Eigenvalues of Structural Matrices Via Gerschgorin Theorem

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Summary

In this paper, we have presented a simple approach for determining eigenvalues for some class of structural matrices. It is based on Gerschgorin theorem. The main advantage of the proposed method is that there is no need to use time-consuming iterative numerical techniques for determining eigenvalues. The proposed approach is expected to be applicable in various computer science and control system applications.

Key words:

Eigenvalues, Gerschgorin theorem, structural matrices, trace of the matrix

1. Introduction

The concept of stability plays very important role in the analysis of systems. A system can be modeled in state space form [1]. In this state space form, stability can be determined by computing the eigenvalues of the system matrix *A*. There exist various methods in the literature for the computation of the eigenvalues [2, 3]. Moreover, in engineering applications, some structural matrices have been used and thus their eigenvalues computations are also important. In mathematical literature, we found that that there exists Gerschgorin theorem [4-6], which gives bounds under which, all eigenvalues lie. Now a days, eigenvalues can be calculated easily using Matlab. But, we found that Gerschgorin theorem can be useful for computation of some eignvalues without involving iterative numerical technique and softwares.

In this paper, we have presented numerical efficient method for computation of eigenvalues using Gerschgorin theorem for some class of structural matrices. The proposed idea can be useful in various engineering applications.

2. Gerschgorin Theorem

For a given matrix *A* of order ($n \times n$), let P_k be the sum of the moduli of the elements along the k^{th} row excluding the diagonal elements a_{kk}. Then every eigenvalues of *A* lies inside the boundary of atleast one of the circles

$$\left|\lambda - a_{kk}\right| = P_k \quad (1)$$

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3. Determination of Eigenvalues of Structural Matrices

Consider a structural system matrix *A* as

$$[A] = \begin{bmatrix} a & -b & -b & \dots & -b \\ -b & a & -b & \dots & -b \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -b & -b & -b & \dots & a \end{bmatrix} \quad (2)$$

where,

$$a > 0, b > 0, a = \left|(n-1)b\right| \quad (3)$$

Applying above Gerschgorin theorem to above matrix *A*, we get,

$$\left|\lambda - a_{ii}\right| \le \sum_{\substack{i=1 \\ i \ne j}}^{n} \left|a_{ij}\right| = r_j \quad (4)$$

In above matrix $[A]$, replace

$$a_{ii} = a, r_j = \left|(n-1)b\right| \quad (5)$$

So, from eq. (4) and eq.(5), we get,

$$\left|\lambda - a\right| \le a \quad (6)$$

By removing modulus of above equation, we get,

$$\pm(\lambda - a) \le a \quad (7)$$

So,

$$-(\lambda - a) \le a \quad (8)$$

or

$$(\lambda - a) \le a \quad (9)$$

Now, consider eq.(8) as

$$-\lambda + a \le a \tag{10}$$

From above eq.(10), we get,

$$-\lambda \le 0 \tag{11}$$

Here, $-\lambda < 0$ is rejected, since Gerschgorin bounds are positive as mentioned in eq. (3). So, from above eq.(11), $\lambda = 0$, is one of the eigenvalue of the structural matrix. Thus, for above structural matrix, one of the eigenvalues is at the origin. Let us consider eigenvalue at origin as $\lambda_n = 0$. Now, we calculate remaining eigenvalues. It is given below in following steps.

Step 1: Applying Gerschgorin theorem to matrix [*A*], we get,

$$|\lambda_1 - a| \le |(n-1)b| \tag{12}$$

$$|\lambda_2 - a| \le |(n-1)b| \tag{13}$$

$$\vdots$$

$$\vdots$$

$$|\lambda_{n-1} - a| \le |(n-1)b| \tag{14}$$

By subtracting eq. (13) from eq.(12), we get,

$$|\lambda_1 - \lambda_2| \le 0 \tag{15}$$

In above eq.(15), $|\lambda_1 - \lambda_2| < 0$ is rejected, since the absolute value of any number is always positive. Thus, we get,

$$\lambda_1 - \lambda_2 = 0 \text{, i.e., } \lambda_1 = \lambda_2 \tag{16}$$

Similarly from eq. (13) to eq.(14), we can show that

$$\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_{n-1} = k \tag{17}$$

where, k is the repeated eignvalue.

Step 2: Using definitions of trace of the matrix, i.e.,

$$Trace(A) = \sum_{i=1}^{n} a_{ii} \tag{18}$$

$$Trace(A) = \sum_{i=1}^{n} \lambda_i \tag{19}$$

We calculate,

$$Trace(A) = na \tag{20}$$

$$na = \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_{n-1} + \lambda_n \tag{21}$$

Since $\lambda_n = 0$, we get,

$$na = \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_{n-1} \tag{22}$$

From eq.(17), we have

$$na = k(n-1) \tag{23}$$

i.e.,

$$k = \frac{na}{(n-1)} \tag{24}$$

Thus, the eigenvalues of the system matrix [*A*] are $0, k_1, k_2, \ldots, k_{n-1}$, where $k = k_1 = k_2 = k_{n-1}$.

Remark 1: In matrix $[A]$, if we replace a by $-a$ and $-b$ by b, then eigenvalues for the matrix $[A]$ are $0, k_1, k_2, \ldots, k_{n-1}$, where, $k = k_1 = k_2 = k_{n-1}$ and

$$k = -\frac{na}{(n-1)} \tag{25}$$

4. EXAMPLE

Consider matrix $[A]$ as

$$[A] = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \tag{26}$$

The above structural matrix is similar to matrix as shown in eq. (2). In this above matrix, $n = 4, a = 3$. So, we can directly determine its eigenvalues. From eq. (11), it has one eigenvalue at origin and remaining eigenvalues can be calculated using eq. (24) as follows.

$$k = \frac{4 \times 3}{(4-1)} = 4. \tag{27}$$

Thus, the eigenvalues of the matrix *A* are $0, 4, 4, 4$. In order to verify the proposed result, the eigenvalues of the matrix [*A*] are calculated using Matlab software and we have got the same results.

5. CONCLUSIONS

In this paper, we have proposed a simple technique for calculating eigenvalues of the structural matrices. The beauty of the proposed method is that there is no need to use iterative methods and software, instead of that, a simple formula is needed to calculate the eigenvalues. In future, some more new techniques can also be developed for determining eigenvalues of various other structural matrices.

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