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On the Orbifold Structure of the Moduli Space of Riemann Surfaces of Genera Four and Five

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Abstract

The moduli space $\mathcal{M}_g$, of compact Riemann surfaces of genus g has orbifold structure since $\mathcal{M}_g$ is the quotient space of the Tiechmüller space by the action of the mapping class group. Using uniformization of Riemann surfaces by Fuchsian groups and the equisymmetric stratification of the branch locus of the moduli space we find the orbifold structure of the moduli spaces of Riemann surfaces of genera 4 and 5.

1 Introduction

The moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g, $g \geq 4$, being the quotient of the Teichmüller space by the discontinuous action of the mapping class group, has the structure of a complex orbifold, whose set of singular points is called the *branch locus* $\mathcal{B}_g$.

In order to study Riemann surfaces our main tool will be their uniformization by Fuchsian groups. Given a Riemann surface X of genus $g > 1$, we consider the universal covering $\mathcal{H} \overset{\pi_1(X)}{\to} X$, where $\mathcal{H}$ is the hyperbolic plane. Hence there is a representation $r : \pi_1(X) \to Isom^+(\mathcal{H}) = PSL(2, \mathbb{R})$ such that $X = \mathcal{H}/r(\pi_1(X))$ and $r(\pi_1(X))$ is a discrete subgroup of $PSL(2, \mathbb{R})$ (i.e. a Fuchsian group).

If there is $\gamma \in PSL(2, \mathbb{R})$, such that $r_1(\pi_1(X)) = \gamma(r_2(\pi_1(X)))\gamma^{-1}$, clearly the Fuchsian groups $r_1(\pi_1(X))$ and $r_2(\pi_1(X))$ uniformize the same Riemann surface. The space:

$$\{r : \pi_1(X) \to PSL(2, \mathbb{R}) : \mathcal{H}/r(\pi_1(X)) \text{ is a genus } g \text{ surface}\}/\text{conjugation in } PSL(2, \mathbb{R})$$

is the Teichmüller space $\mathbf{T}_g$. The Teichmüller space $\mathbf{T}_g$ has complex structure of dimension $3g - 3$ and it is simply connected.

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The group $\mathrm{Aut}^+(\pi_1(X))/\mathrm{Inn}(\pi_1(X)) = \mathrm{Mod}_g$ is the modular group or mapping class group, acting by composition on $\mathbf{T}_g$. Now we define the *moduli space* by $\mathcal{M}_g = \mathbf{T}_g/\mathrm{Mod}_g$.

The projection $\mathbf{T}_g \to \mathcal{M}_g = \mathbf{T}_g/\mathrm{Mod}_g$ is a regular branched covering with *branch locus* $\mathcal{B}_g$, in other words, $\mathcal{M}_g$ is an orbifold with singular locus $\mathcal{B}_g$ (see [Na]). The branch locus $\mathcal{B}_g$ consists of the Riemann surfaces with symmetry, i. e. Riemann surfaces with non-trivial automorphism group (except when $g = 2$, where $\mathcal{B}_2$ consists of the surfaces with automorphisms different from the hyperelliptic involution and the identity).

Harvey ([H]) alluded to the existence of the *equisymmetric stratification* of the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g, each stratum is formed by the points in the moduli space corresponding to equisymmetric surfaces. Two closed Riemann surfaces X, Y of genus g are called *equisymmetric* if their automorphism groups determine conjugate finite subgroups in the modular group of genus g, i.e. the actions of their automorphism groups are topologically equivalent. The branch locus of the moduli space is formed by the strata corresponding to surfaces of genus $g > 2$ admitting non-trivial automorphisms. Broughton ([B1]) showed that the equisymmetric stratification is indeed a stratification of $\mathcal{M}_g$ by irreducible algebraic subvarieties whose interior, if it is non-empty, is a smooth, connected, locally closed algebraic subvariety of $\mathcal{M}_g$, Zariski dense in the stratum. In this way we can equip the moduli space with a structure of complex of groups. One can also use techniques as in [GK] to obtain results about the homology of the space.

Here we calculate the orbifold structure of $\mathcal{M}_4$ and $\mathcal{M}_5$. As the equisymmetric stratification of the branch locus, $\mathcal{B}_g$ and the relations between its strata provide the orbifold structure of the moduli space $\mathcal{M}_g$, to determine the orbifold structure of $\mathcal{M}_4$ and $\mathcal{M}_5$ we need the topological classification of the maximal actions of automorphism groups of Riemann surfaces of genera 4 and 5.

Izquierdo and Ying ([IY]) gave the orbifold structure of the space of cyclic trigonal Riemann surfaces of genus 4. Bogopolski [Bo] and Kimura [K] found the topological classification of all actions of finite groups on surfaces of genus 4. Costa and Izquierdo [CI2] found the strata of $\mathcal{M}_4$ allowing them to prove that the branch locus of $\mathcal{M}_4$ is connected [CI1], see also [BCIP]. Kuribayashi and Kimura [KK] classified algebraically actions of finite groups on surfaces of genus five up to $GL(5, \mathbb{C})$-conjugacy. We obtain here the corresponding topological classification. The algebraic classification in [KK] coincides with the topological classification with the exception of one case: the alternating group A_4 of degree 4 has two topological classes of actions on surfaces of genus 5 but just one algebraic action, none of these actions is maximal. To distinguish the two actions one must look at the full automorphism group (see [CPo]).

Weaver obtained the equisymmetric stratification for spaces of hyper elliptic Riemann surfaces in [W2] (see also [W1]).

Magaard et al [MSSV] gave the classification of of actions of groups of order at least 16 acting on surfaces of genus up to 10 and strata of dimension 0 or 1.

The topological classification and the study of the maximality of the actions here has been done by calculating all actions and their classification with GAP [gap].

2 Uniformization of Riemann Surfaces by Fuchsian Groups

A compact Riemann surface can be realized as the orbifold $\mathcal{D}/\Gamma$, where Γ is a Fuchsian group. The algebraic structure of Γ and the geometric structure of the orbifold $\mathcal{D}/\Gamma$ are given by the signature of Γ:

$$s(\Gamma) = (g;\ m_1, ..., m_r). \tag{1}$$

The group with the signature (1) has a canonical presentation given by generators:

(a) $x_i,\ i = 1, \ldots, r,$ (elliptic transformations)
(b) $a_i, b_i,\ i = 1, \ldots g$ (hyperbolic translations)

and relations:

$$\begin{array}{ll} (1) & x_i^{m_i} = 1,\ i = 1, \ldots, r, \\ (2) & x_1 \ldots x_r a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1. \end{array} \tag{2}$$

A Fuchsian group Γ without elliptic elements is called a surface Fuchsian group. The orbifold $\mathcal{D}/\Gamma$ has no cone points.

Given a subgroup Γ' of index N in a Fuchsian group Γ, one can calculate the signature of Γ' by:

Theorem 1 *([S1]) Let Γ be a Fuchsian group with signature (1) and canonical presentation (2). Then Γ contains a subgroup Γ' of index N with signature*

$$s(\Gamma') = (h; m'_{11}, m'_{12}, ..., m'_{1s_1}, ..., m'_{r1}, ..., m'_{rs_r}).$$

if and only if there exists a transitive permutation representation $\theta : \Gamma \to \Sigma_N$ satisfying the following conditions:

1. The permutation $\theta(x_i)$ has precisely s_i cycles of lengths less than m_i, the lengths of these cycles being $m_i/m'_{i1}, ..., m_i/m'_{is_i}$.

2. The Riemann-Hurwitz formula

$$\mu(\Gamma')/\mu(\Gamma) = N.$$

where $\mu(\Gamma), \mu(\Gamma')$ are the hyperbolic areas of the surfaces $\mathcal{D}/\Gamma$, $\mathcal{D}/\Gamma'$.

Given a Riemann surface $X = \mathcal{D}/\Gamma$, with Γ a surface Fuchsian group, a finite group G is a group of automorphisms of X if and only if there exists a Fuchsian group Δ and an epimorphism $\theta : \Delta \to G$ with $ker(\theta) = \Gamma$.

We shall use the following terminology:

Definition 2 *A closed Riemann surface X which can be realized as a p-sheeted covering of the Riemann sphere is said to be p-gonal, and such a covering will be called a p-gonal morphism. When $p = 2$, the surface will be called hyperelliptic.*

Let Γ be a Fuchsian group with signature (1). Then the Teichmüller space $T(\Gamma)$ of Γ is homeomorphic to a complex ball of dimension $d(\Gamma) = 3g - 3 + r$ (see [Na]). Let $\Gamma' \leq \Gamma$ be Fuchsian groups, the inclusion mapping $\alpha : \Gamma \to \Gamma'$ induces an embedding $T(\alpha) : T(\Gamma) \to T(\Gamma')$ defined by $[r] \mapsto [r\alpha]$. See [Na] and [S2]. The modular group $Mod(\Gamma)$ of Γ is the quotient $Mod(\Gamma) = Aut^+(\Gamma)/Inn(\Gamma)$. The *moduli space* of Γ is the quotient $\mathcal{M}(\Gamma) = T(\Gamma)/Mod(\Gamma)$ endowed with the quotient topology.

Definition 3 *As an application of Nielsen Realisation Theorem one can identify the branch locus of the action of* $Mod(\Gamma)$ *on* $T(\Gamma)$ *as the set* $\mathcal{B}_g = \{X \in \mathcal{M}_g : Aut(X) \neq 1_d\}$, *for* $g \geq 3$.

The branch locus is formed by the singular points of the orbifold $\mathcal{M}_g$.

A Fuchsian group Γ such that there does not exist any other Fuchsian group containing it with finite index is called a *finite maximal* Fuchsian group. To decide whether a given finite group can be the full group of automorphisms of some compact Riemann surface we will need all pairs of signatures $s(\Gamma)$ and $s(\Gamma')$ for some Fuchsian groups Γ and Γ' such that $\Gamma' \leq \Gamma$ and $d(\Gamma) = d(\Gamma')$. The full list of such pairs of signatures was obtained by Singerman in [S2].

An *(effective and orientable) action* of a finite group G on a Riemann surface X is a representation $\theta : G \to \mathrm{Aut}(X)$. Two actions θ, θ' of G on a Riemann surface X are *(weakly) topologically equivalent* if there is an $w \in \mathrm{Aut}(G)$ and an $h \in Hom^+(X)$ such that $\theta'(g) = h\theta w(g)h^{-1}$. If $\theta(G)$ is the full group of automorphisms of X we shall say that θ is a maximal action. The equisymmetric strata are in correspondence with topological equivalence classes of orientation preserving actions of a finite group G on a surface X, see [B1]. Let $\mathcal{M}^{G,\theta}$ denote the stratum of surfaces with full automorphism group the conjugacy class of the finite group G in the modular group and let $\overline{\mathcal{M}}^{G,\theta}$ denote the set of surfaces such that the automorphism group contains a subgroup in the class defined by G. $\overline{\mathcal{M}}^{G,\theta}$ is the image of a cell in $\mathbf{T}_g$ so we shall called it a cell sometimes.

We have the following theorem:

Theorem 4 *([B1]) Let* $\mathcal{M}_g$ *be the moduli space of Riemann surfaces of genus* g, G *a finite subgroup of the corresponding modular group* Mod_g. *Then:*

(1) $\overline{\mathcal{M}}^{G,\theta}$ *is a closed, irreducible algebraic subvariety of* $\mathcal{M}_g$.

(2) $\mathcal{M}^{G,\theta}$, *if it is non-empty, is a smooth, connected, locally closed algebraic subvariety of* $\mathcal{M}_g$, *Zariski dense in* $\overline{\mathcal{M}}^{G,\theta}$.

Each non-empty stratum or cell corresponds with a topological type of maximal actions on Riemann surfaces.

By Nielsen Realisation Theorem each equisymmetric stratum forms a stratum (or cell) of the orbifold structure of $\mathcal{M}_g$.

In terms of Fuchsian groups each action of a finite group G on a surface X_g is determined by an epimorphism $\theta : \Delta \to G$ from a Fuchsian group Δ such that $ker(\theta) = \Gamma_g$, where $X_g = \mathcal{H}/\Gamma$ and Γ_g is a surface Fuchsian group. The condition Γ_g to be a surface Fuchsian group imposes that the order of the image under θ of an elliptic generator x_i of Δ is the same as the order of x_i. Two epimorphisms $\theta_1, \theta_2 : \Delta \to G$ define two topologically equivalent actions of G on X if and only if there exist automorphisms $\phi : \Delta \to \Delta$, $w : G \to G$ such that $\theta_2 = w \circ \theta_1 \circ \phi^{-1}$. See Proposition 2.2 in [B2] and [Sm].

Let $\overline{\mathcal{B}}$ be the subgroup of $\mathrm{Aut}(\Delta)$ induced by orientation preserving homeomorphisms of the orbifold $\mathcal{D}/\Delta$. Then, two different epimorphisms $\theta_1, \theta_2 : \Delta \to G$ define the same class of G-actions if and only if they lie in the same $\overline{\mathcal{B}} \times \mathrm{Aut}(G)$-class.

Actions of finite groups have been studied recently by [BW], [BCC], [BCGG], [BCG] and [CPa] among other.

To find the structure as complex of groups of $\mathcal{M}_4$ and $\mathcal{M}_5$ we use Theorem 1 ([S1]) in the following algorithm:

1. First we find the possible signatures of Fuchsian groups uniformizing orbifolds X_4/G and X_5/G, where X_4 and X_5 are Riemann surfaces of genus four respectively five and G is a group of automorphisms of X_4, respectively X_5. Using Singerman's list ([S2]) we mark which signatures are non-maximal.

2. We calculate epimorphisms $\theta : \Delta \to G$ with $ker(\theta) = \Gamma$ is a surface Fuchsian group uniformizing a surface of genus four respectively five. We will label such epimorphisms by the list of genrators of G which are images by θ of generators of Δ: the **generating vectors**.

   This step was done by Bogopolski [Bo], Kimura [K] and Costa-Izquierdo [CI2] for genus four.

3. We classified topologically the actions obtained above by finding the classes of generating vectors under the action of $\overline{\mathcal{B}} \times \mathrm{Aut}(G)$.

4. We obtain the actions that correspond to actions of full automorphisms groups of Riemann surfaces of genera four and five. These actions are called maximal actions. Each maximal action θ corresponds with one non-empty stratum $\mathcal{M}_4^{G,\theta}$ (respectively $\mathcal{M}_5^{G,\theta}$) of the orbifold $\mathcal{M}_4$, respectively $\mathcal{M}_5$.

   For genus four the strata were found in [CI2].

5. Finally we calculate the inclusion - and intersection- relations between the different strata by the use of Theorem 1 ([S1]).

Steps two to five are calculated with help of GAP. Notice that steps four and five can be done simultaneously by using Theorem 1 according to the following diagram:

$$\begin{array}{ccc} \Delta & \stackrel{\theta}{\to} & G \\ \uparrow & & \uparrow \\ \Delta_1 & \stackrel{\theta_i}{\to} & G_i \\ \uparrow & & \uparrow \\ \Gamma & \stackrel{\theta,\theta_i}{\to} & Id \end{array}$$

We can extend the diagram above to find $\overline{\mathcal{M}}_g^{G_1,\theta_1} \cap \overline{\mathcal{M}}_g^{G_2,\theta_2}$ using the following tower of coverings:

$$\begin{array}{ccccc} & f_{\theta_1} & Y = \mathcal{H}/\Gamma_g & f_{\theta_2} & \\ & \swarrow & & \searrow & \\ \mathcal{H}/\Delta_1 & & \downarrow f_\theta & & \mathcal{H}/\Delta_2 \\ & \searrow & & \swarrow & \\ & & \mathcal{H}/\Delta & & \end{array}$$

with θ, θ_1 and θ_2 are action where we have extensions Δ of $\Delta_1 = \theta^{-1}(G_1)$ and $\Delta_2 = \theta^{-1}(G_2)$ and G of G_1 and G_2 such that $\theta|_{\Delta_1} = \theta_1$ and $\theta|_{\Delta_2} = \theta_2$. To calculate this we use Theorem 1. Now, $\theta_i, i = 1, 2$ is a non-maximal action if the Teichmüller dimensions of Δ and Δ_i $i = 1, 2$ are equal, otherwise we get that the stratum induced by θ_i contains the stratum induced by θ.

In the following we will consider only strata belonging to the branch locus, we will not list the cell containing the regular points of the orbifolds $\mathcal{M}_4$ and $\mathcal{M}_5$ respectively.
Since we use standard methods in Fuchsian groups and GAP we will illustrate the calculations only in a couple of examples to shorten the lenght of this work.
Using the Riemann-Hurwitz formula we obtain the following possible signatures for Fuchsian groups Δ uniformizing orbifolds X_4/G and X_5/G, where X_4 and X_5 are Riemann surfaces of genus four, respectively five, and G is a group of automorphisms of X_4, respectively X_5.
Observe that a Riemann surface of genus four only admits actions of prime order two, three or five.

Lemma 5 *[CI2] Let Δ be a Fuchsian group and G a finite group. If there exist an epimorphism $\theta : \Delta \to G$ such that $ker(\theta)$ is a surface group of genus 4. Then Δ has one of the following signatures.*

$\|G\|$	*signature*	$\|G\|$	*signature*	$\|G\|$	*signature*
2	$(0; 2, \overset{10}{\ldots}, 2)$	2	$(1; 2, \overset{6}{\ldots}, 2)$	2	$(2; 2, 2)$
3	$(2; -)$	3	$(1; 3, 3, 3)$	3	$(0; 2, \overset{6}{\ldots}, 2)$
4	$(0; 2, 2, 2, 2, 4, 4)$	4	$(1; 4, 4)^*$	4	$(0; 2, 4, 4, 4, 4)$
4	$(1; 2, 2, 2)$	4	$(0; 2, \overset{7}{\ldots}, 2)$	5	$(0; 5, 5, 5, 5)^*$
6	$(0; 2, \overset{6}{\ldots}, 2)$	6	$(0; 2, 6, 6, 6)$	6	$(1; 2, 2)^*$
6	$(0; 2, 2, 2, 3, 6)$	6	$(0; 3, 3, 6, 6)^*$	6	$(0; 2, 2, 3, 3, 3)$
8	$(0; 2, 2, 2, 2, 4)$	8	$(0; 2, 4, 4, 4)$	8	$(0; 2, 2, 8, 8)^*$
9	$(0; 9, 9, 9)$	9	$(0; 3, 3, 6, 6)^*$	10	$(0; 5, 10, 10)^*$
10	$(0; 2, 2, 5, 5)^*$	12	$(1; 2)^*$	12	$(0; 6, 6, 6)^*$
12	$(0; 2, 2, 4, 4)^*$	12	$(0; 3, 12, 12)^*$	12	$(0; 6, 4, 12)$
12	$(0; 2, 2, 2, 2, 2)$	12	$(0; 2, 2, 3, 6)^*$	12	$(0; 2, 3, 3, 3)$
15	$(0; 5, 5, 5)^*$	15	$(0; 3, 5, 15)^*$	18	$(0; 2, 9, 18)^*$
18	$(0; 2, 2, 3, 3)^*$	18	$(0; 2, 2, 2, 6)$	20	$(0; 2, 2, 2, 5)$
20	$(0; 2, 10, 10)^*$	20	$(0; 4, 4, 5)^*$	24	$(0; 6, 3, 4)$
24	$(0; 2, 2, 2, 4)$	24	$(0; 4, 4, 4)^*$	24	$(0; 2, 8, 8)^*$
24	$(0; 3, 12, 12)^*$	24	$(0; 2, 6, 12)^*$	27	$(0; 3, 3, 9)^*$
27	$(0; 2, 6, 9)^*$	30	$(0; 2, 5, 10)^*$	32	$(0; 2, 4, 16)$
36	$(0; 3, 3, 6)^*$	36	$(0; 2, 6, 6)^*$	36	$(0; 2, 2, 2, 3)$
36	$(0; 3, 4, 4)^*$	40	$0; 2, 4, 10)$	45	$(0; 3, 3, 5)^*$
48	$(0; 2, 3, 24)$	48	$(0; 2, 4, 8)^*$	54	$(0; 2, 3, 18)$
60	$(0; 2, 3, 15)$	60	$(0; 2, 5, 5)^*$	72	$(0; 3, 3, 4)^*$
72	$(0; 2, 4, 6)$	72	$(0; 2, 3, 12)$	90	$(0; 2, 3, 10)$
108	$(0; 2, 3, 9)$	120	$(0; 2, 4, 5)$	144	$(0; 2, 3, 8)$

* *Non-maximal signature*

Lemma 6 *[KK] Let Δ be a Fuchsian group and G a finite group. If there exist an epimorphism $\theta : \Delta \to G$ such that $ker(\theta)$ is a surface group of genus 5. Then Δ has one of the following signatures.*

$\|G\|$	*signature*	$\|G\|$	*signature*	$\|G\|$	*signature*
2	$(0;2,\overset{12}{\ldots},2)$	10	$(0;2,2,2,2,5)$		$(0;2,12,12)^*$
	$(1;2,\overset{8}{\ldots},2)$		$(0;2,2,10,10)^*$	30	$(0;2,6,15)$
	$(2;2,2,2,2)$		$(1;5)$	32	$(0;2,2,2,4)$
	$(3;-)$	11	$(0;11,11,11)$		$(0;4,4,4)^*$
3	$(0;3,\overset{7}{\ldots},3)$	12	$(0;2,2,2,2,3)$		$(0;2,8,8)^*$
	$(1;3,3,3,3)$		$(0;3,3,3,3)^*$	40	$(0;2,4,20)$
4	$(0;2,\overset{8}{\ldots},2)$		$(0;2,3,4,4)$	48	$(0;2,2,2,3)$
	$(0;2,\overset{5}{\ldots},2,4,4)$		$(0;2,2,6,6)^*$		$(0;3,4,4)^*$
	$(0;2,2,4,4,4,4)$		$(0;6,12,12)^*$		$(0;2,6,6)^*$
	$(1;2,2,2,2)$		$(1;3)^*$		$(0;2,4,12)$
	$(1;2,4,4)$	15	$(0;3,15,15)^*$	60	$(0;3,3,5)^*$
	$(2;-)^*$	16	$(0;2,2,2,2,2)$	64	$(0;2,4,8)^*$
5	$(1;5,5)^*$		$(0;2,2,4,4)^*$	80	$(0;2,5,5)^*$
6	$(0;2,2,2,2,3,3)$		$(0;4,8,8)^*$	96	$(0;3,3,4)^*$
	$(0;2,3,3,3,6)$		$(1;2)^*$		$(0;2,4,6)$
	$(0;2,2,3,6,6)$	20	$(0;2,2,2,10)$	120	$(0;2,3,10)$
	$(0;6,6,6,6)^*$		$(0;4,4,10)^*$	160	$(0;2,4,5)$
	$(1;3,3)^*$		$(0;2,20,20)^*$	192	$(0;2,3,8)$
8	$(0;2,\overset{6}{\ldots},2)$	22	$0;2,11,22$		
	$(0;2,2,2,4,4)$	24	$(0;2,2,3,3)^*$		
	$(0;4,4,4,4)^*$		$(0;2,2,2,6)$		
	$(0;2,4,8,8)$		$(0;4,4,6)^*$		
	$(1;2,2)^*$		$(0;3,6,6)^*$		

** Non-maximal signature*

We list some groups that we will use in the next sections. First of all, C_n is the cyclic group of order n, S_n the symmetric group of degree n and A_n the alternating group of degree n

$D_n = \langle s, a \,:\, s^2 = a^n = (sa)^2 = 1\rangle$

$Q = \langle i, j \,:\, i^2 = j^2, iji = j\rangle$

$C_3 \times C_4 = \langle s, t \,:\, s^4 = t^3 = 1, s^3ts = t^2\rangle$

$C_8 \rtimes_3 C_2 = \langle s, a \,:\, s^2 = a^8 = 1, sa^3 = as\rangle$

$C_8 \rtimes_5 C_2 = \langle s, a \,:\, s^2 = a^8 = 1, sa^5 = as\rangle$

$C_4 \rtimes C_4 = \langle s, t \,:\, s^4 = t^4 = 1, ts^3 = st\rangle$

$G_{4,4} = \langle a, b \,:\, b^2 = a^4 = (ab)^4 = [a^2, b] = 1\rangle$

$C_5 \rtimes C_4 = \langle b, a \,:\, b^5 = a^4 = 1, a^3ba = b^4\rangle$

$C_5 \rtimes_4 C_4 = \langle t, a \,:\, t^5 = a^4 = 1, t^3at = a^2\rangle$

$(C_n \times C_2) \rtimes C_2 = \langle a, b \,:\, a^2 = b^n = (ab)^4 = ab^2ab^2 = 1,\ n = 6, 12\rangle$

$QD_{16} = \langle s, a \,:\, s^2 = a^{16} = sasa^7 = 1\rangle$

$C_2^4 \rtimes C_2 = \langle a, b, c \,:\, C^2 = b^2 = a^2 = (ab)^4 = (ac)^4 = (bc)^2 = (abac)^2 = 1\rangle$

$((C_4 \times C_2) \rtimes C_2) \rtimes C_2 = \langle s, t \,:\, t^2 = s^4 = (st)^4 = [s, t]^2 = 1\rangle$

$(D_4 \times C_2) \rtimes C_2 = \langle a, b, c \,:\, c^2 = a^2 = b^2 = (ab)^8 = (ac)^4 = (bc)^2 = (abac)^2 = 1\rangle$

$(C_4 \rtimes C_2) \rtimes C_2 = \langle s, t \,:\, t^2 = s^8 = (s^2t)^2 = [st]^2 = 1\rangle$

$((C_4 \rtimes C_2) \rtimes C_2) \rtimes C_2 = \langle a, s, t \,:\, s^2 = t^2 = a^8 = (a^2s)^2 = [as]^2 = (st)^2 = tatsa^7 = 1\rangle$

$A_4 \rtimes C_4 = \langle a, b \,:\, a^3 = b^4 = 1, (b^2a)^2 = b^3ab\rangle$

$(A_4 \times C_2 \times C_2) \rtimes C_2 = \langle a, b, c \,:\, c^2 = a^6 = b^2 = (bc)^4 = 1, (ab)^2 = (ba)^2, (ca)^3 = a^3\rangle$

$((C_2^4 \rtimes C_5) \rtimes C_2) \rtimes C_2 = \langle a, b \,:\, a^5 = b^4 = (ab^3)^4 = (ab)^2 = 1\rangle$

$(((C_4 \times C_2) \rtimes C_4) \rtimes C_3) \rtimes C_2 = \langle s, t \,:\, t^8 = (ts)^4 = b(st)^4 = (st^2)^3 = (st^5)^2 = (st^6s^2t^7)^2 = 1\rangle$.

3 Equisymmetric Stratification

Costa-Izquierdo [CI2] gave the equisymmetric stratification of the moduli space of Riemann sufaces of genus four in the following:

Theorem 7 *[CI2] The equisymmetric strata of the branch locus $\mathcal{B}_4$ are the following.*

Strata	*dim.*	*Strata*	*dim.*	*Strata*	*dim.*	*Strata*	*dim.*
$\mathcal{M}_4^{C_2,0}$	7	$\mathcal{M}_4^{C_2,1}$	6	$\mathcal{M}_4^{C_2,0}$	5	$\mathcal{M}_4^{C_3,1}$	3
$\mathcal{M}_4^{C_3,01}$	3	$\mathcal{M}_4^{C_3,02}$	3	$\mathcal{M}_4^{C_2\times C_2,02}$	4	$\mathcal{M}_4^{C_2\times C_2,12}$	4
$\mathcal{M}_4^{C_2\times C_2,1}$	3	$\mathcal{M}_4^{C_4,0}$	3	$\mathcal{M}_4^{C_4,1}$	2	$\mathcal{M}_4^{C_5}$	1
$\mathcal{M}_4^{D_3,1}$	3	$\mathcal{M}_4^{C_6,0}$	2	$\mathcal{M}_4^{C_6,01}$	2	$\mathcal{M}_4^{D_3,2}$	2
$\mathcal{M}_4^{C_6,1}$	1	$\mathcal{M}_4^{D_4,02}$	2	$\mathcal{M}_4^{D_4,12}$	2	$\mathcal{M}_4^{Q}$	1
$\mathcal{M}_4^{D_5}$	1	$\mathcal{M}_4^{C_{10}}$	0	$\mathcal{M}_4^{D_6,1}$	2	$\mathcal{M}_4^{D_6,2}$	1
$\mathcal{M}_4^{C_6\times C_2}$	1	$\mathcal{M}_4^{A_4}$	1	$\mathcal{M}_4^{C_{12}}$	0	$\mathcal{M}_4^{C_{15}}$	0
$\mathcal{M}_4^{D_8}$	1	$\mathcal{M}_4^{C_3\times D_3}$	1	$\mathcal{M}_4^{C_{18}}$	0	$\mathcal{M}_4^{D_{10}}$	1
$\mathcal{M}_4^{S_4}$	1	$\mathcal{M}_4^{Q\rtimes C_3}$	0	$\mathcal{M}_4^{QD_{16}}$	0	$\mathcal{M}_4^{D_3\times D_3}$	1
$\mathcal{M}_4^{D_3\times C_6}$	0	$\mathcal{M}_4^{C_5\rtimes D_4}$	0	$\mathcal{M}_4^{S_4\times C_3}$	0	$\mathcal{M}_4^{(C_3\times C_3)\rtimes D_4}$	0
$\mathcal{M}_4^{S_5}$	0						

Again we will study the structure of $\mathcal{B}_5$ omitting the regular points of $\mathcal{M}_5$. We recall that to obtain the strata of $\mathcal{B}_5$ we first calculate and classify all the actions of finite groups on Riemann surfaces of genus five and secondly we determine wich actions are maximal. We will represent an action $\theta : \Delta \to G$ by a generating vector for the action, that is a list of the images by θ of the canonical generators $(a_i, b_i; x_j)$, $i = 1, \ldots, h$, $j = 1, \ldots, r$ of Δ with signature $(h; m_1, \ldots m_r)$ that generate G.

Lemma 8 *The actions of orders* 2, 4 *and* 8 *are*

$\theta_{2,0} : \Delta(0; 2, \overset{12}{\ldots}, 2) \to C_2$	$(a, a, a, a, a, a, a, a, a, a, a, a)$
$\theta_{2,1} : \Delta(1; 2, \overset{8}{\ldots}, 2) \to C_2$	$(1, 1; a, a, a, a, a, a, a, a)$
$\theta_{2,2} : \Delta(2; 2, 2, 2, 2) \to C_2$	$(1, 1, 1, 1; a, a, a, a)$
$\theta_{2,3} : \Delta(3; -) \to C_2$	$(a, 1, 1, 1, 1, 1; -)$
$\theta_{4,1} : \Delta(0; 2, 2, 4, 4, 4, 4) \to C_4$	(a^2, a^2, a, a, a, a)
$\theta_{4,2} : \Delta(0; 2, 2, 4, 4, 4, 4) \to C_4$	$(a^2, a^2, a, a, a^{-1}, a^{-1})$
$\theta_{4,3} : \Delta(0; 2, 2, 2, 2, 2, 4, 4) \to C_4$	$(a^2, a^2, a^2, a^2, a^2, a, a)$
$\theta_{4,4} : \Delta(1; 2, 4, 4) \to C_4$	$(1, 1; a^2, a, a)$
$\theta_{4,5} : \Delta(1; 2, 2, 2, 2) \to C_4$	$(a, 1; a^2, a^2, a^2, a^2)$
$\theta_{4,6} : \Delta(2; -) \to C_4$	$(a, 1, 1, 1; -)$
$\theta_{4,7} : \Delta(0; 2, \overset{8}{\ldots}, 2) \to C_2 \times C_2$	$(a, \overset{6}{\ldots}, a, b, b)$
$\theta_{4,8} : \Delta(0; 2, \overset{8}{\ldots}, 2) \to C_2 \times C_2$	(a, a, a, a, b, b, b, b)
$\theta_{4,9} : \Delta(0; 2, \overset{8}{\ldots}, 2) \to C_2 \times C_2$	$(a, a, a, a, b, b, ab, ab)$
$\theta_{4,10} : \Delta(1; 2, 2, 2, 2) \to C_2 \times C_2$	$(1, 1; a, a, b, b)$

$\theta_{4,11} : \Delta(1; 2, 2, 2, 2) \to C_2 \times C_2$ $(b, 1; a, a, a, a)$
$\theta_{4,12} : \Delta(2; -) \to C_2 \times C_2$ $(a, b, 1, 1; -)$
$\theta_{4,13} : \Delta(2; -) \to C_2 \times C_2$ $(a, 1, b, 1; -)$
$\theta_{8,1} : \Delta(0; 2, 4, 8, 8) \to C_8$ (a^4, a^2, a, a)
$\theta_{8,2} : \Delta(0; 2, 4, 8, 8) \to C_8$ (a^4, a^2, a^3, a^{-1})
$\theta_{8,3} : \Delta(1; 2, 2) \to C_8$ $(a, 1; a^4, a^4)$
$\theta_{8,4} : \Delta(0; 2, 2, 2, 4, 4) \to C_4 \times C_2$ (b, b, b, ba^3, a)
$\theta_{8,5} : \Delta(0; 2, 2, 2, 4, 4) \to C_4 \times C_2$ (b, b, a^2b, ab, a)
$\theta_{8,6} : \Delta(0; 2, 2, 2, 4, 4) \to C_4 \times C_2$ (b, b, a^2, a, a)
$\theta_{8,7} : \Delta(0; 2, 2, 2, 4, 4) \to C_4 \times C_2$ (a^2, a^2b, b, a, a^3)
$\theta_{8,8} : \Delta(0; 2, 2, 2, 4, 4) \to C_4 \times C_2$ (a^2, a^2, b, a^3b, a)
$\theta_{8,9} : \Delta(0; 4, 4, 4, 4) \to C_4 \times C_2$ (a, a, ab, ab)
$\theta_{8,10} : \Delta(0; 4, 4, 4, 4) \to C_4 \times C_2$ (a, a^3, ab, a^3b)
$\theta_{8,11} : \Delta(1; 2, 2) \to C_4 \times C_2$ $(a, 1; b, b)$
$\theta_{8,12} : \Delta(1; 2, 2) \to C_4 \times C_2$ $(a, b; a^2, a^2)$
$\theta_{8,13} : \Delta(0; 2, \overset{6}{\ldots}, 2) \to C_2 \times C_2 \times C_2$ (a, a, a, ab, bc, c)
$\theta_{8,14} : \Delta(0; 2, \overset{6}{\ldots}, 2) \to C_2 \times C_2 \times C_2$ (a, a, b, b, c, c)
$\theta_{8,15} : \Delta(0; 2, \overset{6}{\ldots}, 2) \to C_2 \times C_2 \times C_2$ (a, a, b, ab, bc, abc)
$\theta_{8,16} : \Delta(1; 2, 2) \to C_2 \times C_2 \times C_2$ $(b, c; a)$
$\theta_{8,17} : \Delta(0; 2, \overset{6}{\ldots}, 2) \to D_4$ (s, s, s, s, sa, sa)
$\theta_{8,18} : \Delta(0; 2, \overset{6}{\ldots}, 2) \to D_4$ (s, s, sa, sa, a^2, a^2)
$\theta_{8,19} : \Delta(0; 2, 2, 2, 4, 4) \to D_4$ (s, s, a^2, a, a)
$\theta_{8,20} : \Delta(1; 2, 2) \to D_4$ $(t, 1; s, s)$
$\theta_{8,21} : \Delta(0; 4, 4, 4, 4) \to Q$ (i, i, j, j)

where $\theta_{4,6}$, $\theta_{4,12}$, $\theta_{4,13}$, $\theta_{8,3}$, $\theta_{8,9}$, $\theta_{8,10}$, $\theta_{8,11}$, $\theta_{8,12}$, $\theta_{8,16}$, $\theta_{8,20}$ *and* $\theta_{8,21}$ *are non-maximal actions.*

Proof. We show the detailed calculations to find actions of Q as an example of the algorithm. There are six classes of epimorphisms up to the action of $Aut(Q) \simeq S_4$. Those give the following generating vectors, (i, i, j, j), $(i, j, -i, j)$, (i, j, j, i), $(i, j, i, -j)$, $(i, j, -j, -i)$ and $(i, -i, -j, j)$. Now let B_i be the elements in $\overline{\mathcal{B}}$ defined by $x_i \mapsto x_{i+1}$, $x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}$. We get $B_2(i, i, j, j) = (i, j, -jij, j) = (i, j, -i, j)$, $B_3(i, j, -i, j) = (i, j, j, -j-ij) = (i, j, j, i)$, $B_3(i, j, j, i) = (i, j, i, -iji) = (i, j, i, -j)$, $B_3(i, j, i, -j) = (i, j, -j, ji-j) = (i, j, -j, -i)$, $B_2(i, j, -i, j) = (i, -i, ij-i, j) = (i, -i, -j, j)$. Thus there is one class of epimorphisms $\theta_{8,21} : \Delta(0; 4, 4, 4, 4) \to Q$.
Here we prove non-maximality. We need to consider only epimorphisms from Fuchsian groups with marked non-maximal signature.
(1) $\theta_{4,6}$ extends to $\theta_{8,17}$ where $C_4 \simeq \langle a \rangle \subset D_4$.
(2) $\theta_{4,12}$ extends to $\theta_{8,13}$ where $C_2 \times C_2 \simeq \langle b, abc \rangle \subset C_2 \times C_2 \times C_2$.
(3) $\theta_{4,13}$ extends to $\theta_{8,14}$ where $C_2 \times C_2 \simeq \langle ab, bc \rangle \subset C_2 \times C_2 \times C_2$.
(4) For $\theta_{8,9}$ consider the isomorphism $C_4 \times C_2 \simeq \left\langle ac, (ac)^2(bc)^2 \right\rangle \subset (C_4 \times C_2 \times C_2) \rtimes C_2 = G'$ and the action $\theta_{32,2} : \Delta'(0; 2, 2, 2, 4) \to G'$ with generating vector (ba, b, c, ac). The action $\delta : G' \to S_4$ of G' on the $C_4 \times C_2$-cosets defined by $\delta(ba) = (1, 4)(2, 3)$, $\delta(b) = (1, 2)(4, 3)$, $\delta(c) = (1, 3)(2, 4)$ and $\delta(ac) = (1)(4)(2)(3)$ is the monodromy of the desired covering $\mathcal{H}/\Delta \to \mathcal{H}/\Delta'$, where $s(\Delta) = (0; 4, 4, 4, 4)$ and the generating vector of the restriction to Δ of $\theta_{32,2}$ is $(ac, ac, ac(bc)^2, ac(bc)^2)$ as wanted.
(5) Similarly for $\theta_{8,10}$ consider the isomorphism $C_4 \times C_2 \simeq \left\langle ac, (ab)^2 \right\rangle \subset (C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2 = G'$ and the action $\theta_{32,1} : \Delta'(0; 2, 2, 2, 4) \to G'$ with generating vector (cac, b, bc, ac). The action $\delta : G' \to S_4$ of G' on the $\left\langle ac, (bc)^2 \right\rangle$-cosets defined by $\delta(cac) = (1, 3)(2, 4)$, $\delta(b) = (1, 2)(4, 3)$, $\delta(bc) = (1, 4)(2, 3)$ and $\delta(ac) = (1)(4)(2)(3)$ is the monodromy of the desired covering

$\mathcal{H}/\Delta \to \mathcal{H}/\Delta'$, where $s(\Delta) = (0; 4, 4, 4, 4)$ and the generating vector of the restriction to Δ of $\theta_{32,1}$ is $(ac, (ac)^3, ac(bc)^2, (ac)^3(bc)^2)$ as wanted.
(6) For $\theta_{8,21}$ consider the isomorphism $Q \simeq \langle ac, (ab)^2 \rangle \subset (D_4 \times C_2) \rtimes C_2 = G'$ and the action $\theta_{32,3} : \Delta'(0; 2, 2, 2, 4) \to G'$ with generating vector (abac, a, b, ac). The action $\delta : G' \to S_4$ of G' on the $\langle ac, (ab)^2 \rangle$-cosets defined by $\delta(abac) = (1, 4)(2, 3)$, $\delta(a) = (1, 3)(4, 2)$, $\delta(b) = (1, 3)(2, 4)$ and $\delta(ac) = (1)(4)(2)(3)$ is the monodromy of the desired covering $\mathcal{H}/\Delta \to \mathcal{H}/\Delta'$, where $s(\Delta) = (0; 4, 4, 4, 4)$ and the generating vector of the restriction to Δ of $\theta_{32,1}$ is $(ac = i, ac, (ab)^2 = j, (ab)^2)$ as wanted.
(7) With similar calculations to the ones above we see that $\theta_{8,3}$ extends to $\theta_{16,4}$ by $C_8 \simeq \langle a \rangle \subset D_8$. $\theta_{8,11}$, respectively $\theta_{8,12}$, extends to $\theta_{16,5}$, respectively $\theta_{16,6}$, by $C_4 \times C_2 \simeq \langle a, b \rangle \subset D_4 \times C_2$. $\theta_{8,16}$ extends to $\theta_{16,3}$ by $C_2 \times C_2 \times C_2 \simeq \langle a, bc, cd \rangle \subset C_2^4$. $\theta_{8,20}$ extends to $\theta_{16,5}$ by $C_4 \times C_2 \simeq \langle a, sb \rangle \subset D_4 \times C_2$. ■

Lemma 9 *The actions of order* 16, 32 *and* 64 *are:*

$\theta_{16,1} : \Delta(0; 4, 8, 8) \to C_8 \times C_2$	(a^6b, a, ab)
$\theta_{16,2} : \Delta(0; 2, 2, 4, 4) \to C_4 \times C_2 \times C_2$	(b, c, a, a^3bc)
$\theta_{16,3} : \Delta(0; 2, 2, 2, 2, 2) \to C_2^4$	$(a, b, c, d, abcd)$
$\theta_{16,4} : \Delta(0; 2, 2, 2, 2, 2) \to D_8$	$(s, sa, sa^7, sa^2, (a)^4)$
$\theta_{16,5} : \Delta(0; 2, 2, 2, 2, 2) \to D_4 \times C_2$	(s, s, sa, sab, b)
$\theta_{16,6} : \Delta(0; 2, 2, 2, 2, 2) \to D_4 \times C_2$	$(s, sa, sb, sab, (a)^2)$
$\theta_{16,7} : \Delta(0; 2, 2, 4, 4) \to D_4 \times C_2$	$(s, sb, ab, (a)^3)$
$\theta_{16,8} : \Delta(0; 2, 2, 4, 4) \to C_8 \rtimes_3 C_2$	$(s, s, as, (as)^3)$
$\theta_{16,9} : \Delta(0; 2, 2, 4, 4) \to (C_4 \times C_2) \rtimes C_2$	(b, c, ab, ac)
$\theta_{16,10} : \Delta(0; 4, 8, 8) \to C_8 \rtimes_5 C_2$	(asa, a, as)
$\theta_{16,11} : \Delta(1; 2) \to C_8 \rtimes_5 C_2$	$(a, s; a^4)$
$\theta_{16,12} : \Delta(1; 2) \to C_4 \rtimes C_4$	$(s, t; s^2)$
$\theta_{16,13} : \Delta(0; 2, 2, 4, 4) \to G_{4,4}$	(b, a^2, a, ab)
$\theta_{16,14} : \Delta(0; 2, 2, 4, 4) \to G_{4,4}$	(a^2b, b, a, a)
$\theta_{16,15} : \Delta(0; 2, 2, 4, 4) \to G_{4,4}$	(b, b, a, a^{-1})
$\theta_{16,16} : \Delta(0; 2, 2, 4, 4) \to G_{4,4}$	$(aba, aba^{-1}b, a, ab)$
$\theta_{16,17} : \Delta(1; 2) \to G_{4,4}$	$(a, b; aba^{-1}b)$
$\theta_{32,1} : \Delta(0; 2, 2, 2, 4) \to C_2^4 \rtimes C_2$	(cac, b, bc, ac)
$\theta_{32,2} : \Delta(0; 2, 2, 2, 4) \to (C_4 \times C_2 \times C_2) \rtimes C_2$	(ba, b, c, ac)
$\theta_{32,3} : \Delta(0; 2, 2, 2, 4) \to (D_4 \times C_2) \rtimes C_2$	$(abac, a, b, ac)$
$\theta_{32,4} : \Delta(0; 4, 4, 4) \to (C_4 \times C_2) \rtimes C_4$	(s, t, t^3s^3)
$\theta_{32,5} : \Delta(0; 4, 4, 4) \to ((C_4 \times C_2) \rtimes C_2) \rtimes C_2$	(s, st, ts^2)
$\theta_{32,6} : \Delta(0; 2, 8, 8) \to (C_8 \times C_2) \rtimes C_2$	$(t, s, s^{-1}t)$
$\theta_{32,7} : \Delta(0; 2, 8, 8) \to (C_8 \rtimes C_2) \rtimes C_2$	$(t, s, s^{-1}t)$
$\theta_{64,1} : \Delta(0; 2, 4, 8) \to ((C_8 \times C_2) \rtimes C_2) \rtimes C_2$	(t, ta^{-1}, a)
$\theta_{64,2} : \Delta(0; 2, 4, 8) \to ((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2$	(t, ta^{-1}, a)

where $\theta_{16,1}$, $\theta_{16,2}$, $\theta_{16,7}$, $\theta_{16,8}$, $\theta_{16,9}$, $\theta_{16,10}$, $\theta_{16,11}$, $\theta_{16,12}$, $\theta_{16,14}$, $\theta_{16,15}$, $\theta_{16,17}$, $\theta_{32,4}$, $\theta_{32,5}$, $\theta_{32,6}$, $\theta_{32,7}$ *and* $\theta_{64,1}$ *are non-maximal actions.*

Proof. Here we show non-maximality:
(1) $\theta_{16,1}$ extends to θ_{192} where $C_8 \times C_2 \simeq \langle t, (st^7s)^2 \rangle \subset (((C_4 \times C_2) \rtimes C_4) \rtimes C_3) \rtimes C_2$.
(2) $\theta_{16,2}$ extends to $\theta_{32,2}$ where $C_4 \times C_2 \times C_2 \simeq \langle a, bcb, c \rangle \subset (C_4 \times C_2 \times C_2) \rtimes C_2$.
(3) $\theta_{16,7}$ extends to $\theta_{32,1}$ where $D_4 \times C_2 \simeq \langle ac, c, (ab)^2 \rangle \subset (C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$.
(4) $\theta_{16,8}$ extends to $\theta_{32,3}$ where $C_8 \rtimes_3 C_2 \simeq \langle ab, bc \rangle \subset (D_4 \times C_2) \rtimes C_2$.
(5) $\theta_{16,9}$ extends to $\theta_{32,3}$ where $(C_4 \times C_2) \rtimes C_2 \simeq \langle (ab)^2, a, c \rangle \subset (D_4 \times C_2) \rtimes C_2$.

(6) $\theta_{16,10}$ extends to $\theta_{64,2}$ where $C_8 \rtimes_5 C_2 \simeq \langle a, [a,s] \rangle \subset ((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2$.
(7) $\theta_{16,11}$ extends to $\theta_{32,3}$ where $C_8 \rtimes_5 C_2 \simeq \langle ab, c \rangle \subset (D_4 \times C_2) \rtimes C_2$.
(8) $\theta_{16,12}$ extends to $\theta_{32,2}$ where $C_4 \rtimes C_4 \simeq \langle a, bc \rangle \subset (C_4 \times C_2 \times C_2) \rtimes C_2$.
(9) $\theta_{16,14}$ extends to $\theta_{32,2}$ where $G_{4,4} \simeq \langle ac, b \rangle \subset (C_4 \times C_2 \times C_2) \rtimes C_2$.
(10) $\theta_{16,15}$ extends to $\theta_{32,1}$ where $G_{4,4} \simeq \langle ac, b \rangle \subset (C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$.
(11) $\theta_{16,17}$ extends to $\theta_{32,1}$ where $G_{4,4} \simeq \langle ab, c \rangle \subset (C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$.
(12) $\theta_{32,4}$ extends to θ_{192} where $(C_4 \times C_2) \rtimes C_4 \simeq \langle (t^7 s)^2, (st^7)^2 \rangle \subset (((C_4 \times C_2) \rtimes C_4) \rtimes C_3) \rtimes C_2$.
(13) $\theta_{32,5}$ extends to $\theta_{64,2}$ where $((C_4 \times C_2) \rtimes C_2) \simeq \langle at, s \rangle \subset ((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2$.
(14) $\theta_{32,6}$ extends to θ_{192} where $(C_8 \times C_2) \rtimes C_2 \simeq \langle t, (st)^2 t^4 \rangle \subset (((C_4 \times C_2) \rtimes C_4) \rtimes C_3) \rtimes C_2$.
(15) $\theta_{32,7}$ extends to $\theta_{64,2}$ where $(C_8 \rtimes C_2) \rtimes C_2 \simeq \langle a^2 s, a \rangle \subset ((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2$.
(16) $\theta_{64,1}$ extends to θ_{192} where $((C_8 \times C_2) \rtimes C_2) \rtimes C_2 \simeq \langle st^2 s^2 t, st^6 s^2 \rangle \subset (((C_4 \times C_2) \rtimes C_4) \rtimes C_3) \rtimes C_2$. ■

Lemma 10 *The actions of order* $3, 6$ *and* 12 *are*

$\theta_{3,0} : \Delta(0;3,3,3,3,3,3,3) \to C_3$	(a,a,a,a,a,a^2,a^2)
$\theta_{3,1} : \Delta(1;3,3,3,3) \to C_3$	$(1,1;a,a,a^2,a^2)$
$\theta_{6,1} : \Delta(0;2,2,2,2,3,3) \to C_6$	$(a^3,a^3,a^3,a^3,a^2,a^4)$
$\theta_{6,2} : \Delta(0;2,3,3,3,6) \to C_6$	(a^3,a^2,a^2,a^4,a)
$\theta_{6,3} : \Delta(0;2,2,3,6,6) \to C_6$	(a^3,a^3,a^4,a,a)
$\theta_{6,4} : \Delta(0;6,6,6,6) \to C_6$	(a,a,a^5,a^5)
$\theta_{6,5} : \Delta(1;3,3) \to C_6$	$(a,1;a^2,a^4)$
$\theta_{6,6} : \Delta(0;2,2,2,2,3,3) \to D_3$	(s,s,s,s,a,a^2)
$\theta_{6,7} : \Delta(1;3,3) \to D_3$	$(s,1;a,a^2)$
$\theta_{12,1} : \Delta(0;6,12,12) \to C_{12}$	(a^{10},a,a)
$\theta_{12,2} : \Delta(0;2,2,6,6) \to C_6 \times C_2$	(b,b,a,a^{-1})
$\theta_{12,3} : \Delta(0;2,2,6,6) \to C_6 \times C_2$	(b,a^3,a,ba^2)
$\theta_{12,4} : \Delta(0;2,2,2,2,3) \to D_6$	(s,s,sa^3,sa,a^2)
$\theta_{12,5} : \Delta(0;2,2,2,2,3) \to D_6$	(a^3,a^3,s,sa^4,a^2)
$\theta_{12,6} : \Delta(0;2,2,6,6) \to D_6$	(sa^2,s,a,a)
$\theta_{12,7} : \Delta(1;3) \to D_6$	$(s,a;a^2)$
$\theta_{12,8} : \Delta(0;3,3,3,3) \to A_4$	(a,a^2,b,b^2)
$\theta_{12,9} : \Delta(0;3,3,3,3) \to A_4$	(a,a,ab,b^2)
$\theta_{12,10} : \Delta(0;2,3,4,4) \to C_3 \rtimes C_4$	(s^2,t,s,st^2)
$\theta_{12,11} : \Delta(1;3) \to C_3 \rtimes C_4$	$(t,s;t)$

where $\theta_{6,4}$, $\theta_{6,5}$, $\theta_{6,7}$, $\theta_{12,1}$, $\theta_{12,2}$, $\theta_{12,3}$, $\theta_{12,6}$, $\theta_{12,7}$, $\theta_{12,8}$, $\theta_{12,9}$ *and* $\theta_{12,11}$ *are non-maximal actions.*

Proof. We see non-maximality.
(1) $\theta_{6,4}$ extends to $\theta_{24,6}$ where $C_6 \simeq \langle a^5 b \rangle \subset D_6 \times C_2$.
(2) $\theta_{6,5}$ extends to $\theta_{12,4}$ where $C_6 \simeq \langle a \rangle \subset D_6$.
(3) $\theta_{6,7}$ extends to $\theta_{12,5}$ where $D_3 \simeq \langle a^2, sa \rangle \subset D_6$.
(4) $\theta_{12,1}$ extends to $\theta_{48,1}$ where $C_{12} \simeq \langle a \rangle \subset (C_{12} \times C_2) \rtimes C_2$.
(5) $\theta_{12,2}$ extends to $\theta_{24,6}$ where $C_6 \times C_2 \simeq \langle a, b \rangle \subset D_6 \times C_2$.
(6) $\theta_{12,3}$ extends to $\theta_{24,7}$ where $C_6 \times C_2 \simeq \langle b, (ab)^2 \rangle \subset (C_6 \times C_2) \rtimes C_2$.
(7) $\theta_{12,6}$ extends to $\theta_{24,6}$ where $D_6 \simeq \langle ab, sb \rangle \subset D_6 \times C_2$.
(8) $\theta_{12,7}$ extends to $\theta_{24,6}$ where $D_6 \simeq \langle a^2 b, sa^3 \rangle \subset D_6 \times C_2$.
(9) $\theta_{12,8}$ extends to $\theta_{48,5}$ where $A_4 \simeq \langle b^3 ab^3, a \rangle \subset S_4 \times C_2$.
(10) $\theta_{12,9}$ extends to $\theta_{24,2}$ where $A_4 \simeq \langle (ab)^2, b^2 \rangle \subset S_4 \times C_2$.

(11) $\theta_{12,11}$ extends to $\theta_{24,7}$ where $C_3 \rtimes C_4 \simeq \langle ab^3, b^2 \rangle \subset (C_6 \times C_2) \rtimes C_2$. ∎

Remark 11 *We note that there are two topologically non-equivalent actions of A_4 induced by the signature $(0;3,3,3,3)$. Indeed by considering all actions $\overline{\mathcal{B}}$ on (a, a^2, b, b^2), 218 in total, we note that none of the resulting generating vectors contain a pair of equal elements. These groups have the same representation up to $GL(5,\mathbb{C})$-conjugacy, see [KK], this provides a new counterexample to Theorem 1 in [Br]. These actions of A_4 are extensions fixed point free actions of $C_2 \times C_2$ of different topological type, see [CPo] and [Sm].*

Lemma 12 *Actions of order* $24, 48, 96$ *and* 192 *are:*

$\theta_{24,1} : \Delta(0;2,12,12) \to C_{12} \times C_2$	$(b, a, a^{-1}b)$
$\theta_{24,2} : \Delta(0;2,2,3,3) \to A_4 \times C_2$	(ab^3, b^3, b^2, b^4a)
$\theta_{24,3} : \Delta(0;2,2,3,3) \to A_4 \times C_2$	(ab^3, ab^3, b^2, b^4)
$\theta_{24,4} : \Delta(0;3,6,6) \to A_4 \times C_2$	(b^4, ba, ab)
$\theta_{24,5} : \Delta(0;2,2,3,3) \to S_4$	(ab, ab, a, a^{-1})
$\theta_{24,6} : \Delta(0;2,2,2,6) \to D_6 \times C_2$	(s, a^3, sa^4b, a^5b)
$\theta_{24,7} : \Delta(0;2,2,2,6) \to (C_6 \times C_2) \rtimes C_2$	(a, b^3, ab^4, aba)
$\theta_{24,8} : \Delta(0;4,4,6) \to (C_3 \rtimes C_4) \times C_2$	(s, st, s^2t^5)
$\theta_{48,1} : \Delta(0;2,4,12) \to (C_{12} \times C_2) \rtimes C_2$	(b, ba, a^{11})
$\theta_{48,2} : \Delta(0;2,6,6) \to A_4 \times C_2 \times C_2$	(b, a, a^5b)
$\theta_{48,3} : \Delta(0;3,4,4) \to A_4 \rtimes C_4$	(a, b, b^3a^2)
$\theta_{48,4} : \Delta(0;3,4,4) \to A_4 \rtimes C_4$	(a, b^3, ba^2)
$\theta_{48,5} : \Delta(0;2,2,2,3) \to S_4 \times C_2$	(ab, b^2c, ca^2b^3, a)
$\theta_{96,1} : \Delta(0;2,4,6) \to (A_4 \times C_2 \times C_2) \rtimes C_2$	(b, bac, ca^5)
$\theta_{96,2} : \Delta(0;3,3,4) \to ((C_4 \times C_2) \rtimes C_4) \rtimes C_3$	(s, s^2t^7, t)
$\theta_{192} : \Delta(0;2,3,8) \to (((C_4 \times C_2) \rtimes C_4) \rtimes C_3) \rtimes C_2$	(st^5, s, t^5)

where $\theta_{24,1}$, $\theta_{24,3}$, $\theta_{24,4}$, $\theta_{24,5}$, $\theta_{24,8}$, $\theta_{48,2}$, $\theta_{48,3}$ and $\theta_{96,2}$ are non-maximal actions.

Proof. We show the non-maximality:
(1) $\theta_{24,1}$ extends to $\theta_{48,1}$ where $C_{12} \times C_2 \simeq \langle a, (ab)^2 \rangle \subset (C_{12} \times C_2) \rtimes C_2$.
(2) $\theta_{24,3}$ extends to $\theta_{48,5}$ where $A_4 \times C_2 \simeq \langle ab^2a^2, ac \rangle \subset S_4 \times C_2$.
(3) $\theta_{24,4}$ extends to $\theta_{96,1}$ where $A_4 \times C_2 \simeq \langle [a,b], a \rangle \subset A_4 \times C_2 \times C_2$
(4) $\theta_{24,5}$ extends to $\theta_{48,5}$ where $S_4 \simeq \langle a, b \rangle \subset S_4 \times C_2$.
(5) $\theta_{24,1}$ extends to $\theta_{48,1}$ where $(C_3 \rtimes C_4) \times C_2 \simeq \langle ab, a^2 \rangle \subset (C_{12} \times C_2) \rtimes C_2$.
(6) $\theta_{48,2}$ extends to $\theta_{96,1}$ since $A_4 \times C_2 \times C_2 \subset (A_4 \times C_2 \times C_2) \rtimes C_2$
(7) $\theta_{48,3}$ extends to $\theta_{96,1}$ where $A_4 \rtimes C_4 \simeq \langle a^4, a^3cb \rangle \subset (A_4 \times C_2 \times C_2) \rtimes C_2$.
(8) $\theta_{96,2}$ extends to θ_{192}. ∎

Lemma 13 *Actions of order* 5, 10, 15, 20, 30, 40, 60, 80, 120 *and* 160 *are*

$\theta_5 : \Delta(1;5,5) \to C_5$	$(1, 1; a, a^4)$
$\theta_{10,1} : \Delta(0;2,2,10,10) \to C_{10}$	(a^5, a^5, a, a^9)
$\theta_{10,2} : \Delta(0;2,2,2,2,5) \to D_5$	(s, s, s, sa^5, a)
$\theta_{10,3} : \Delta(1;5) \to D_5$	$(s, a; a^2)$
$\theta_{15} : \Delta(0;3,15,15) \to C_{15}$	(a^5, a^4, a^8)
$\theta_{20,1} : \Delta(0;2,20,20) \to C_{20}$	(a^{10}, a, a^9)
$\theta_{20,2} : \Delta(0;2,2,2,10) \to D_{10}$	(a^5, s, sa^2, a^3)
$\theta_{20,3} : \Delta(0;4,4,10) \to C_5 \rtimes C_4$	(ab, a, a^2b)

$\theta_{30} : \Delta(0; 2, 6, 15) \to D_5 \times C_3 \qquad (b^3, ba, bab)$
$\theta_{40} : \Delta(0; 2, 4, 20) \to D_5 \times C_4 \qquad (b, ba, a^{-1})$
$\theta_{60} : \Delta(0; 3, 3, 5) \to A_5 \qquad (b, bab, b^2a^4b)$
$\theta_{80} : \Delta(0; 2, 5, 5) \to (C_2^4) \rtimes C_5 \qquad (b, a, a^{-1}b)$
$\theta_{120} : \Delta(0; 2, 3, 10) \to A_5 \times C_2 \qquad (ba^2c, b, a^2c)$
$\theta_{160} : \Delta(0; 2, 4, 5) \to ((C_2^4) \rtimes C_5) \rtimes C_2 \qquad (ab, b^{-1}, a^{-1})$

where θ_5, $\theta_{10,1}$, $\theta_{10,3}$, θ_{15}, $\theta_{20,1}$, $\theta_{20,3}$, θ_{60} *and* θ_{80} *are non-maximal actions.*

Proof. Clearly $\theta_{20,2}$ induces $\theta_{10,3}$, where D_5 is the subgroup $\langle a^2, sa\rangle$ of D_{10}. By [S2] θ_5, $\theta_{10,1}$, θ_{15}, $\theta_{20,1}$, $\theta_{20,3}$, θ_{60} and θ_{80} are non-maximal actions since there is only one action for each group and there exist actions of orders 30, 40, 120 and 160. ■

Lemma 14 *The actions of order* 11 *and* 22 *are*
$\theta_{11,1} : \Delta(0; 11, 11, 11) \to C_{11} \qquad (a, a^2, a^8)$
$\theta_{11,2} : \Delta(0; 11, 11, 11) \to C_{11} \qquad (a, a, a^9)$
$\theta_{22} : \Delta(0; 2, 11, 22) \to C_{22} \qquad (a^{11}, a^{10}, a)$

where $\theta_{11,2}$ *is a non-maximal action.*

Proof. The action $\theta_{11,2}$ extends to $\phi : \Delta(0; 2, 11, 22) \to C_{22}$ defined by (b^{11}, b^{10}, b).
■

Now each maximal action corresponds to a non-empty equisymmetric strata (see [B2]). Letting the strata corresponding to a maximal action $\theta_{i,j}$ be denoted $\mathcal{M}_5^{i,j}$ we have from Lemma 8 through Lemma 14 above the following theorem.

Theorem 15 *The equisymmetric strata of the branch locus* $\mathcal{B}_5$ *are the following.*

Strata	*dim.*	*Strata*	*dim.*	*Strata*	*dim.*	*Strata*	*dim.*
$\mathcal{M}_5^{2,0}$	9	$\mathcal{M}_5^{6,1}$	3	$\mathcal{M}_5^{8,19}$	2	$\mathcal{M}_5^{24,7}$	1
$\mathcal{M}_5^{2,1}$	8	$\mathcal{M}_5^{6,2}$	2	$\mathcal{M}_5^{10,2}$	2	$\mathcal{M}_5^{30}$	0
$\mathcal{M}_5^{2,2}$	7	$\mathcal{M}_5^{6,3}$	2	$\mathcal{M}_5^{11,1}$	0	$\mathcal{M}_5^{32,1}$	1
$\mathcal{M}_5^{2,3}$	6	$\mathcal{M}_5^{6,6}$	3	$\mathcal{M}_5^{12,4}$	2	$\mathcal{M}_5^{32,2}$	1
$\mathcal{M}_5^{3,0}$	4	$\mathcal{M}_5^{8,1}$	1	$\mathcal{M}_5^{12,5}$	2	$\mathcal{M}_5^{32,3}$	1
$\mathcal{M}_5^{3,1}$	4	$\mathcal{M}_5^{8,2}$	1	$\mathcal{M}_5^{12,10}$	1	$\mathcal{M}_5^{40}$	0
$\mathcal{M}_5^{4,1}$	3	$\mathcal{M}_5^{8,4}$	2	$\mathcal{M}_5^{16,3}$	2	$\mathcal{M}_5^{48,1}$	0
$\mathcal{M}_5^{4,2}$	3	$\mathcal{M}_5^{8,5}$	2	$\mathcal{M}_5^{16,4}$	2	$\mathcal{M}_5^{48,4}$	0
$\mathcal{M}_5^{4,3}$	4	$\mathcal{M}_5^{8,6}$	2	$\mathcal{M}_5^{16,5}$	2	$\mathcal{M}_5^{48,5}$	1
$\mathcal{M}_5^{4,4}$	3	$\mathcal{M}_5^{8,7}$	2	$\mathcal{M}_5^{16,6}$	2	$\mathcal{M}_5^{64,2}$	0
$\mathcal{M}_5^{4,5}$	4	$\mathcal{M}_5^{8,8}$	2	$\mathcal{M}_5^{16,13}$	1	$\mathcal{M}_5^{120}$	0
$\mathcal{M}_5^{4,7}$	5	$\mathcal{M}_5^{8,13}$	3	$\mathcal{M}_5^{16,16}$	1	$\mathcal{M}_5^{96,1}$	0
$\mathcal{M}_5^{4,8}$	5	$\mathcal{M}_5^{8,14}$	3	$\mathcal{M}_5^{20,2}$	1	$\mathcal{M}_5^{160}$	0
$\mathcal{M}_5^{4,9}$	5	$\mathcal{M}_5^{8,15}$	3	$\mathcal{M}_5^{22}$	0	$\mathcal{M}_5^{196}$	0
$\mathcal{M}_5^{4,10}$	4	$\mathcal{M}_5^{8,17}$	3	$\mathcal{M}_5^{24,2}$	1		
$\mathcal{M}_5^{4,11}$	4	$\mathcal{M}_5^{8,18}$	3	$\mathcal{M}_5^{24,6}$	1		

4 Orbifold Structure of the Moduli Spaces of Riemann Surfaces of Genera Four and Five

Here we will compute the structure of the equisymmetric stratification of the branch locus. Remember that $G = Aut(X)$ determines a conjugacy class of subgroups of $M(\Gamma)$, defining a symmetry type of X [B2]. $\mathcal{M}_g^{G,\theta}$ is the set of classes of surfaces with full automorphism group inducing the action (G,θ) and $\overline{\mathcal{M}}_g^{G,\theta}$ is the set of surfaces such that the automorphism group contains a subgroup yielding the action (G,θ). If, on some surface, (G,θ) is an extension of the action (G',θ') then $\overline{\mathcal{M}}_g^{G} \subset \overline{\mathcal{M}}_g^{G'}$. The action (G,θ) determines a Fuchsian group Δ and an epimorphism $\theta : \Delta \to G$. Letting $\Delta' = \theta'^{-1}(G')$ and examining the monodromy of the covering $\mathcal{H}/\Delta' \to \mathcal{H}/\Delta$ we can find the induced action $\theta : \Delta \to H$ by using of Theorem 1. Now as we have a one to one correspondence of the cosets of H and the cosets of Δ it is enough to study the permutation representations in the finite group G. As in Lemma 8 through Lemma 14, all the group symbolic calculations in finding the induced actions have been done with GAP.

We begin with theorems giving the structure of the branch locus in terms of strata corresponding to actions of prime order:

Theorem 16 *[CI2] The branch locus $\mathcal{B}_4$ is contained in*

$$\overline{\mathcal{M}}_4^{C_2,0} \cup \overline{\mathcal{M}}_4^{C_2,1} \cup \overline{\mathcal{M}}_4^{C_2,2} \cup \overline{\mathcal{M}}_4^{C_3,1} \cup \overline{\mathcal{M}}_4^{C_3,01} \cup \overline{\mathcal{M}}_4^{C_3,02} \cup \overline{\mathcal{M}}_4^{C_5}.$$

With the notation in Theorem 15 with have:

Theorem 17 *[BI] The branch locus $\mathcal{B}_5$ is contained in*

$$\overline{\mathcal{M}}_5^{2,0} \cup \overline{\mathcal{M}}_5^{2,1} \cup \overline{\mathcal{M}}_5^{2,2} \cup \overline{\mathcal{M}}_5^{2,3} \cup \overline{\mathcal{M}}_5^{3,0} \cup \overline{\mathcal{M}}_5^{3,1} \cup \overline{\mathcal{M}}_5^{11,1}.$$

Proof. **(1)** $\mathcal{M}_5^{2,0}$, $\mathcal{M}_5^{2,1}$, $\mathcal{M}_5^{2,2}$ and $\mathcal{M}_5^{2,3}$ correspond to epimorphisms $\theta : \Delta \to C_2$ with signatures $s(\Delta_0) = (0; 2, \overset{12}{\ldots}, 2)$, $s(\Delta_1) = (1; 2, \overset{8}{\ldots}, 2)$, $s(\Delta_2) = (2; 2, 2, 2, 2)$ and $s(\Delta_3) = (3; -)$ respectively.
(2) The strata $\mathcal{M}_5^{3,0}$ and $\mathcal{M}_5^{3,1}$ correspond to epimorphisms $\theta : \Delta \to C_3$ where $s(\Delta_0) = (0; 3, \overset{7}{\ldots}, 3)$ and $s(\Delta_1) = (1; 3, 3, 3, 3)$ respectively.
(3) $\overline{\mathcal{M}}_5^{5}$ is induced by non-maximal epimorphisms $\theta : \Delta \to C_5$, $s(\Delta) = (1; 5, 5)$. They extend to surface kernel epimorphism $\phi : \Delta' \to D_5 = \langle a, s | a^5 = s^2 = (sa)^2 = 1 \rangle$, $s(\Delta') = (0; 2, 2, 2, 2, 5)$, defined by (s, s, s, sa, a^{-1}). We see that $s(\phi^{-1}\langle a \rangle) = (1; 5, 5)$ and $s(\phi^{-1}\langle s \rangle) = (2; 2, 2, 2, 2)$. Thus $\overline{\mathcal{M}}_5^{5} \equiv \overline{\mathcal{M}}_5^{D_5,\theta} \subset \overline{\mathcal{M}}_5^{2,2}$.
(4) Signature $(0; 11, 11, 11)$. There are two classes of actions of C_{11} with representatives $\theta_{11,1} : \Delta \to C_{11}$, defined by (a, a^2, a^{-3}) and $\theta_{11,2} : \Delta \to C_{11}$, defined by (a, a, a^{-2}). Now $\theta_{11,2}$ extends to $\phi : \Delta(0; 2, 11, 22) \to C_{22}$ defined by (b^{11}, b^{10}, b). By Theorem 1 $\phi^{-1}\langle b^2 \rangle$ is a group with signature $(0; 11, 11, 11)$ and the images of the elliptic generators by ϕ (with the isomorphism $b^2 \to a$) are a, a and a^{-2}. So $\overline{\mathcal{M}}_5^{11,2} \equiv \overline{\mathcal{M}}_5^{22}$. By Theorem 1 $s(\phi_2^{-1}\langle b^{11} \rangle) = (0; 2, \overset{12}{\ldots}, 2)$, thus $\overline{\mathcal{M}}_5^{22} \subset \overline{\mathcal{M}}_5^{C_2,0}$. The epimorphism $\theta_{11,1}$ yields a maximal action of C_{11} in $\mathcal{M}_5$ producing an isolated point $\overline{\mathcal{M}}_5^{11,1}$. ∎
We show the inclusion relations of the strata in decreasing order of the number of automorphisms of the surfaces.

Theorem 18

$$\overline{\mathcal{M}}_4^{S_5} \in \overline{\mathcal{M}}_4^{S_4} \cap \overline{\mathcal{M}}_4^{D_5}$$

Proof. The Riemann surface $\overline{\mathcal{M}}_4^{S_5}$, known as Bring's curve is given by the action $\theta : \Delta(0;2,4,5) \to S_5$ defined by $\theta(x_1) = (4,5)$, $\theta(x_2) = (1,4,3,2)$ and $\theta(x_3) = (1,2,3,4,5)$. To prove the stament we consider the maximal subgroups of S_5: $A_5 = \langle 1(,3)(4,2),(1,2,3,4,5)\rangle$, $S_4 = \langle (1,2),(1,2,3,4)\rangle$ and $C_5 \rtimes_2 C_4 = \langle (1,2,5,4),(1,2,3,4,5)\rangle$. First, $\theta^{-1}(A_5) = \Lambda_1(0;2,5,5)$ with induced action $\phi_1 : \Lambda_1 \to A_5$ defined by $\phi_1(y_1) = (1,3)(4,2)$, $\phi_1(y_2) = (1,2,3,4,5)$ and $\phi_1(y_3) = (1,4,3,2,5)$. This action is non-maximal ([CI2]). Secondly, $\theta^{-1}(S_4) = \Lambda_2(0;2,2,2,4)$ with induced action $\phi_2 : \Lambda_2 \to S_4$ defined by $\phi_2(z_1) = (1,2)$, $\phi_2(z_2) = (3,4)$ $\phi_2(z_3) = (2,4)$ and $\phi_2(z_4) = (1,2,3,4)$, given in Theorem 7. Now, $\theta^{-1}(C_5 \rtimes_2 C_4) = \Lambda_3(0;4,4,5)$ with induced action $\phi_3 : \Lambda_3 \to C_5 \rtimes_4 C_4$ defined by $\phi_3(w_1) = (1,2,4,3)$, $\phi_3(w_2) = (1,2,5,4)$ and $\phi_3(w_3) = (1,2,3,4,5)$. This action is non-maximal ([CI2]).

We shall study the maximal subgroups of $C_5 \rtimes_4 C_4$: $D_5 = \langle (1,5)(2,4),(1,2,3,4,5)\rangle$ and $C_4 = \langle (1,2,5,4)\rangle$. $\theta^{-1}(D_5) = \Lambda_4(0;2,2,5,5)$ with induced action $\phi_4 : \Lambda_4 \to D_5$ defined by $\phi_4(\overline{x}_1) = (1,5)(2,4)$, $\phi_4(\overline{x}_2) = (2,5)(3,4)$ $\phi_4(\overline{x}_3) = (1,2,3,4,5)$ and $\phi_4(\overline{x}_4) = (1,4,2,5,3)$, given in Theorem 7. Finally $\theta^{-1}(C_4) = \Lambda_5(1;4,4)$ with induced action $\phi_5 : \Lambda_5 \to C_4$ defined by $\phi_5(\overline{a}_1) = \phi_5(\overline{b}_1) = Id$, $\phi_5(\overline{y}_1) = (1,2,5,4)$ and $\phi_5(\overline{y}_2) = (1,4,5,2)$. This action is non-maximal ([CI2]). ∎

Theorem 19

$$\overline{\mathcal{M}}_4^{(C_3\times C_3)\rtimes D_4} \in \overline{\mathcal{M}}_4^{D_3\times D_3} \cap \overline{\mathcal{M}}_4^{D_6,2}$$
$$\overline{\mathcal{M}}_4^{S_4\times C_3} \in \overline{\mathcal{M}}_4^{S_4} \cap \overline{\mathcal{M}}_4^{C_3\times D_3} \cap \overline{\mathcal{M}}_4^{C_6\times C_2}$$
$$\overline{\mathcal{M}}_4^{C_5\rtimes D_4} \in \overline{\mathcal{M}}_4^{D_{10}}$$
$$\overline{\mathcal{M}}_4^{D_3\times D_3} \subset \overline{\mathcal{M}}_4^{D_3,2} \cap \overline{\mathcal{M}}_4^{C_2\times C_2,1}$$
$$\overline{\mathcal{M}}_4^{C_6\times D_3} \in \overline{\mathcal{M}}_4^{C_3\times D_3}$$
$$\overline{\mathcal{M}}_4^{QD_{16}} \in \overline{\mathcal{M}}_4^{Q} \cap \overline{\mathcal{M}}_4^{D_8}$$

Theorem 20

$$\overline{\mathcal{M}}_4^{Q\rtimes C_3} \in \overline{\mathcal{M}}_4^{Q} \cap \overline{\mathcal{M}}_4^{C_6,0} \cap \overline{\mathcal{M}}_4^{C_4,0}$$
$$\overline{\mathcal{M}}_4^{S_4} \subset \overline{\mathcal{M}}_4^{D_4,12} \cap \overline{\mathcal{M}}_4^{D_3,1}$$
$$\overline{\mathcal{M}}_4^{D_{10}} \subset \overline{\mathcal{M}}_4^{C_2,0} \cap \overline{\mathcal{M}}_4^{C_2,2}$$
$$\overline{\mathcal{M}}_4^{C_3\times D_3} \subset \overline{\mathcal{M}}_4^{C_6,01} \cap \overline{\mathcal{M}}_4^{D_3,1} \cap \overline{\mathcal{M}}_4^{C_6,1}$$
$$\overline{\mathcal{M}}_4^{C_{18}} \in \overline{\mathcal{M}}_4^{C_6,0}$$
$$\overline{\mathcal{M}}_4^{D_8} \subset \overline{\mathcal{M}}_4^{D_4,02}$$

Theorem 21

$$\overline{\mathcal{M}}_4^{C_{15}} \in \overline{\mathcal{M}}_4^{C_5} \cap \overline{\mathcal{M}}_4^{C_3,01}$$
$$\overline{\mathcal{M}}_4^{D_6,2} \subset \overline{\mathcal{M}}_4^{D_3,2} \cap \overline{\mathcal{M}}_4^{C_2\times C_2,1}$$
$$\overline{\mathcal{M}}_4^{D_6\times C_2} \subset \overline{\mathcal{M}}_4^{C_2\times C_2,12} \cap \overline{\mathcal{M}}_4^{C_6,01}$$
$$\overline{\mathcal{M}}_4^{A_4} \subset \overline{\mathcal{M}}_4^{C_3,1} \cap \overline{\mathcal{M}}_4^{C_2\times C_2,1}$$
$$\overline{\mathcal{M}}_4^{C_{12}} \in \overline{\mathcal{M}}_4^{C_6,1} \cap \overline{\mathcal{M}}_4^{C_4,1}$$
$$\overline{\mathcal{M}}_4^{D_6,1} \subset \overline{\mathcal{M}}_4^{D_3,1} \cap \overline{\mathcal{M}}_4^{C_2\times C_2,12}$$

Theorem 22

$$\overline{\mathcal{M}}_4^{C_{10}} \in \overline{\mathcal{M}}_4^{C_5} \cap \overline{\mathcal{M}}_4^{C_2,2}$$
$$\overline{\mathcal{M}}_4^{D_5} \subset \overline{\mathcal{M}}_4^{C_2,2}$$
$$\overline{\mathcal{M}}_4^{Q} \subset \overline{\mathcal{M}}_4^{C_4,0}$$
$$\overline{\mathcal{M}}_4^{D_4,02} \subset \overline{\mathcal{M}}_4^{C_2\times C_2,02} \cap \overline{\mathcal{M}}_4^{C_4,0}$$
$$\overline{\mathcal{M}}_4^{D_4,12} \subset \overline{\mathcal{M}}_4^{C_2\times C_2,1} \cap \overline{\mathcal{M}}_4^{C_2\times C_2,12}$$

Theorem 23

$$\overline{\mathcal{M}}_4^{C_6,1} \subset \overline{\mathcal{M}}_4^{C_2,1} \cap \overline{\mathcal{M}}_4^{C_3,1}$$
$$\overline{\mathcal{M}}_4^{C_6,01} \subset \overline{\mathcal{M}}_4^{C_2,1} \cap \overline{\mathcal{M}}_4^{C_3,01}$$
$$\overline{\mathcal{M}}_4^{C_6,0} \subset \overline{\mathcal{M}}_4^{C_2,0} \cap \overline{\mathcal{M}}_4^{C_3,1}$$
$$\overline{\mathcal{M}}_4^{D_3,2} \subset \overline{\mathcal{M}}_4^{C_2,2} \cap \overline{\mathcal{M}}_4^{C_3,02}$$
$$\overline{\mathcal{M}}_4^{D_3,1} \subset \overline{\mathcal{M}}_4^{C_2,1}$$

Theorem 24

$$\overline{\mathcal{M}}_4^{C_2\times C_2,1} \subset \overline{\mathcal{M}}_4^{C_2,2}$$
$$\overline{\mathcal{M}}_4^{C_2\times C_2,02} \subset \overline{\mathcal{M}}_4^{C_2,2} \cap \overline{\mathcal{M}}_4^{C_2,0}$$
$$\overline{\mathcal{M}}_4^{C_2\times C_2,12} \subset \overline{\mathcal{M}}_4^{C_2,1} \cap \overline{\mathcal{M}}_4^{C_2,2}$$
$$\overline{\mathcal{M}}_4^{C_4,0} \subset \overline{\mathcal{M}}_4^{C_2,0}$$
$$\overline{\mathcal{M}}_4^{C_4,1} \subset \overline{\mathcal{M}}_4^{C_2,1}$$

The hyperelliptic locus of $\mathcal{M}_4$ is the subspace $\overline{\mathcal{M}}_4^{C_2,0} \supset \overline{\mathcal{M}}_4^{C_2\times C_2,02} \cup \overline{\mathcal{M}}_4^{C_4,0} \cup \overline{\mathcal{M}}_4^{C_6,0} \cup \overline{\mathcal{M}}_4^{D_4,02} \cup \overline{\mathcal{M}}_4^{Q} \cup \overline{\mathcal{M}}_4^{D_8} \cup \overline{\mathcal{M}}_4^{C_{18}} \cup \overline{\mathcal{M}}_4^{D_{10}} \cup \overline{\mathcal{M}}_4^{Q\times C_3} \cup \overline{\mathcal{M}}_4^{QD_{16}} \cup \overline{\mathcal{M}}_4^{C_5\times D_4}$, the cyclic trigonal locus is the subspace $\overline{\mathcal{M}}_4^{C_3,01} \cup \overline{\mathcal{M}}_4^{C_3,02}$ and the cyclic pentagonal locus is the subspace $\overline{\mathcal{M}}_4^{C_5} \cup \overline{\mathcal{M}}_4^{D_5} \cup \overline{\mathcal{M}}_4^{D_{10}}$. The stratum $\overline{\mathcal{M}}_4^{S_5}$ is formed by one curve called Bring's curve: the point in $\mathcal{M}_4$ with biggest number of automorphisms, 120. Bring's curve is a non-cyclic trigonal and cyclic pentagonal curve. The curve admits six pentagonal morphisms and the highest order of an automorphism is five (see [IY] also[CI3]).

Theorem 25

$$\overline{\mathcal{M}}_5^{64,2} \subset \overline{\mathcal{M}}_5^{32,1} \cap \overline{\mathcal{M}}_5^{16,16} \cap \overline{\mathcal{M}}_5^{8,2} \cap \overline{\mathcal{M}}_5^{8,14} \cap \overline{\mathcal{M}}_5^{4,2} \cap \overline{\mathcal{M}}_5^{4,5}$$
$$\overline{\mathcal{M}}_5^{32,1} \subset \overline{\mathcal{M}}_5^{16,3} \cap \overline{\mathcal{M}}_5^{16,5} \cap \overline{\mathcal{M}}_5^{8,14} \cap \overline{\mathcal{M}}_5^{8,19} \cap \overline{\mathcal{M}}_5^{4,2} \cap \overline{\mathcal{M}}_5^{4,5}$$
$$\overline{\mathcal{M}}_5^{32,2} \subset \overline{\mathcal{M}}_5^{16,5} \cap \overline{\mathcal{M}}_5^{16,6} \cap \overline{\mathcal{M}}_5^{8,6} \cap \overline{\mathcal{M}}_5^{8,14} \cap \overline{\mathcal{M}}_5^{8,15} \cap \overline{\mathcal{M}}_5^{4,2} \cap \overline{\mathcal{M}}_5^{4,5}$$
$$\overline{\mathcal{M}}_5^{32,3} \subset \overline{\mathcal{M}}_5^{16,6} \cap \overline{\mathcal{M}}_5^{8,7} \cap \overline{\mathcal{M}}_5^{8,18} \cap \overline{\mathcal{M}}_5^{8,19} \cap \overline{\mathcal{M}}_5^{4,2} \cap \overline{\mathcal{M}}_5^{4,5}$$

Theorem 26

$$\overline{\mathcal{M}}_5^{16,3} \subset \overline{\mathcal{M}}_5^{8,14} \cap \overline{\mathcal{M}}_5^{4,11}$$
$$\overline{\mathcal{M}}_5^{16,4} \subset \overline{\mathcal{M}}_5^{8,18} \cap \overline{\mathcal{M}}_5^{4,5}$$
$$\overline{\mathcal{M}}_5^{16,5} \subset \overline{\mathcal{M}}_5^{8,14} \cap \overline{\mathcal{M}}_5^{8,17} \cap \overline{\mathcal{M}}_5^{4,11}$$
$$\overline{\mathcal{M}}_5^{16,6} \subset \overline{\mathcal{M}}_5^{8,14} \cap \overline{\mathcal{M}}_5^{8,18} \cap \overline{\mathcal{M}}_5^{4,5} \cap \overline{\mathcal{M}}_5^{4,11}$$
$$\overline{\mathcal{M}}_5^{16,13} \subset \overline{\mathcal{M}}_5^{8,4} \cap \overline{\mathcal{M}}_5^{8,8} \cap \overline{\mathcal{M}}_5^{8,13}$$
$$\overline{\mathcal{M}}_5^{16,16} \subset \overline{\mathcal{M}}_5^{8,5} \cap \overline{\mathcal{M}}_5^{8,15}$$

Theorem 27 *The subvariety inclusions for strata corresponding to automorphism groups of order* 8 *are:*

$\overline{\mathcal{M}}_5^{8,1} \subset \overline{\mathcal{M}}_5^{4,1}$ $\quad\overline{\mathcal{M}}_5^{8,2} \subset \overline{\mathcal{M}}_5^{4,2}$
$\overline{\mathcal{M}}_5^{8,4} \subset \overline{\mathcal{M}}_5^{4,4} \cap \overline{\mathcal{M}}_5^{4,7}$ $\quad\overline{\mathcal{M}}_5^{8,5} \subset \overline{\mathcal{M}}_5^{4,4} \cap \overline{\mathcal{M}}_5^{4,9}$
$\overline{\mathcal{M}}_5^{8,6} \subset \overline{\mathcal{M}}_5^{4,1} \cap \overline{\mathcal{M}}_5^{4,8}$ $\quad\overline{\mathcal{M}}_5^{8,7} \subset \overline{\mathcal{M}}_5^{4,2} \cap \overline{\mathcal{M}}_5^{4,9}$
$\overline{\mathcal{M}}_5^{8,8} \subset \overline{\mathcal{M}}_5^{4,3} \cap \overline{\mathcal{M}}_5^{4,7}$ $\quad\overline{\mathcal{M}}_5^{8,13} \subset \overline{\mathcal{M}}_5^{4,7} \cap \overline{\mathcal{M}}_5^{4,10} \cap \overline{\mathcal{M}}_5^{2,3}$
$\overline{\mathcal{M}}_5^{8,14} \subset \overline{\mathcal{M}}_5^{4,8} \cap \overline{\mathcal{M}}_5^{4,11} \cap \overline{\mathcal{M}}_5^{2,3}$ $\quad\overline{\mathcal{M}}_5^{8,15} \subset \overline{\mathcal{M}}_5^{4,9} \cap \overline{\mathcal{M}}_5^{4,10} \cap \overline{\mathcal{M}}_5^{4,11}$
$\overline{\mathcal{M}}_5^{8,17} \subset \overline{\mathcal{M}}_5^{4,10}$ $\quad\overline{\mathcal{M}}_5^{8,18} \subset \overline{\mathcal{M}}_5^{4,5} \cap \overline{\mathcal{M}}_5^{4,9}$
$\overline{\mathcal{M}}_5^{8,19} \subset \overline{\mathcal{M}}_5^{4,2} \cap \overline{\mathcal{M}}_5^{4,9} \cap \overline{\mathcal{M}}_5^{4,11}$

Theorem 28 *The subvariety inclusions for strata corresponding to automorphism groups of order* 4 *are:*

$\overline{\mathcal{M}}_5^{4,1} \subset \overline{\mathcal{M}}_5^{2,1}$ $\quad\overline{\mathcal{M}}_5^{4,2} \subset \overline{\mathcal{M}}_5^{2,1}$
$\overline{\mathcal{M}}_5^{4,3} \subset \overline{\mathcal{M}}_5^{2,0}$ $\quad\overline{\mathcal{M}}_5^{4,4} \subset \overline{\mathcal{M}}_5^{2,2}$
$\overline{\mathcal{M}}_5^{4,5} \subset \overline{\mathcal{M}}_5^{2,1}$ $\quad\overline{\mathcal{M}}_5^{4,7} \subset \overline{\mathcal{M}}_5^{2,0} \cap \overline{\mathcal{M}}_5^{2,2} \cap \overline{\mathcal{M}}_5^{2,3}$
$\overline{\mathcal{M}}_5^{4,8} \subset \overline{\mathcal{M}}_5^{2,1} \cap \overline{\mathcal{M}}_5^{2,3}$ $\quad\overline{\mathcal{M}}_5^{4,9} \subset \overline{\mathcal{M}}_5^{2,1} \cap \overline{\mathcal{M}}_5^{2,2}$
$\overline{\mathcal{M}}_5^{4,10} \subset \overline{\mathcal{M}}_5^{2,2} \cap \overline{\mathcal{M}}_5^{2,3}$ $\quad\overline{\mathcal{M}}_5^{4,11} \subset \overline{\mathcal{M}}_5^{2,1} \cap \overline{\mathcal{M}}_5^{2,3}$

Theorem 29

$$\overline{\mathcal{M}}_5^{192} \subset \overline{\mathcal{M}}_5^{48,5} \cap \overline{\mathcal{M}}_5^{32,2} \cap \overline{\mathcal{M}}_5^{8,1}$$

$$\overline{\mathcal{M}}_5^{96,1} \subset \overline{\mathcal{M}}_5^{48,5} \cap \overline{\mathcal{M}}_5^{32,1} \cap \overline{\mathcal{M}}_5^{24,7} \cap \overline{\mathcal{M}}_5^{24,2} \cap \overline{\mathcal{M}}_5^{16,3} \cap \overline{\mathcal{M}}_5^{16,13} \cap \overline{\mathcal{M}}_5^{12,10} \cap \overline{\mathcal{M}}_5^{6,3}$$

$$\overline{\mathcal{M}}_5^{48,1} \subset \overline{\mathcal{M}}_5^{24,6} \cap \overline{\mathcal{M}}_5^{16,3} \cap \overline{\mathcal{M}}_5^{12,10} \cap \overline{\mathcal{M}}_5^{8,4} \cap \overline{\mathcal{M}}_5^{4,4}$$

$$\overline{\mathcal{M}}_5^{48,4} \subset \overline{\mathcal{M}}_5^{24,2} \cap \overline{\mathcal{M}}_5^{16,13} \cap \overline{\mathcal{M}}_5^{12,10}$$

$$\overline{\mathcal{M}}_5^{48,5} \subset \overline{\mathcal{M}}_5^{16,5} \cap \overline{\mathcal{M}}_5^{12,4} \cap \overline{\mathcal{M}}_5^{8,14} \cap \overline{\mathcal{M}}_5^{3,1}$$

Theorem 30

$$\overline{\mathcal{M}}_5^{24,2} \subset \overline{\mathcal{M}}_5^{8,13} \cap \overline{\mathcal{M}}_5^{6,1} \cap \overline{\mathcal{M}}_5^{3,1}$$

$$\overline{\mathcal{M}}_5^{24,6} \subset \overline{\mathcal{M}}_5^{12,5} \cap \overline{\mathcal{M}}_5^{12,4} \cap \overline{\mathcal{M}}_5^{8,13} \cap \overline{\mathcal{M}}_5^{6,6} \cap \overline{\mathcal{M}}_5^{4,7} \cap \overline{\mathcal{M}}_5^{4,10} \cap \overline{\mathcal{M}}_5^{3,1} \cap \overline{\mathcal{M}}_5^{2,2}$$

$$\overline{\mathcal{M}}_5^{24,7} \subset \overline{\mathcal{M}}_5^{12,4} \cap \overline{\mathcal{M}}_5^{8,17} \cap \overline{\mathcal{M}}_5^{6,3} \cap \overline{\mathcal{M}}_5^{4,8}$$

$$\overline{\mathcal{M}}_5^{12,4} \subset \overline{\mathcal{M}}_5^{6,6} \cap \overline{\mathcal{M}}_5^{4,10} \cap \overline{\mathcal{M}}_5^{3,1} \cap \overline{\mathcal{M}}_5^{2,3}$$

$$\overline{\mathcal{M}}_5^{12,5} \subset \overline{\mathcal{M}}_5^{6,1} \cap \overline{\mathcal{M}}_5^{6,6} \cap \overline{\mathcal{M}}_5^{4,7} \cap \overline{\mathcal{M}}_5^{3,1} \cap \overline{\mathcal{M}}_5^{2,3}$$

$$\overline{\mathcal{M}}_5^{12,10} \subset \overline{\mathcal{M}}_5^{6,1} \cap \overline{\mathcal{M}}_5^{4,3}$$

Theorem 31 *The subvariety inclusions for strata corresponding to automorphism groups of order* 6 *are:*

$\overline{\mathcal{M}}_5^{6,1} \subset \overline{\mathcal{M}}_5^{3,1} \cap \overline{\mathcal{M}}_5^{2,0}$ $\quad\overline{\mathcal{M}}_5^{6,2} \subset \overline{\mathcal{M}}_5^{3,0} \cap \overline{\mathcal{M}}_5^{2,2}$
$\overline{\mathcal{M}}_5^{6,3} \subset \overline{\mathcal{M}}_5^{3,1} \cap \overline{\mathcal{M}}_5^{2,1}$ $\quad\overline{\mathcal{M}}_5^{6,6} \subset \overline{\mathcal{M}}_5^{3,1} \cap \overline{\mathcal{M}}_5^{2,2}$

Theorem 32

$$\overline{\mathcal{M}}_5^{160} \subset \overline{\mathcal{M}}_5^{32,1} \cap \overline{\mathcal{M}}_5^{16,3} \cap \overline{\mathcal{M}}_5^{10,2}$$

$$\overline{\mathcal{M}}_5^{120} \subset \overline{\mathcal{M}}_5^{24,2} \cap \overline{\mathcal{M}}_5^{12,4} \cap \overline{\mathcal{M}}_5^{20,2}$$

$$\overline{\mathcal{M}}_5^{40} \subset \overline{\mathcal{M}}_5^{20,2} \cap \overline{\mathcal{M}}_5^{8,8} \cap \overline{\mathcal{M}}_5^{4,3}$$

$$\overline{\mathcal{M}}_5^{30} \subset \overline{\mathcal{M}}_5^{10,2} \cap \overline{\mathcal{M}}_5^{6,2} \cap \overline{\mathcal{M}}_5^{3,0}$$

$$\overline{\mathcal{M}}_5^{20,2} \subset \overline{\mathcal{M}}_5^{10,2} \cap \overline{\mathcal{M}}_5^{4,7} \cap \overline{\mathcal{M}}_5^{2,0} \cap \overline{\mathcal{M}}_5^{2,3}$$

$$\overline{\mathcal{M}}_5^{10,2} \subset \overline{\mathcal{M}}_5^{2,2}$$

Theorem 33 *$\overline{\mathcal{M}}_5^{11,1}$ is an isolated point of the branch locus and $\overline{\mathcal{M}}_5^{22} \subset \overline{\mathcal{M}}_5^{2,0}$.*

Proof. This was proved in Theorem 17. ∎

The next corollary was proved by Bartolini and Izquierdo [BI], but here we state it for the sake of completeness.

Corollary 34 *The branch locus $\mathcal{B}_5$ is connected with the exception of a single point.*

The conditions for existence of isolated points in the branch locus are given by Kulkarni [Ku] (see [BCI] for a geometric point of view). In [CI1] general conditions on connectedness of branch loci are given.

Remark 35 *From the theorems above we see that the hyperelliptic locus of $\overline{\mathcal{M}}_5$ contains:*

1. *a Riemann surface $\overline{\mathcal{M}}_5^{120}$ with automorphism group $A_5 \times C_2$ given in Lemma 12,*
2. *a Riemann surface $\overline{\mathcal{M}}_5^{96,1}$ with automorphism group $(A_4 \times C_2 \times C_2) \rtimes C_2$ given in Lemma 12,*
3. *a Riemann surface $\overline{\mathcal{M}}_5^{48,1}$ with automorphism group $(C_{12} \times C_2) \rtimes C_2$ given in Lemma 12,*
4. *a Riemann surface $\overline{\mathcal{M}}_5^{48,4}$ with automorphism group $A_4 \rtimes C_4$ given in Lemma 12,*
5. *a Riemann surface $\overline{\mathcal{M}}_5^{40}$ with automorphism group $D_5 \times C_4$ given in lemma 13,*
6. *a uniparametric family of Riemann surfaces $\overline{\mathcal{M}}_5^{24,2}$ with automorphism group $A_4 \times C_2$ given in Lemma 12,*
7. *a single Riemann surface $\overline{\mathcal{M}}_5^{22}$, which is also 11-gonal, with automorphism group C_{22} given in Lemma 14,*
8. *a uniparametric family of Riemann surfaces $\mathcal{M}_5^{20,2}$ with automorphism group D_{10} given in Lemma 13,*
9. *a uniparametric family of Riemann surfaces $\mathcal{M}_5^{16,13}$ with automorphism group$G_{4,4}$ given in Lemma 9,*
10. *a uniparametric family of Riemann surfaces $\mathcal{M}_5^{12,10}$ with automorphism group $C_3 \rtimes C_4$ given in Lemma 10,*
11. *a family of Riemann surfaces $\mathcal{M}_5^{8,4}$ with automorphism group $C_4 \times C_2$ given in Lemma 8,*
12. *a family of Riemann surfaces $\mathcal{M}_5^{8,8}$ with automorphism group $C_4 \times C_2$ given in Lemma 8,*
13. *a family of Riemann surfaces $\mathcal{M}_5^{8,13}$ with automorphism group $C_2 \times C_2 \times C_2$ given in Lemma 8,*

14. a family of Riemann surfaces $\mathcal{M}_5^{6,1}$ with automorphism group C_6 given in Lemma 10,

15. a family of Riemann surfaces $\mathcal{M}_5^{4,3}$ with C_4 given in Lemma 8

16. a family of Riemann surfaces $\mathcal{M}_5^{4,7}$ with automorphism group $C_2 \times C_2$ given in Lemma 8.

Remark 36 *From Theorems 29, 30, 31 and 32 it follows that a cyclic trigonal Riemann surface of genus* 5 *has at most* 30 *automorphisms and with exception of one surface, X_{30}, at most six automorphisms. The structure of the trigonal surfaces in $\mathcal{M}_5$ is*

$$\overline{\mathcal{M}}_5^{30} \in \overline{\mathcal{M}}_5^{6,2} \subset \overline{\mathcal{M}}_5^{3,0}$$

where $\overline{\mathcal{M}}_5^{30}$ is a single surface with automorphism group $D_5 \times C_3$ given in Theorem 13. Observe that if $g \not\equiv 2$ mod 3 there are cyclic trigonal surfaces of genus g having $C_{3(g+1)}$ as a group of automorphisms (See [CI3]).

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5 Appendix: GAP-codes

Some GAP-codes to find and classify actions of groups on Riemann surfaces, beginning with a code to find admissible signatures.

```
findorders:= function(genus, index)
local orders;
orders:=[];
for d in DivisorsInt(index) do
 if d>1 then
  if d<4*genus+3 then
   Add(orders,d);
  fi;
 fi;
od;
return orders;
end;;


signcalc:= function(orders,position,sign,signatures,n,g,g0)
local tempsign, sigsum;
 tempsign:=ShallowCopy(sign);

 #Display([position,sign,tempsign]);

 if position < Length(orders) + 1 then
  sigsum:=[];

  Add(tempsign, orders[position]);

  for order in tempsign do
   Add(sigsum,1-1/order);
  od;

  if n*(Sum(sigsum)-2+2*g0)=2*g-2 then
   Add(signatures,[g0,tempsign]);
```

```
  elif n*(Sum(sigsum)-2+g0)<2*g-2 then
   #Display("recursion list position:");
   #Display([position,sign,tempsign]);
   signcalc(orders,position+1,sign,signatures,n,g,g0);
   #Display([position,sign,tempsign]);
   #Display("recursion number of elements:");
   signcalc(orders,position,tempsign,signatures,n,g,g0);

  fi;

 fi;

 return signatures;
end;;


findsignatures:= function(genus,index)
local signs;
 signs:=[];
 for a in [0.. genus] do
  Append(signs,signcalc(findorders(genus,index),
          1,[],[],index,genus,a));
  if index*(-2+2*a)=2*genus-2 then
   Add(signs,[a,[]]);
  fi;
 od;
 return signs;
end;;
```

Next one finds and classifies group actions with signature $(0; m_1, \ldots, m_k)$. A slightly modified code is used when $g = 1$.

```
elementproducts:=function(elements,position,products)
local templist, templists1, templists2;
templist:=[];
templists1:=[];
templists2:=[];
if position>Size(elements) then
 return products;
else
 templists1:=elementproducts(elements,position+1,products);
 for number in elements[position] do
  for list in templists1 do
   templist:=ShallowCopy(list);
   #Display(templist);
   Add(templist,number,1);
   Add(templists2,templist);
   #Display(templists1);
  od;
 od;
 return templists2;
fi;
end;;

#########################################
findepi:=function(group,signature)
local orders, elementsoforder, epimorphisms, autorbits,
      braidorbits, totalorbits, epiclasses, templist;
orders:=[];
elementsoforder:=[];
epimorphisms:=[];
epiclasses:=[];
templist:=[];
autorbits:=[];
braidorbits:=[];
```

```
totalorbits:=[];
for order in signature[2] do
 if (order in orders)<>true then
  Add(orders,order);
 fi;
od;
for order in orders do
 templist:=[];
 for g in group do if Order(g)=order then Add(templist,g); fi; od;
 Add(elementsoforder,[order,templist]);
od;
templist:=[];
for order in signature[2] do
 for set in elementsoforder do
  if set[1]=order then
   Add(templist, set[2]);
  fi;
 od;
od;
for product in elementproducts(templist,1,[[]]) do
 if (Product(product)=Identity(group)) and
    (Size(Subgroup(group,product))=Size(group)) then
  Add(epimorphisms,product);
 fi;
od;
autorbits:=OrbitsDomain(AutomorphismGroup(group),
                        epimorphisms, OnTuples);
#Display(["auto",autorbits]);
if Size(autorbits)=1 then
 epiclasses:=[Representative(autorbits[1])];
else
 for epiclass in autorbits do
  Add(braidorbits,braidorbit(Representative(epiclass)));
 od;
 totalorbits:=[1..Size(autorbits)];
 for braidclass in [1..Size(autorbits)-1] do
  for element in braidorbits[braidclass] do
   for class in [braidclass..Size(autorbits)-1] do
    if (element in autorbits[class+1]) then
     #Display([braidclass,element,class]);
     totalorbits[class+1]:=0;
    fi;
   od;
  od;
 od;
 for n in totalorbits do
  if n > 0 then
   Add(epiclasses,Representative(autorbits[n]));
  fi;
 od;
fi;
#Display(totalorbits);
return epiclasses;
end;;

#########################################

braid:=function(genvector,position)
local tempvector;
tempvector:=ShallowCopy(genvector);
Remove(tempvector,position);
Add(tempvector,genvector[position+1],position);
Remove(tempvector,position+1);
Add(tempvector,
```

```
    genvector[position+1]^-1*genvector[position]*genvector[position+1],
    position+1);
return tempvector;
end;;

#####################

braidorbit:=function(genvector)
local orbit;
orbit:=[];
Add(orbit, genvector);
for point in orbit do
 for position in [1..Size(genvector)-1] do
  if (braid(point,position) in orbit) <> true then
   Add(orbit, braid(point,position));
  fi;
 od;
od;
return orbit;
end;;
```

To calculate the signatures of subgroups of Fuchsian groups:

```
subgroupsignatures:=function(signature, epimorphism, orggroup)
local group, subgroup, indsig, subgroupinfo, cycles, order, i, j,
      k, sigsum1, sigsum2, genus, sggens;
indsig:=[];
cycles:=[];
order:=[];
subgroupinfo:=[];
sggens:=[];
group:=GroupWithGenerators(epimorphism);
for subgroup in MaximalSubgroupClassReps(group) do
  indsig:=[];
  PermG:=Action(group,RightCosets(group,subgroup),OnRight);
  Permgens:=GeneratorsOfGroup(PermG);
  #Display(Permgens);
  for k in [1..Size(Permgens)] do
   cycles:=CycleStructurePerm(Permgens[k]);
   for j in [1..Size(cycles)] do
    if IsBound(cycles[j]) then
     for i in [1..cycles[j]] do
      #Display([sign[2][k],(j+1)]);
      order:=sign[2][k]/(j+1);
      if order>1 then
       Add(indsig,sign[2][k]/(j+1));
      fi;
     od;
    fi;
   od;
   if Index(group,subgroup)>NrMovedPoints(Permgens[k]) then
    for i in [1..Index(group,subgroup)-NrMovedPoints(Permgens[k])] do
     Add(indsig,sign[2][k]);
    od;
   fi;
  od;
  sigsum1:=[];
  sigsum2:=[];
  for order in indsig do
   Add(sigsum1,1-1/order);
  od;
  for order in signature[2] do
   Add(sigsum2,1-1/order);
  od;
  genus:=1+(signature[1]-1+Sum(sigsum2)/2)*Index(group,subgroup)
```

```
          -Sum(sigsum1)/2;
  Sort(indsig);
  sggens:=[];
  for g in GeneratorsOfGroup(subgroup) do
   Add(sggens,Factorization(orggroup, g));
  od;
  Add(subgroupinfo,[IdSmallGroup(subgroup),StructureDescription(subgroup),
                    [genus,indsig],sggens,Permgens,
                    List(RightTransversal(group,subgroup))]);
od;
return subgroupinfo;
end;;
```