

$$[u_0, u_1; f] = \frac{f(u_1) - f(u_0)}{u_1 - u_0}.$$

The values of the divided difference are independent of the order of the points u_0, u_1 and may be extended to include the case when the points are equal, that is

$$[u_0, u_0; f] = \lim_{u_1 \to u_0} [u_0, u_1; f] = f'(u_0),$$

provided that f' exists.

REMARK 3.6. One can note that if for all u_0, $u_1 \in I$ holds $[u_0, u_1; f] \geqslant 0$, then f is increasing on I.

We consider the functional

$$\Phi(f) = A(f(a), f(b)) - A(f; a, b) \tag{14}$$

where $A(f; a, b)$ and $A(f(a), f(b))$ are defined in (1).

REMARK 3.7. Under the assumptions of Theorem 1.3, if f is a starshaped function on $[a, b]$ then $\Phi(f) \geqslant 0$.

To define different families of functions let $[a,b], J \subseteq \mathbb{R}$ be intervals. For distinct points $u_0, u_1 \in [a,b]$, we suppose

$$\mathbf{E}_1 = \Big\{ f_t : [a,b] \to \mathbb{R} \mid t \in J \text{ and } t \mapsto [u_0, u_1; F_t] \text{ is } n\text{-exponentially convex in the Jensen sense on } J, \text{ where } F_t(u) = \frac{f_t(u) - f_t(a)}{u-a} \Big\}.$$

$$\mathbf{E}_2 = \Big\{ f_t : [a,b] \to \mathbb{R} \mid t \in J \text{ and } t \mapsto [u_0, u_1; F_t] \text{ is exponentially convex in the Jensen sense on } J, \text{ where } F_t(u) = \frac{f_t(u) - f_t(a)}{u-a} \Big\}.$$

$$\mathbf{E}_3 = \Big\{ f_t : [a,b] \to \mathbb{R} \mid t \in J \text{ and } t \mapsto [u_0, u_1; F_t] \text{ is 2-exponentially convex in the Jensen sense on } J, \text{ where } F_t(u) = \frac{f_t(u) - f_t(a)}{u-a} \Big\}.$$

THEOREM 3.8. *Let $\Phi(f)$ be linear functional defined as in (14) and $f_t \in \mathbf{E}_1$. Then $t \mapsto \Phi(f_t)$ is an n-exponentially convex function in the Jensen sense on J. If the function $t \mapsto \Phi(f_t)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. Consider the families of functions $\mathbf{E}_1$, and for $\xi_i \in \mathbb{R}$, $i = 1, \ldots, n$, and $t_i \in J$, $i = 1, \ldots, n$, define the function

$$h(u) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{t_i+t_j}{2}}(u). \tag{15}$$

We have

$$[u_0, u_1; H] = \sum_{i,j=1}^{n} \xi_i \xi_j \left[u_0, u_1; F_{\frac{t_i+t_j}{2}} \right],$$

where $H(u) = \frac{h(u)-h(a)}{u-a}$ and $F_t(u) = \frac{f_t(u)-f_t(a)}{u-a}$.

Since $t \mapsto [u_0, u_1; F_t]$ is n-exponentially convex in the Jensen sense on J, right hand side of the above expression is nonnegative, which implies by Remark 3.6, that $\frac{h(u)-h(a)}{u-a}$ is an increasing function on $[a,b]$.

Thus by Remark 3.7, we have

$$\Phi(h) \geqslant 0,$$

thus

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Phi\left(f_{\frac{t_i+t_j}{2}} \right) \geqslant 0.$$

Hence, we conclude that the function $t \mapsto \Phi(f_t)$ is n-exponentially convex in the Jensen sense on J.

If the function $t \mapsto \Phi(f_t)$ is also continuous on J then $t \mapsto \Phi(f_t)$ is n-exponentially convex by definition. □

The following corollary is an immediate consequence of the above theorem.

COROLLARY 3.9. *Let $\Phi(f)$ be linear functional defined as in (14) and $f_t \in \mathbf{E}_2$. Then $t \mapsto \Phi(f_t)$ is an exponentially convex function in the Jensen sense on J. If the function $t \mapsto \Phi(f_t)$ is continuous on J then it is exponentially convex on J.*

Proof. For any $n \in \mathbb{N}$, applying the same steps as in above theorem. □

COROLLARY 3.10. *Let $\Phi(f)$ be linear functional defined as in (14) and $f_t \in \mathbf{E}_3$. Then the following statements hold:*

(i) If the function $t \mapsto \Phi(f_t)$ is continuous on J then it is 2-exponentially convex function on J. If the function $t \mapsto \Phi(f_t)$ is additionally strictly positive, then it is also log-convex on J, and for $r,s,t \in J$ such that $r < s < t$, we have

$$(\Phi(f_s))^{t-r} \leqslant (\Phi(f_r))^{t-s}(\Phi(f_t))^{s-r}. \tag{16}$$

(ii) If the function $t \mapsto \Phi(f_t)$ is strictly positive and differentiable on J then for every $t,r,u,v \in J$ such that $t \leqslant u$, $r \leqslant v$, we have

$$\mathfrak{B}(t,r;\Phi) \leqslant \mathfrak{B}(u,v;\Phi),$$

where

$$\mathfrak{B}(t,r;\Phi) = \begin{cases} \left(\frac{\Phi(f_t)}{\Phi(f_r)}\right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{\frac{d}{dt}(\Phi(f_t))}{\Phi(f_t)}\right), & t = r. \end{cases} \tag{17}$$

Proof. (i) The first part is an immediate consequence of Theorem 3.8 and in second part log-convexity on J is a consequence of Lemma 3.3. Since $t \mapsto \Phi(f_t)$ is strictly positive, so for $r,s,t \in J$ such that $r < s < t$ with $f(t) = \log\Phi(f_t)$ in Lemma 3.4 gives

$$(t-s)\log\Phi(f_r) + (r-t)\log\Phi(f_s) + (s-r)\log\Phi(f_t) \geqslant 0.$$

This is equivalent to inequality (16).

(ii) By (i) the function $t \mapsto \Phi(f_t)$ is log-convex on J, that is, the function $t \mapsto \log\Phi(f_t)$ is convex on J. Thus, by using Lemma 3.5 with $t \leqslant u$, $r \leqslant v$, $t \neq r$, $u \neq v$, we get

$$\frac{\log\Phi(f_t) - \log\Phi(f_r)}{t-r} \leqslant \frac{\log\Phi(f_u)) - \log\Phi(f_v)}{u-v}, \tag{18}$$

concluding

$$\mathfrak{B}(t,r;\Phi) \leqslant \mathfrak{B}(u,v;\Phi).$$

Now, if $t = r \leqslant u$, we apply $\lim_{r \longrightarrow t}$, concluding

$$\mathfrak{B}(t,t;\Phi)) \leqslant \mathfrak{B}(u,v;\Phi).$$

Other possible cases are treated similarly. □

REMARK 3.11. The results given in Theorem 3.8, Corollary 3.9 and Corollary 3.10 are still true when the points u_0, $u_1 \in I$ coincide, say $u_1 = u_0$, for a family of differentiable function f_t such that the function $t \mapsto [u_0, u_1; F_t]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense).

4. Means

Let $r, s \in \mathbb{R}$ and let $a, b > 0$. The Stolarsky mean $E(a,b;r,s)$ of order (r,s) of a and b with $a \neq b$ are defined as

$$
\begin{aligned}
E(a,b;r,s) &= \left\{\frac{r(b^s-a^s)}{s(b^r-a^r)}\right\}^{\frac{1}{s-r}}, \quad \text{for } r \neq s,\ rs \neq 0;\\
E(a,b;r,0) = E(a,b;0,r) &= \left\{\frac{b^r-a^r}{r(\log b-\log a)}\right\}^{1/r}, \quad \text{for } r \neq 0;\\
E(a,b;r,r) &= e^{-\frac{1}{r}}\left(\frac{a^{a^r}}{b^{b^r}}\right)^{1/(a^r-b^r)}, \quad \text{for } r \neq 0;\\
E(a,b;0,0) &= \sqrt{ab}.
\end{aligned}
$$

Stolarsky [6] in 1975 (see also [5, page 120]) introduced these means. He also proved that the function $E(r,s;a,b)$ is increasing in both r and s. One can note that these means are symmetric with respect to the variable a and b. In [3] and [4], new classes of symmetric means of Stolarsky type are introduced. In this section we consider a class of starshaped functions to introduce means of Stolarsky type with functional due to the difference of Hermite-Hadamard inequality.

For all $t \in \mathbb{R}$, let $f_t : (0,\infty) \to \mathbb{R}$ be the function defined as

$$
f_t(x) = \begin{cases} \frac{(x-a)x^t}{t}, & t \neq 0;\\ (x-a)\log x, & t = 0. \end{cases}
$$

Then $F_t(x) := (f_t(x) - f_t(a))/(x-a)$ is strictly increasing for $x \in (0,\infty)$ and for each $t \in \mathbb{R}$. One can note that $t \mapsto [u_0, u_0; F_t]$ is log-convex for all $t \in \mathbb{R}$ and hence $t \mapsto \Phi(f_t)$ is log-convex. Also for $r < s < t$, where $r, s, t \in \mathbb{R}$, we have

$$
(\Phi(f_s))^{t-r} \leqslant (\Phi(f_r))^{t-s}(\Phi(f_t))^{s-r}. \tag{19}
$$

From Corollary 3.10, we can define, for $t \neq r$ and $t, r \neq 0, -1, -2$,

$$
\mathfrak{B}(t,r;\Phi) = \left(\frac{r(r+1)(r+2)}{t(t+1)(t+2)} \cdot \frac{\left(B_1 t^2 + (B_2 - a^2)t + 2a^2\right)b^t - 2a^{t+2}}{\left(B_1 r^2 + (B_2 - a^2)r + 2a^2\right)b^r - 2a^{r+2}}\right)^{\frac{1}{t-r}} \tag{20}
$$

where $B_1 = (b-a)^2$ and $B_2 = (b-2a)^2$, and for $t = r$ and $t \neq 0, -1, -2$,

$$
\mathfrak{B}(t,t;\Phi) = \exp\left(-\frac{3t^2+6t+2}{t(t+1)(t+2)} + \frac{\left(2B_1 t + B_2^2 - a^2\right)b^t + \left(B_1 t^2 + (B_2 - a^2)t + 2a^2\right)b^t \log b - 2a^{t+2}\log a}{\left(B_1 t^2 + (B_2 - a^2)t + 2a^2\right)b^t - 2a^{t+2}}\right).
$$

However, to get the continuous extension of (20) in order to cover all choices of r and t, we consider the following.

For $t \neq 0,-1,-2$,

$$\mathfrak{B}(t,0;\Phi)=\mathfrak{B}(0,t;\Phi)=\left(\frac{2}{t(t+1)(t+2)}\frac{\left(B_1t^2+(B_2-a^2)t+2a^2\right)b^t-2a^{t+2}}{B_2-a^2(1-2\log b+2\log a)}\right)^{\frac{1}{t}},$$

$$\mathfrak{B}(t,-1;\Phi)=\mathfrak{B}(-1,t;\Phi)=\left(\frac{-b}{t(t+1)(t+2)}\frac{\left(B_1t^2+(B_2-a^2)t+2a^2\right)b^t-2a^{t+2}}{(\log b-2)B_1+(1-\log b)B_2-a^2(1-3\log b)-2ab\log a}\right)^{\frac{1}{t+1}},$$

$$\mathfrak{B}(t,-2;\Phi)=\mathfrak{B}(-2,t;\Phi)=\left(\frac{2b^2}{t(t+1)(t+2)}\frac{\left(B_1t^2+(B_2-a^2)t+2a^2\right)b^t-2a^{t+2}}{4(\log b-1)B_1+(1-2\log b)B_2-a^2(1-4\log b)-2b^2\log a}\right)^{\frac{1}{t+2}},$$

$$\mathfrak{B}(0,0;\Phi)=\exp\left(\frac{2B_1+(2\log b-3)B_2+a^2\left(3-8\log b+6\log a+2(\log b)^2-2(\log a)^2\right)}{2\left(B_2-a^2(1+2\log a-2\log b)\right)}\right),$$

$$\mathfrak{B}(-1,-1;\Phi)=\exp\left(\frac{\left(2-4\log b+(\log b)^2\right)B_1+(2-\log b)B_2\log b+a((3\log b-2)a\log b-2b(\log a)^2)}{2\left[(\log b-2)B_1+(1-\log b)B_2-a^2(1-3\log b)-2ab\log a\right]}\right),$$

$$\mathfrak{B}(-2,-2;\Phi)$$
$$=\exp\left(\frac{2\left(-5+2\log b+2(\log b)^2\right)B_1+\left(3-4\log b-2(\log b)^2\right)B_2+a^2\left(-3+10\log b+4(\log b)^2\right)-2b^2(3+\log a)\log a}{2\left(4(\log b-1)B_1+(1-2\log b)B_2+a^2(-1+4\log b)-2b^2\log a\right)}\right).$$

Also note that if the function $t\mapsto\Phi(\varphi_t)$ is positive and differentiable on $\mathbb{R}$ then for every $t,r,u,v\in\mathbb{R}$ such that $t\leqslant u$, $r\leqslant v$, we have

$$\mathfrak{B}(t,r;\Phi)\leqslant\mathfrak{B}(u,v;\Phi). \tag{21}$$

If we apply Theorem 2.3 on functions $f=f_t$ and $g=f_r$, where $t\neq r$, we get that there exists some $\xi\in[a,b]$ such that

$$\frac{A(f_t(a),f_t(b))-A(f_t;a,b)}{A(f_r(a),f_r(b))-A(f_r;a,b)}=\xi^{t-r}.$$

Since the function $\xi\mapsto\xi^{t-r}$ is invertible for $t\neq r$, we then have

$$a\leqslant\left(\frac{A(f_t(a),f_t(b))-A(f_t;a,b)}{A(f_r(a),f_r(b))-A(f_r;a,b)}\right)^{\frac{1}{t-r}}\leqslant b,$$

that is

$$a\leqslant B(t,r;\Phi)\leqslant b,$$

which together with the fact that $B(t,r;\Phi)$ is continuous and monotonous with respect to its both arguments t and r, shows that $B(t,r;\Phi)$ are means of a and b for all $t,r\in\mathbb{R}$. These means are non-symmetric with respect to its variable a and b.

REFERENCES

[1] A. M. BRUCKNER, E. OSTROW, *Some function classes related to the class of convex functions*, Pacific J. Math., **12** (4) (1962), 1203–1215.

[2] A. M. FINK, D. S. MITRINOVIĆ AND J. PEČARIĆ, *Classical and new inequalities in analysis*, Kluwer Academic Publishers, The Netherlands, 1993.

[3] J. JAKŠETIĆ, J. PEČARIĆ AND ATIQ UR REHMAN, *On Stolarsky and related means*, Math. Inequal. Appl., **13** (4) (2010), 899–909.

[4] J. PEČARIĆ AND J. PERIĆ, *Improvements of the Giaccardi and the Petrović inequalty and related Stolarsky type means*, An. Univ. Craiova Ser. Mat. Inform. **39** (1) (2012), 65–75.

[5] J. PEČARIĆ, F. PROSCHAN AND Y. L. TONG, *Convex functions, Partial Orderings and Statistical Applications*, vol. 187 of Mathematics in Science and Engineering, Academic Press, Boston, Mass, USA, 1992.

[6] K. B. STOLARSKY, *Generalization of the logarithmic mean*, Math. Mag., **48** (1975), 87–92.

[7] GH. TOADER, *Superaddivtivity and Hermite-Hadamard's inequalites*, Studia Univ. Babeş-Bolyai Math., **39** (1994), 27–32.

(Received March 5, 2012)

Saad Ihsan Butt
Abdus Salam School of Mathematical Sciences
GC University
Lahore, Pakistan
e-mail: saadihsanbutt@gmail.com

Josip Pečarić
Abdus Salam School of Mathematical Sciences
GC University
Lahore, Pakistan
University of Zagreb, Faculty of Textile Technology
Croatia
e-mail: pecaric@mahazu.hazu.hr

Atiq ur Rehman
Department of Mathematics
University of Sargodha
Sargodha
e-mail: atiq@mathcity.com