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# Replenishment Decision Making with Permissible Shortage, Repairable Nonconforming Products and Random Equipment Failure

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**Abstract:** This study is concerned with replenishment decision making with repairable nonconforming products, backordering and random equipment failure during production uptime. In real world manufacturing systems, due to different factors generation of nonconforming items and unexpected machine breakdown are inevitable. Also, in certain business environments various situations between vendor and buyer, the backordering of shortage stocks sometimes is permissible with extra cost involved. This study incorporates backlogging, random breakdown and rework into a production system, with the objective of determination of the optimal replenishment lot size and optimal level of backordering that minimizes the long-run average system costs. Mathematical modeling along with the renewal reward theorem is employed for deriving system cost function. Hessian matrix equations are used to prove its convexity. Research result can be directly adopted by practitioners in the production planning and control field to assist them in making their own robust production replenishment decision.

**Keywords:** Backordering, equipment failure, production planning and control, repairable defects

## INTRODUCTION

Addressing the problem on he Economic Production Quantity (EPQ) can be traced back to the study by Taft (1918) several decades ago. The EPQ model guides manufacturing firms in determining the optimal production lot size that minimizes the long-run average production-inventory costs. Although assumptions in the classic EPQ model are relatively simple or unrealistic, the EPQ model remains to be the basis for analyzing more complex systems (Wagner and Whitin, 1958; Hadley and Whitin, 1963; Hutchings, 1976; Schneider, 1979; Schwaller, 1988; Silver *et al*., 1998; Tripathy *et al*., 2003; Nahmias, 2009; Chen, 2011).

One of the assumptions in EPQ model is that all manufactured items are of perfect quality. However, owing to many unpredictable factors, generating the nonconforming items seems inevitable. The defective items issues and its consequence quality assurance matters have been broadly studied (Bielecki and Kumar, 1988; Lee and Rosenblatt, 1987; Cheng, 1991; Chern and Yang, 1999; Boone *et al*., 2000; Teunter and Flapper, 2003; Chiu *et al*., 2011b, 2012a; Amirteimoori and Emrouznejad, 2011; Pandey *et al*., 2011). In real world the stock-out situations may arise occasionally due to unexpected excess demands and in certain business environments various situations between vendor and buyer, the backordering of shortage items sometimes is permissible. They are commonly satisfied in the very next replenishment and in this case extra backordering cost is involved (Chiu, 2003; Chiu and Chiu, 2006; Drake *et al*., 2011).

Production equipment failure is another reliability factor that troubles the production practitioners most. Therefore, to effectively control and manage the disruption caused by random breakdown, so the overall production costs can be minimized, becomes a critical task to most production planners. It is not surprising that such an issue has received extensive attentions from researchers during past decades (Widmer and Solot, 1990; Groenevelt *et al*., 1992; Kuhn, 1997; Makis and Fung, 1998; Giri and Dohi, 2005; Chiu *et al*., 2010, 2012b; Chiu *et al*., 2011a, 2012b; Das *et al*., 2011). Widmer and Solot (1990) examined breakdown and maintenance operation problem using queuing network theory. They presented an easy way of modeling these perturbations so that they can be taken into account when evaluating the performances of an FMS (production rate, machine utilization, etc.). A comparison between the analytical and simulation results was provided to demonstrate the accuracy of their proposed modeling technique. Groenevelt *et al*. (1992) studied effects of machine breakdown and corrective maintenance on economic lot sizing decisions. Two different control policies: the No-Resumption (NR) and Abort-Resume (AR) were examined. NR policy assumes that production of the interrupted lots is not resumed after

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a breakdown, while AR policy assumes that production is immediately resumed after a breakdown, if the current on-hand inventory is below a certain threshold level. They showed that this control structure is optimal among all stationary policies and provided exact optimal and closed form approximate lot sizing formulas and bounds on average cost per unit time for the approximations. Makis and Fung (1998) examined an EMQ model with inspections and random machine failures. Effects of breakdowns on the optimal lot size and optimal number of inspections were studied. The formula for the long-run expected average cost per unit time was obtained and the optimal production and inspection policy that minimize the expected average costs are derived. Giri and Dohi (2005) presented the exact formulation of stochastic EMQ model for an unreliable production system. Their EMQ model was formulated based on the Net Present Value (NPV) approach and by taking limitation on the discount rate the traditional long-run average cost model was obtained. The criteria for the existence and uniqueness of the optimal production time and its computational results were provided to show that the optimal decision based on the NPV approach is superior to that based on the long-run average cost approach. Chiu *et al*. (2012b) studies the optimal replenishment run time for a production system with stochastic machine breakdown and failure in rework. They assumed that a production system is subject to Poisson breakdowns (under no-resumption policy) and an imperfect reworking of defective items. Mathematical modeling was used and the production-inventory cost function was derived. Conditional proof of theorem and proposition was presented with the objective of determining the optimal replenishment run time that minimizes the expected costs per unit time. This paper incorporates the backlogging, reworking of nonconforming items and random equipment failure into the EPQ model, with the objective of determining the optimal replenishment lot size and maximal level of backordering that minimizes the long-run average cost for such a realistic system. Because little attention has been paid to the aforementioned area, this research intends to bridge the gap.

## METHODOLOGY

**Mathematical modeling and formulation:** Consider in a production system the annual demand rate for a specific item is $\lambda$ and this item can be produced at a rate $P$ per year, where $P$ is much larger than $\lambda$. All products produced are screened and the unit inspection cost is included in unit manufacturing cost $C$. Let $x$ be the random nonconforming rate and $d$ denotes the rate of making imperfect quality items, where, $d = Px$. All nonconforming items produced are assumed to be 100% repairable during the rework process (Fig. 1) and it is further assumed that the production rate of perfect quality items must always be greater than the sum of the demand rate $\lambda$ and the defective rate $d$. That is $(P-d-\lambda)>0$.

Due to the long-term relationships between manufacturer and its clients, when demand occasionally exceeds supply, shortages are allowed and backordered. These items will be satisfied when the next replenishment production cycle starts. The imperfect quality items are assumed to be all repairable through a rework process. Further, according to the Mean Time Between Failures (MTBF) analysis, a Poisson distributed breakdown may occur during the on-hand inventory piling time (Fig. 1). When a machine failure happens, the abort/resume inventory control policy is adopted in this study. Under such a policy, when a breakdown takes place the machine is under repair immediately and a constant repair time is assumed. Further, the interrupted lot will be resumed right after the production equipment is fixed and put back to use.

It is also assumed that during the setup time, prior to the production uptime, the working status of machine is fully checked and confirmed. Hence, the chance of breakdown in a very short period of time when production begins is small. It is also assumed that due to tight preventive maintenance schedule, the probability of more than one machine breakdown occurrences in a production cycle is very small. However, if it does happen, safety stock will be used to satisfy the demand during machine repairing time. Therefore, multiple machine failures are assumed to have insignificant effect on the proposed model. Figure 1 depicts the level of on-hand inventory of perfect quality items in proposed model.

The related system cost parameters include: unit production cost $C$, setup cost $K$, unit repair cost for each defective item reworked $C_R$, cost for repairing machine $M$, unit holding cost $h$, unit holding cost per reworked item $h$ and unit shortage backordering cost $b$. Additional notation has:

- $Q$ = Production replenishment lot size for each cycle, to be determined by this study
- $B$ = The maximum backorder level allowed for each cycle, to be determined by this study
- $T$ = Production cycle length
- $T_1$ = Production run time to be determined by the proposed study
- $H_1$ = Level of on-hand inventory when machine breakdown occurs
- $H_2$ = Level of on-hand inventory when machine is repaired and restored
- $H_3$ = Level of on-hand inventory when the remaining regular production uptime ends
- $H_4$ = The maximum level of perfect quality inventory when rework finishes
- t = Production time before a random breakdown occurs

$t_r$ = Time required for repairing and restoring the machine
$t_2$ = Time needed to rework the defective items
$t_3$ = Time required for depleting all available perfect quality on-hand items,
$t_4$ = Shortage permitted time
$t_5$ = Time required for filling the backorder quantity $B$
$I(t)$ = On-hand inventory of perfect quality items in time $t$
$I_d(t)$ = On-hand inventory of defective items in time $t$
$TC(T_1,B)$ = Total production-inventory costs per cycle
$TCU(T_1,B)$ = Total production-inventory costs per unit time
$E[TCU(T_1,B)]$ = The expected total production-inventory costs per unit time

From Fig. 1, the following basic formulas can be directly obtained: different levels of on-hand perfect products during production uptime; production run time $T_1$; the cycle length $T$; time for rework $t_2$; time required to deplete all available on-hand items $t_3$; shortage allowed time $t_4$, time for refilling backlogging $B$ (maximum backordering quantity) $t_5$ and the levels of on-hand inventory $H_1$, $H_2$, $H_3$ and $H_4$:

$$H_1 = (P - d - \lambda)t \quad (1)$$

$$H_2 = H_1 - t_r \lambda = H_1 - g\lambda \quad (2)$$

$$H_3 = H_2 + (P - d - \lambda).(T_1 - t_5 - t) \quad (3)$$

$$H_4 = H_3 + (P_1 - \lambda)t_2 \quad (4)$$

$$T_1 = Q/P \quad (5)$$

$$T = T_1 + t_2 + t_3 + t_4 + t_r \quad (6)$$

$$t_2 = d.T_1/P_1 \quad (7)$$

$$t_3 = H_4/\lambda \quad (8)$$

$$t_4 = B/\lambda \quad (9)$$

$$t_5 = B/(P - d - \lambda) \quad (10)$$

where, the repair time for equipment is assumed to be a constant $t_r = g$ and $d = Px$.

In real life situation as well as in the present study, it is conservatively assumed that if a failure of a machine cannot be fixed within a certain allowable amount of time, then a spare machine will be in place to avoid further delay of production. The level of on-hand nonconforming products for the proposed system is depicted in Fig. 2.

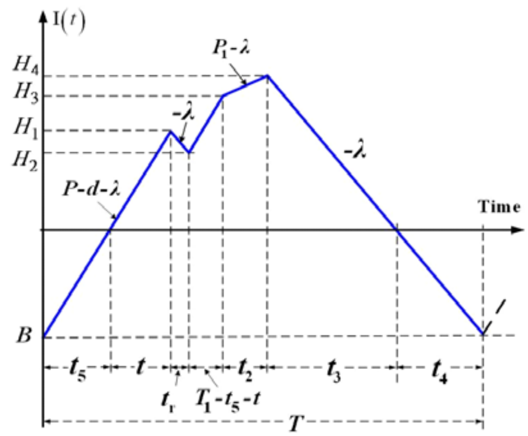


Fig. 1: On-hand inventory of perfect products in the proposed model with backlogging, repairable defects and breakdown taking place during stock piling time

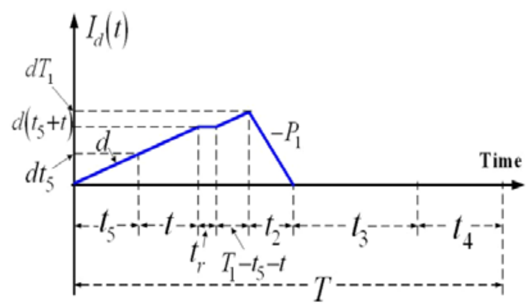


Fig. 2: On-hand inventory of repairable nonconforming items in the proposed production sy

Total imperfect quality items produced during the production run time $T_1$ are:

$$d.T_1 = x.\ Q \quad (11)$$

**Cost analysis for the proposed system:** Form the above equations and Fig. 1 and 2, one obtains the total production-inventory cost per cycle $TC(T_1, B)$ as follows:

$$\begin{aligned} TC(T_1,B) &= K + M + C.(PT_1) + C_R.[PT_1.x] \\ &+ h\left[\frac{H_1(t)}{2} + \frac{H_1+H_2}{2}(t_r) + \frac{H_2+H_3}{2}(T_1 - t_5 - t) + \frac{H_3+H_4}{2}(t_2) + \frac{H_4(t_3)}{2}\right] \\ &+ h\left[\frac{d(t_5+t)}{2}(t_5+t) + (t_5+t)t_r + \frac{(t_5+t)+dT_1}{2}(T_1 - t_5 - t)\right] \\ &+ h_1\left[\frac{P_1 t_2}{2}(t_2)\right] + b\left[\frac{B}{2}(t_5) + \frac{B}{2}(t_4)\right] \end{aligned} \quad (12)$$

Substituting all parameters from Eq. (1) to (11) in (12), $TC(T_1, B)$ becomes:

$$\begin{aligned} TC(T_1,B) &= C\cdot P\cdot T_1 + K + M + C_R\cdot T1\cdot P\cdot x \\ &+ \frac{h}{2}\left\{\frac{P^2}{\lambda}T_1^2 + \frac{(1-x)}{\lambda(1-x-\lambda/P)}B^2 - PT_1^2\right\} \\ &+ \frac{b(1-x)}{2\lambda(1-x-\lambda/P)}B^2 + \frac{P_1x_2}{2P_1}\left[h_1-h\right]T_1^2 - h\frac{P}{\lambda}T_1B \\ &+ \frac{hg}{(1-x-\lambda/P)}B - hPgT_1 + hPgt + \frac{hg^2\lambda}{(1-x-\lambda/P)} \end{aligned} \tag{13}$$

One notes that the expected values of $x$ can be employed to take into account random nonconforming rate in the production-inventory cost analysis. Further, because the machine is subject to Poisson machine breakdown rate (with mean equals to $\beta$ per unit time), one can use integration of $TC(T_1, B)$ to deal with such a random failure distribution. Therefore, the long-run expected costs per unit time $E[TCU(T_1, B)]$ can be calculated as follows:

$$E\left[TCU\left(T_1,B\right)\right] = \frac{E\int_0^{T_1-t_5}TC\left(T_1,B\right)\cdot f\left(t\right)dt}{E\left[T\right]} = \frac{E\left[\int_0^{T_1-t_5}TC\left(T_1,B\right)\left(\beta e^{-\beta t}\right)dt\right]}{\left[T_1P/\lambda\right]\cdot\left(1-e^{-\beta(T_1-t_5)}\right)} \tag{14}$$

Substituting all related parameters from Eq. (1) to (13) in the numerator of (14) one has:

$$E\left[\int_0^{T1-t5}TC(T1,B)\,f(t)dt\right] = E\left[\begin{array}{l} (1-e^{-\beta(T_1-t_5)})\left\{\begin{array}{l} K+M+P\cdot T_1\cdot\left(C+C_R\cdot x\right) \\ +\frac{h}{2}\left[\frac{P^2}{\lambda}T_1^2+\frac{(1-x)}{\lambda(1-x-\lambda/P)}B^2-PT_1^2\right] \\ +\frac{b(1-x)}{2\lambda(1-x-\lambda/P)}\cdot B^2+\frac{P^2x^2(h_1-h)}{2P_1}T_1^2-h\cdot\frac{P}{\lambda}T_1B \\ +\frac{hg}{(1-x-\lambda/P)}B-hPgT_1+hPg\left(T_1-\frac{(g\lambda+B)}{P(1-x-\lambda/P)}+\frac{1}{\beta}\right)+\frac{hg^2\lambda}{(1-x-\lambda/P)} \end{array}\right\} \\ -hPg\left(T_1-\frac{(g\lambda+B)}{P(1-x-\lambda/P)}\right) \end{array}\right] \tag{15}$$

With further derivations, the numerator of Eq. (14) becomes:

$$\begin{aligned} E\left[\int_0^{T_1-t_5}TC\left(T_1,B\right)F(t)dt\right] &= -hPgT_1 + h\left(g^2\lambda+gB\right)E\left[\frac{1}{1-x-\lambda/P}\right] \\ &+\left[1-e^{-\beta\left(T_1-t_5\right)}\right]\cdot\left\{\begin{array}{l} K+M+P\cdot T_1\cdot\left[C+C_R\cdot E[x]\right]+hPg/\beta \\ +\frac{h}{2}\left[\frac{P^2}{\lambda}T_1^2-PT_1^2\right]-\frac{hPT_1B}{\lambda} \\ +\frac{B^2}{2\lambda}(b+h)E\left[\frac{1-x}{1-x-\lambda/P}\right]+\frac{P^2T_1^2\left(E[x]^2\right)}{2P_1}\left[h_1-h\right] \end{array}\right\} \end{aligned} \tag{16}$$

Substituting Eq. (16) in (14) one has $E[TCU(T_1, B)]$ as follows:

$$\begin{aligned} E\left[TCU\left(T_1,B\right)\right] &= \frac{h\lambda\left(g^2\lambda+gB\right)}{T_1P\left(1-e^{-\beta\left(T_1-t_5\right)}\right)}E\left[\frac{1}{1-x-\lambda/P}\right]-\frac{hg\lambda}{\left(1-e^{-\beta\left(t_1-t_5\right)}\right)} \\ &+\left\{\begin{array}{l} \frac{\lambda(K+M)}{T_1P}+\lambda\left[C+C_R\cdot E[x]\right]+\frac{hg\lambda}{T_1\beta}+\frac{h}{2}\left[PT_1-T_1\lambda\right]-hB \\ +\frac{B^2}{2PT_1}(b+h)E\left[\frac{1-x}{1-x-\lambda/P}\right]+\frac{PT_1\lambda\left(E[x]\right)^2}{2P_1}\left[h_1-h\right] \end{array}\right\} \end{aligned} \tag{17}$$

$$Let\ E_1 = E[x]; E_2 = (E[x])^2; E_3 = E\left[\frac{1-x}{1-x-\lambda/P}\right]; E_4 = E\left[\frac{1}{1-x-\lambda/P}\right] \tag{18}$$

Substituting Eq. (18) in (17) one has:

$$E\left[TCU(T_1,B)\right] = \frac{h\lambda(g^2\lambda+gB)}{T_1P\left(1-e^{-\beta(T_1-t_5)}\right)}E_4 - \frac{hg\lambda}{(1-e^{-\beta(T_1-t_5)})} + \frac{\lambda(K+M)}{T_1P} + \lambda\left[C + C_R E_1\right] + \frac{hg\lambda}{T_1\beta} + \frac{hT_1}{2}(P-\lambda) - hB + \frac{B^2}{2PT_1}(b+h)E_3 + \frac{PT_1\lambda}{2P_1}(h_1-h)E_2 \tag{19}$$

**Convexity and the optimal operating decisions:** In order to find the optimal production lot size, one should first prove the convexity of $E[TCU(T_1,B)]$. Hessian matrix equations (Rardin, 1998; Hillier and Lieberman, 2001) can be employed for the proof:

$$\begin{bmatrix}T_1 & B\end{bmatrix}\cdot\begin{pmatrix}\frac{\partial^2 E\left[TCU(T_1,B)\right]}{\partial T_1^2} & \frac{\partial^2 E\left[TCU(T_1,B)\right]}{\partial T_1\partial B}\\ \frac{\partial^2 E\left[TCU(T_1,B)\right]}{\partial T_1\partial B} & \frac{\partial^2 E\left[TCU(T_1,B)\right]}{\partial B^2}\end{pmatrix}\cdot\begin{bmatrix}T_1\\ B\end{bmatrix} > 0 \tag{20}$$

$E[TCU(T_1,B)]$ is strictly convex only if Eq. (21) is satisfied for all $T_1$ and $B$ different from zero. With further derivation one obtains (Appendix):

$$\begin{bmatrix}T1 & B\end{bmatrix}\cdot\begin{pmatrix}\frac{\partial^2 E[TCU(T_1,B)]}{\partial T_1^2} & \frac{\partial^2 E[TCU(T1,B)]}{\partial T_1\partial B}\\ \frac{\partial^2 E[TCU(T_1,B)]}{\partial T_1\partial B} & \frac{\partial^2 E[TCU(T_1,B)]}{\partial B^2}\end{pmatrix}\cdot\begin{bmatrix}T_1\\ B\end{bmatrix} = \frac{2(K+M)\lambda}{T_1P} + \frac{2hg\lambda}{T_1\beta} + \frac{2hg^2\lambda^2E_4}{T_1P\left(1-e^{-\beta(T_1-t_5)}\right)} > 0 \tag{21}$$

Equation (21) is resulting positive because all parameters are positive. Hence, $E[TCU(T_1,B)]$ is a strictly convex function. It follows that for the optimal uptime $T_1$ and the optimal backordering level $B$, one differentiates $E[TCU(T_1,B)]$ with respect to $T_1$ and with respect to $B$ and solve the linear systems of Eq. (22) and (23) by setting these partial derivatives equal to zero:

$$\frac{\partial E[TCU(T_1,B)]}{\partial T_1} = \left\{\begin{aligned}&-\frac{\lambda(K+M)}{T_1^2P} + \frac{h}{2}(P-\lambda) - \frac{B^2}{2PT_1^2}(b+h)E_3\\ &\frac{P\lambda}{2P_1}(h_1-h)E_2 - \frac{hg\lambda}{T_1^2\beta} - \frac{h\lambda(g^2\lambda+gB)}{T_1^2P(1-e^{-\beta(T_1-t_5)})}E_4\end{aligned}\right\} = 0 \tag{22}$$

$$\frac{\partial E[TCU(T_1,B)]}{\partial B} = \left\{\frac{B}{T_1P}(b+h)E_3 - h + \frac{hg\lambda}{T_1P(1-e^{-\beta(t_1-t_5)})}E_4\right\} = 0 \tag{23}$$

From Eq. (23) one has:

$$\therefore\ B^* = \left(\frac{h}{b+h}\right)\left(\frac{1}{E_3}\right)\left[PT_1 - \frac{g\lambda E_4}{\left(1-e^{-\beta(T_1-t_5)}\right)}\right] \tag{24}$$

With further derivations Eq. (22) becomes:

$$\frac{1}{T_1^2}\left[\frac{\lambda(K+M)}{P}+\frac{B^2}{2P}(b+h)E_3+\frac{hg\lambda}{\beta}+\frac{h\lambda(g^2\lambda+gB)}{P(1-e^{-\beta(T_1-t_5)})}E_4\right] = \frac{h}{2}(P-\lambda)+\frac{P\lambda}{2P_1}[h1-h]\cdot E_2 \tag{25}$$

Substituting Eq. (24) in (25) one has:

$$\frac{1}{T_1^2}\left[\begin{array}{l}\frac{\lambda(K+M)}{P}+\frac{1}{2P}(b+h)E_3\left[\frac{hPT_1}{(b+h)E_3}-\frac{hg\lambda E_4}{(b+h)E_3(1-e^{-\beta(T_1-t_5)})}\right]^2+\frac{hg\lambda}{\beta}\\ +\frac{hg^2\lambda^2}{P(1-e^{-\beta(T_1-t_5)})}E_4+\frac{hg\lambda}{P(1-e^{-\beta(T_1-t_5)})}E_4\left[\frac{hPT_1}{(b+h)E_3}-\frac{hg\lambda E_4}{(b+h)E_3\left[1-e^{-\beta(T_1-t_5)}\right]}\right]\end{array}\right] = \frac{h}{2}(P-\lambda)+\frac{P\lambda}{2P_1}\left[h_1-h\right].E_2 \tag{26}$$

Therefore, the optimal replenishment run time is:

$$T_1^* = \frac{1}{P}\sqrt{\frac{2\lambda(K+M)-\dfrac{h^2g^2\lambda^2E_4^2}{(b+h)E_3[1-e^{-\beta(T_1-t_5)}]^2}+\dfrac{2hg^2\lambda^2}{\left[1-e^{-\beta(T1-t5)}\right]}E_4+\dfrac{2Phg\lambda}{\beta}}{h\left(1-\dfrac{\lambda}{P}\right)+\dfrac{\lambda}{P_1}\left(h_1-h\right)E_2-\dfrac{h^2}{(b+h)E_3}}} \tag{27}$$

Substituting Eq. (18) in (27) and let:

$$\pi_1 = E[1/(1-x-\lambda/P)] \text{ and } \pi_2 = (b+h).\ E[(1-x)/(1-x-\lambda/P)] \tag{28}$$

## RESULTS AND DISCUSSION

The optimal solutions in terms of production run time and lot size are obtained as follows:

$$T_1^* = \frac{1}{P}\sqrt{\frac{2\lambda(K+M)-(h^2g^2\lambda^2\pi_1^2)/\left[\left(1-e^{-\beta(T_1-t_5)^2}\pi_2\right)\right]+\left(2hg^2\lambda^2\pi_1\right)/\left[1-e^{-\beta(T_1-t_5)}\right]+\left(2Phg\lambda\right)/\beta}{h\left[1-\dfrac{\lambda}{P}\right]+\dfrac{\lambda}{P_1}\left[h_1-h\right]\left(E[x]\right)^2-\dfrac{h^2}{\pi_2}}} \tag{29}$$

$$Q^* = \sqrt{\frac{2\lambda(K+M)-\left(h^2g^2\lambda^2\pi_1^2\right)/\left[\left(1-e^{\left(-\beta(T_1-t_5)\right)}\right)^2\pi_2\right]+\left(2hg^2\lambda^2\pi_1\right)/\left(1-e^{-\beta(T_1-t_5)}\right)+\left(2Phg\lambda\right)/\beta}{h\left[1-\dfrac{\lambda}{P}\right]+\dfrac{\lambda}{P_1}\left[h_1-h\right]\left(E[x]\right)^2-\dfrac{h_2}{\pi_2}}} \tag{30}$$

If production equipment failure factor is not an issue at all, then machine repairing cost and time are both zero (i.e., $M$ = 0 and $g$ = 0), Eq. (30) and (24) become the same as were given in Chiu (2003) as follows:

$$Q^* = \sqrt{\frac{2K\lambda}{h\left[1-\dfrac{\lambda}{P}\right]+\dfrac{\lambda}{P_1}\left[h_1-h\right]\left(E[x]\right)^2-\dfrac{h^2}{(b+h)\cdot E\left[(1-x)/(1-x-\lambda/P)\right]}}} \tag{31}$$

$$B^* = \left(\frac{h}{b+h}\right)\left(\frac{1}{E\left[\frac{1-x}{1-x-\lambda/P}\right]}\right)PT_1 \tag{32}$$

or

$$B^* = \left(\frac{h}{b+h}\right)\left(\frac{1}{E\left[\frac{1-x}{1-x-\lambda/P}\right]}\right)Q^* \tag{33}$$

Further, if the nonconforming rate is zero (i.e., $x = 0$), then Eq. (31) and (33) become the same equations as those in classic EPQ model with shortages backordered (Hillier and Lieberman, 2001):

$$Q^* = \sqrt{\frac{2K\lambda}{h\left(1-\frac{\lambda}{P}\right)}}\cdot\sqrt{\frac{b+h}{b}} \tag{34}$$

$$B^* = \left[\frac{h}{(b+h)}\left(1-\frac{\lambda}{P}\right)\right]\cdot Q^* \tag{35}$$

**Numerical example with further discussion:** Consider a product has annual demand 4600 units and its annual production rate $P$ is 11500 units. Production equipment is subject to a random breakdown that follows a Poisson distribution with mean $\beta = 2$ times per year and according to the MTBF analysis, a random breakdown is expected to occur in inventory piling time. Abort/Resume (AR) policy is used when a random breakdown occurs. The percentage $x$ of defective items produced follows a uniform distribution over the interval [0, 0.2]. All nonconforming products are repairable at a rework rate $P_1$= 600 units/year. Additional values of system parameters are

$C$ = $2 per item
$C_R$ = $0.5 for each item reworked
$K$ = $450 for each production run
$h$ = $0.6 per item per unit time
$h_1$ = $0.8 per item per unit time,
$M$ = $500 repair cost for each breakdown
$b$ = $0.2 per item backordered per unit time
$g$ = 0.018 years, time needed to repair and restore the machine

Applying Eq. (29), (30) and (24), one obtains the optimal production run time $T_1^*$ = 0.7454 (years), the optimal replenishment lot-size $Q^*$ = 8572 and the optimal backordering level $B^*$ = 3447. Plugging these decision variables in Eq. (19), the optimal $E[TCU(T_1^*,B^*)]$ = $10,386.06. Figure 3 illustrates variation of the expected values of nonconforming rate E[$x$] effects on E[TCU($Q^*$, $B^*$)]. One notes that as E[$x$] increases, the value of the long-run average cost function E[TCU($Q^*$, $B^*$)] increases significantly.

Figure 4 demonstrates the convexity of the long-run average cost function E[TCU($Q$, $B$)] for the proposed model. Finally, one notes that suppose the research result

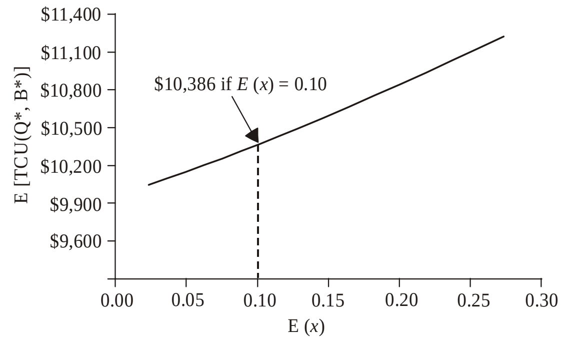


Fig. 3: Variation of the expected values of nonconforming rate E[$x$] effects on E[TCU($Q^*$,$B^*$)]

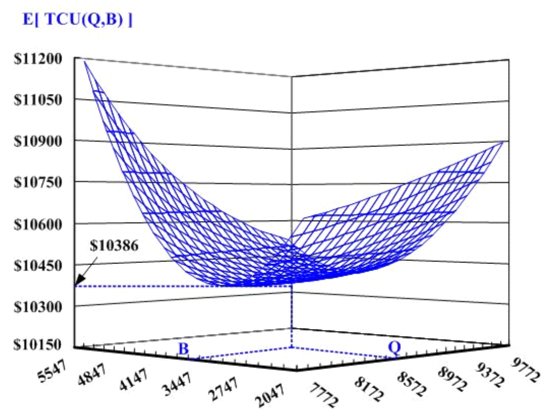


Fig. 4: Demonstration of the convexity of the long-run average cost function E[TCU($Q$,$B$)] for the proposed model

of the present study does not exist, one will probably use a closely related model (Chiu, 2003) to obtain $Q$ = 5601 and $B$ = 2320. Then by plugging $T_1$ and B in Eq. (19) one obtains $E[TCU(Q,B)]$ = $10,577. One notes that it will cost $191 more than the optimal cost we have or 16.1% more on total other related cost (i.e., E[$TCU$($Q$,$B$)]-$\lambda C$).

## CONCLUSION

The present study incorporates the backordering of permissible shortage, the reworking of repairable nonconforming items and random equipment failure into the classic economic production quantity model. In the manufacturing sector, all aforementioned factors are realistic and/or inevitable. Without an in-depth investigation of such a real life production system, the optimal lot-size and level of backordering that minimize total production-inventory costs cannot be obtained. Because little attention has been paid to this area, this research intends to bridge the gap. For future study, one could look into the effect of equipment failure taking place in backorder satisfying time on the replenishment decisions.

**Appendix:**

Proof of convexity of $E[TCU(T_1,B)]$
Applying the Hessian matrix equations (Rardin, 1998) to Eq. (19) one has:

$$\frac{\partial^2 E[TCU(T_1,B)]}{\partial T_1^2} = \frac{2\lambda(K+M)}{T_1^3 P} + \frac{B^2}{T_1^3 P}(b+h)\cdot E_3 + \frac{2hg\lambda}{T_1^3 \beta} + \frac{2h\lambda(g^2\lambda + gB)}{T_1^3 P(1-e^{-\beta(T_1-t_5)})}E_4 \quad \text{(A-1)}$$

$$\frac{\partial^2 E[TCU(T_1,B)]}{\partial T_1 \partial B} = -\frac{B}{T_1^2 P}(b+h)E_3 - \frac{hg\lambda}{T_1^2\left(1-e^{-\beta(T_1-t_5)}\right)}E_4 \quad \text{(A-2)}$$

$$\frac{\partial^2 E[TCU(T_1,B)]}{\partial B^2} = \frac{1}{T_1 P}(b+h)E_3 \quad \text{(A-3)}$$

then,

$$\begin{aligned}
&\begin{bmatrix} T_1 & B \end{bmatrix} \cdot \begin{pmatrix} \frac{\partial^2 E[TCU(T_1,B)]}{\partial T_1^2} & \frac{\partial^2 E[TCU(T_1,B)]}{\partial T_1 \partial B} \\ \frac{\partial^2 E[TCU(T_1,B)]}{\partial T_1 \partial B} & \frac{\partial^2 E[TCU(T_1,B)]}{\partial B^2} \end{pmatrix} \cdot \begin{bmatrix} T_1 \\ B \end{bmatrix} \\
&= \left\{ \begin{array}{l} \frac{2(K+M)\lambda}{T_1 P} + \frac{B^2}{T_1 P}(b+h)E_3 + \frac{2hg\lambda}{T_1\beta} + \frac{2hg^2\lambda^2 E_4}{T_1 P(1-e^{-\beta(T_1-t_5)})} + \frac{2hg\lambda B E_4}{T_1 P(1-e^{-\beta(T_1-t_5)})} \\ -\frac{2B^2}{T_1 P}(b+h)E_3 - \frac{2Bhg\lambda E_4}{T_1 P(1-e^{-\beta(T_1-t_5)})} + \frac{B^2}{T_1 P}(b+h)E_3 \end{array} \right\} \quad \text{(A-4)} \\
&= \frac{2(K+M)\lambda}{T_1 P} + \frac{2hg\lambda}{T_1\beta} + \frac{2hg^2\lambda^2 E_4}{T_1 P\left(1-e^{-\beta(T_1-t_5)}\right)}
\end{aligned}$$

Therefore, one has Eq. (21).

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