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Existence of Common Coupled Fixed Point for a Class of Mappings in Partially Ordered Metric Spaces

Vahid Parvaneh

Department of mathematics
Gilan-E-Gharb Branch, Islamic Azad University
Gilan-E-Gharb, Iran
vahid.parvaneh@kiau.ac.ir

Abstract

In this paper we introduce the class of P-contraction mappings, analogous to the concept of C-contraction [2]. Also, we obtain a fixed point result for this class of contractions in complete metric spaces.

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1 Introduction

In recent years, extension of the Banach's contraction principle [2] has been considered by many authors in different metric spaces. In [3], Bhaskar and Lakshmikantham presented coupled fixed point results for mixed monotone operators in partially ordered metric spaces and in 2009, Lakshmikantham and Ciric [6] proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in this spaces.

2 Main results

Definition 2.1 *([6]) Let* $(X, \preceq, d)$ *be a partially ordered set and* $F : X \times X \to X$ *and* $g : X \to X$ *be two self mappings.* F *has the mixed* g*-monotone property if* F *is monotone* g*-non-decreasing in its first argument and is monotone* g*-non-increasing in its second argument, that is, if for all* $x_1, x_2 \in X$*;* $gx_1 \preceq gx_2$ *implies* $F(x_1, y) \preceq F(x_2, y)$ *for any* $y \in X$ *and for all* $y_1, y_2 \in X$*;* $gy_1 \succeq gy_2$ *implies* $F(x, y_1) \preceq F(x, y_2)$ *for any* $x \in X$*.*

Definition 2.2 *([1]) The mappings $F : X \times X \to X$ and $g : X \to X$ are called w-compatible if $g(F(x, y)) = F(gx, gy)$, whenever $g(x) = F(x, y)$ and $g(y) = F(y, x)$.*

Definition 2.3 *([6], [1]) An element $(x, y) \in X \times X$ is called:*

(1) a coupled coincidence point of mappings $F : X \times X \to X$ and $g : X \to X$ if $g(x) = F(x, y)$ and $g(y) = F(y, x)$, and (gx, gy) is called coupled point of coincidence, and,

(2) a common coupled fixed point of mappings $F : X \times X \to X$ and $g : X \to X$ if $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Theorem 2.4 *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property and satisfy*

$$\begin{aligned} d(F(x,y),F(u,v)) \leq\ & \tfrac{1}{5}(d(gx,gu)+d(F(x,y),gy)+d(F(x,y),gu) \\ & +d(F(u,v),gx)+d(F(u,v),gu)) \\ & -\varphi(d(gx,gu),d(F(x,y),gx),d(F(x,y),gu) \\ & ,d(F(u,v),gx),d(F(u,v),gu)), \end{aligned} \tag{1}$$

for every two pairs $(x, y), (u, v) \in X \times X$ such that $gx \preceq gu$ and $gy \succeq gv$, where $\varphi : [0, \infty)^5 \to [0, \infty)$ be a continuous function such that $\varphi(x, y, z, t, u) = 0$ if and only if $x = y = z = t = u = 0$. Also suppose X has the following properties:

i. If a non-decreasing sequence $x_n \to x$; then $x_n \preceq x$ for all $n \geq 0$.

ii. If a non-increasing sequence $y_n \to y$; then $y_n \succeq y$ for all $n \geq 0$.

Let $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X. If there exists $(x_0, y_0) \in X \times X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then F and g have a coupled coincidence point in X.

Proof 2.5 *Let $x_0, y_0 \in X$ be such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can define $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$, then $gx_0 \preceq F(x_0, y_0) = gx_1$ and $gy_0 \succeq F(y_0, x_0) = gy_1$. Since F has the mixed g-monotone property, we have $F(x_0, y_0) \preceq F(x_1, y_0) \preceq F(x_1, y_1)$ and $F(y_0, x_0) \succeq F(y_1, x_0) \succeq F(y_1, x_1)$. In this way we construct the sequences z_n and t_n inductively as $z_n = gx_n = F(x_{n-1}, y_{n-1})$, and $t_n = gy_n = F(y_{n-1}, x_{n-1})$, for all $n \geq 0$.*

We know that for all $n \geq 0$, $z_{n-1} \preceq z_n$, and $t_{n-1} \succeq t_n$. This can be done as in Theorem 3.1. of [4], so we omit the proof of this part.

Step I. We will prove that $\lim_{n\to\infty} d(z_n, z_{n+1}) = \lim_{n\to\infty} d(t_n, t_{n+1}) = 0$.

Using 1 (which is possible since $gx_{n-1} \preceq gx_n$ and $gy_{n-1} \succeq gy_n$), we obtain that

$$\begin{aligned}
d(z_n, z_{n+1}) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\
&\leq \tfrac{1}{5}(d(gx_{n-1}, gx_n) + d(F(x_{n-1}, y_{n-1}), gx_{n-1}) + d(F(x_{n-1}, y_{n-1}), gx_n) \\
&+d(F(x_n, y_n), gx_{n-1}) + d(F(x_n, y_n), gx_n)) \\
&-\varphi(d(gx_{n-1}, gx_n), d(F(x_{n-1}, y_{n-1}), gx_{n-1}), d(F(x_{n-1}, y_{n-1}), gx_n) \\
&, d(F(x_n, y_n), gx_{n-1}), d(F(x_n, y_n), gx_n)) \\
&= \tfrac{1}{5}(d(z_{n-1}, z_n) + d(z_n, z_{n-1}) + d(z_n, z_n) + d(z_{n+1}, z_{n-1}) + d(z_{n+1}, z_n)) \\
&-\varphi(d(z_{n-1}, z_n), d(z_n, z_{n-1}), d(z_n, z_n), d(z_{n+1}, z_{n-1}), d(z_{n+1}, z_n)) \\
&\leq \tfrac{1}{5}(d(z_{n-1}, z_n) + d(z_n, z_{n-1}) + d(z_n, z_n) + d(z_{n+1}, z_{n-1}) + d(z_{n+1}, z_n)) \\
&\leq \tfrac{1}{5}(3d(z_{n-1}, z_n) + 2d(z_{n+1}, z_n)),
\end{aligned} \tag{2}$$

hence, $d(z_{n+1}, z_n) \leq d(z_n, z_{n-1})$.

Again, since $gy_n \preceq gy_{n-1}$ and $gx_n \succeq gx_{n-1}$,

$$\begin{aligned}
d(t_{n+1}, t_n) &= d(F(y_n, x_n), F(y_{n-1}, x_{n-1})) \\
&\leq \tfrac{1}{5}(d(gy_n, gy_{n-1}) + d(F(y_n, x_n), gy_n) + d(F(y_n, x_n), gy_{n-1}) \\
&+d(F(y_{n-1}, x_{n-1}), gy_n) + d(F(y_{n-1}, x_{n-1}), gy_{n-1})) \\
&-\varphi(d(gy_n, gy_{n-1}), d(F(y_n, x_n), gy_n), d(F(y_n, x_n), gy_{n-1}) \\
&, d(F(y_{n-1}, x_{n-1}), gy_n), d(F(y_{n-1}, x_{n-1}), gy_{n-1})) \\
&= \tfrac{1}{5}(d(t_n, t_{n-1}) + d(t_{n+1}, t_n) + d(t_{n+1}, t_{n-1}) + d(t_n, t_n) + d(t_n, t_{n-1}) \\
&-\varphi(d(t_n, t_{n-1}), d(t_{n+1}, t_n), d(t_{n+1}, t_{n-1}), d(t_n, t_n), d(t_n, t_{n-1})) \\
&\leq \tfrac{1}{5}(3d(t_n, t_{n-1}) + 2d(t_{n+1}, t_n)),
\end{aligned} \tag{3}$$

hence, $d(t_{n+1}, t_n) \leq d(t_n, t_{n-1})$.

It follows that the sequences $d(z_{n+1}, z_n)$ and $d(t_{n+1}, t_n)$ are monotone decreasing sequences of non-negative real numbers and consequently there exist $r, s \geq 0$ *such that* $\lim_{n\to\infty} d(z_{n+1}, z_n) = r$, *and* $\lim_{n\to\infty} d(t_{n+1}, t_n) = s$.

From 2 we have

$$\begin{aligned}
d(z_{n+1}, z_n) &\leq \tfrac{1}{5}(d(z_{n-1}, z_n) + d(z_n, z_{n-1}) + d(z_n, z_n) + d(z_{n+1}, z_{n-1}) + d(z_{n+1}, z_n)) \\
&\leq \tfrac{1}{5}(3d(z_{n-1}, z_n) + 2d(z_{n+1}, z_n)).
\end{aligned} \tag{4}$$

If $n \to \infty$ in 4, we have $r \leq \lim_{n\to\infty} \frac{1}{5}(3r + d(z_{n-1}, z_{n+1})) \leq r$, *hence* $\lim_{n\to\infty} d(z_{n-1}, z_{n+1}) = 2r$.

We have proved in (2)

$$\begin{aligned}
d(z_n, z_{n+1}) &\leq \tfrac{1}{5}(d(z_{n-1}, z_n) + d(z_n, z_{n-1}) + d(z_n, z_n) + d(z_{n+1}, z_{n-1}) + d(z_{n+1}, z_n)) \\
&-\varphi(d(z_{n-1}, z_n), d(z_n, z_{n-1}), d(z_n, z_n), d(z_{n+1}, z_{n-1}), d(z_{n+1}, z_n)) \\
&\leq \tfrac{1}{5}(3d(z_{n-1}, z_n) + 2d(z_{n+1}, z_n)).
\end{aligned} \tag{5}$$

Now, if $n \to \infty$ *and since* φ *is continuous, we can obtain*

$$r \leq r - \varphi(r, r, 0, 2r, r) \leq r.$$

Consequently, $\varphi(r, r, 0, 2r, r) = 0$*. Hence*

$$\lim_{n\to\infty} d(z_{n+1}, z_n) = r = 0. \tag{6}$$

In a same way, we have

$$\lim_{n\to\infty} d(t_{n+1}, t_n) = s = 0. \tag{7}$$

Now, we show that $\{z_n\}$ *and* $\{t_n\}$ *are Cauchy sequences in* X*.*

Let $\{z_n\}$ *is not a Cauchy sequence, then there exists* $\varepsilon > 0$ *for which we can find subsequences* $\{z_{m(k)}\}$ *and* $\{z_{n(k)}\}$ *of* $\{z_n\}$ *such that* $n(k) > m(k) > k$ *and* $d(z_{m(k)}, z_{n(k)}) \geq \varepsilon$*, where* $n(k)$ *is the smallest index with this property, i.e.,*

$$d(z_{m(k)}, z_{n(k)-1}) < \varepsilon. \tag{8}$$

From triangle inequality

$$\begin{aligned} \varepsilon \leq d(z_{m(k)}, z_{n(k)}) &\leq d(z_{m(k)}, z_{n(k)-1}) + d(z_{n(k)-1}, z_{n(k)}) \\ &< \varepsilon + d(z_{n(k)-1}, z_{n(k)}). \end{aligned} \tag{9}$$

If $k \to \infty$*, Since* $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$*, from 9 we can conclude that*

$$\lim_{k\to\infty} d(z_{m(k)}, z_{n(k)}) = \varepsilon. \tag{10}$$

Moreover, we have

$$|d(z_{n(k)}, z_{m(k)+1}) - d(z_{n(k)}, z_{m(k)})| \leq d(z_{m(k)+1}, z_{m(k)}), \tag{11}$$

and

$$|d(z_{n(k)+1}, z_{m(k)}) - d(z_{n(k)}, z_{m(k)})| \leq d(z_{n(k)+1}, z_{n(k)}), \tag{12}$$

and

$$|d(z_{m(k)+1}, z_{n(k)+1}) - d(z_{m(k)+1}, z_{n(k)})| \leq d(z_{n(k)+1}, z_{n(k)}). \tag{13}$$

Since $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$*, and 11, 12 and 13 are hold, we get*

$$\lim_{k\to\infty} d(z_{m(k)+1}, z_{n(k)}) = \lim_{k\to\infty} d(z_{m(k)+1}, z_{n(k)+1}) = \lim_{k\to\infty} d(z_{n(k)+1}, z_{m(k)}) = \varepsilon. \tag{14}$$

Again, as $n(k) > m(k)$*, we have* $gx_{m(k)} \preceq gx_{n(k)}$*, and* $gy_{m(k)} \succeq gy_{n(k)}$*. So, from 1, for all* $k \geq 0$*, we have*

$$\begin{aligned}
&d(z_{m(k)+1}, z_{n(k)+1}) = d(F(x_{m(k)}, y_{m(k)}), F(x_{n(k)}, y_{n(k)}))\\
&\leq \tfrac{1}{5}(d(gx_{m(k)}, gx_{n(k)}) + d(F(x_{m(k)}, y_{m(k)}), gx_{m(k)}) + d(F(x_{m(k)}, y_{m(k)}), gx_{n(k)})\\
&+d(F(x_{n(k)}, y_{n(k)}), gx_{m(k)}) + d(F(x_{n(k)}, y_{n(k)}), gx_{n(k)}))\\
&-\varphi(d(gx_{m(k)}, gx_{n(k)}), d(F(x_{m(k)}, y_{m(k)}), gx_{m(k)}), d(F(x_{m(k)}, y_{m(k)}), gx_{n(k)})\\
&, d(F(x_{n(k)}, y_{n(k)}), gx_{m(k)}), d(F(x_{n(k)}, y_{n(k)}), gx_{n(k)}))\\
&= \tfrac{1}{5}(d(z_{m(k)}, z_{n(k)}) + d(z_{m(k)+1}, z_{m(k)}) + d(z_{m(k)+1}, z_{n(k)})\\
&+d(z_{n(k)+1}, z_{m(k)}) + d(z_{n(k)+1}, z_{n(k)}))\\
&-\varphi(d(z_{m(k)}, z_{n(k)}), d(z_{m(k)+1}, z_{m(k)}), d(z_{m(k)+1}, z_{n(k)})\\
&, d(z_{n(k)+1}, z_{m(k)}), d(z_{n(k)+1}, z_{n(k)})).
\end{aligned} \tag{15}$$

If $k \to \infty$*, from 10 and 14 we have,* $\varepsilon \leq \frac{1}{5}(3\varepsilon) - \varphi(\varepsilon, 0, \varepsilon, \varepsilon, 0)$*, hence, we have* $\varepsilon = 0$*, which is a contradiction and it follows that* $\{z_n\}$ *is a Cauchy sequence in* X*. Analogously, it can be proved that* $\{t_n\}$ *is a Cauchy sequence in* X*.*

Since (X, d) *is complete and* $\{z_n\}$ *is Cauchy, there exists* $z \in X$ *such that* $\lim_{n\to\infty} z_n = \lim_{n\to\infty} gx_n = z$*, and since* $g(X)$ *is closed and* $\{z_n\} \subseteq g(X)$*, we have* $z \in g(X)$ *and hence there exists* $u \in X$ *such that* $z = gu$*. Similarly, there exist* $t, v \in X$ *such that* $t = \lim_{n\to\infty} t_n = \lim_{n\to\infty} gy_n = gv$*.*

We prove that (u, v) *is a coupled coincidence point of* F *and* g*.*

We know that gx_n *and* gy_n *are non-decreasing and non-increasing in* X*, respectively and* $gx_n \to z = gu$ *and* $gy_n \to t = gv$*. From conditions of our theorem,* $gx_n \preceq gu$ *and* $gy_n \succeq gv$*. So, using 1 we obtain that*

$$\begin{aligned}
d(z_{n+1}, F(u, v)) &= d(F(x_n, y_n), F(u, v))\\
&\leq \tfrac{1}{5}(d(gx_n, gu) + d(F(x_n, y_n), gx_n) + d(F(x_n, y_n), gu)\\
&+d(F(u, v), gx_n) + d(F(u, v), gu))\\
&-\varphi(d(gx_n, gu), d(F(x_n, y_n), gx_n), d(F(x_n, y_n), gu)\\
&, d(F(u, v), gx_n), d(F(u, v), gu))\\
&= \tfrac{1}{5}(d(z_n, z) + d(z_{n+1}, z_n) + d(z_{n+1}, z) + d(F(u, v), z_n) + d(F(u, v), z))\\
&-\varphi(d(z_n, z), d(z_{n+1}, z_n), d(z_{n+1}, z), d(F(u, v), z_n), d(F(u, v), z)).
\end{aligned} \tag{16}$$

If in (16) $n \to \infty$*,*

$$\begin{aligned}
d(z, F(u, v)) &\leq \tfrac{1}{5}(d(z, z) + d(z, z) + d(z, z) + d(F(u, v), z) + d(F(u, v), z))\\
&-\varphi(d(z, z), d(z, z), d(z, z), d(F(u, v), z), d(F(u, v), z)),
\end{aligned} \tag{17}$$

and hence $\varphi(0, 0, 0, d(F(u, v), z), d(F(u, v), z)) \leq -\frac{3}{5}d(z, F(u, v)) \leq 0$*, and therefore,* $d(z, F(u, v)) = 0$*. So,* $F(u, v) = z = g(u)$ *and in a similar way* *we can obtain that $F(v,u) = t = g(v)$. That is, g and F have a coupled coincidence point.*

Theorem 2.6 *Adding the following conditions to the hypotheses of Theorem 2.4, we obtain the existence of the common coupled fixed point of F and g.*

(i) If any nondecreasing sequence z_n in X converges to z, then we assume $gz \preceq z$, and also, if any nonincreasing sequence t_n in X converges to t, then we assume $gt \succeq t$.

(ii) g and F be w-compatible continuous mappings.

Proof 2.7 *We know that the nondecreasing sequence $gx_n = z_n \to z$ and by our assumptions $gz_n \preceq gz \preceq z = gu$.*

Also, the noninreasing sequence $gy_n = t_n \to t$ and by our assumptions $gt_n \succeq gt \succeq t = gv$.

So, from (1) we have

$$\begin{aligned} d(F(z_n,t_n),F(u,v)) \ &\leq \tfrac{1}{5}(d(gz_n,gu)+d(F(z_n,t_n),gz_n)+d(F(z_n,t_n),gu) \\ &\quad +d(F(u,v),gz_n)+d(F(u,v),gu)) \\ &\quad -\varphi(d(gz_n,gu),d(F(z_n,t_n),gz_n),d(F(z_n,t_n),gu) \\ &\quad ,d(F(u,v),gz_n),d(F(u,v),gu)). \end{aligned} \tag{18}$$

Since F and g are w-compatible, and $F(u,v) = gu = z$ and $F(v,u) = gv = t$ we have that $gz = g(gu) = g(F(u,v)) = F(gu,gv) = F(z,t)$.

Now, if in (18), $n \to \infty$, we obtain

$$\begin{aligned} d(gz,z) \ &\leq \tfrac{1}{5}(d(gz,z)+d(gz,gz)+d(gz,z)+d(z,gz)+d(z,z)) \\ &\quad -\varphi(d(gz,z),d(gz,gz),d(gz,z),d(z,gz),d(z,z)). \end{aligned} \tag{19}$$

Hence, $\varphi(d(gz,z),d(gz,gz),d(gz,z),d(z,gz),d(z,z)) = 0$ and so, $d(gz,z) = 0$. Therefore $gz = z$ and from $F(z,t) = gz$, we conclude that $F(z,t) = gz = z$.

Analogously, we can prove that $F(z,t) = gt = t$.

Note that if $(X,\preceq)$ be a partially ordered set, then we endow $X \times X$ with the following partial order relation:

$$(x,y) \preceq (u,v) \Longleftrightarrow x \preceq u\ ,\ y \succeq v.$$

for all $(x,y),(u,v) \in X \times X$. ([7])

Theorem 2.8 *Let all the conditions of theorem 2.6 be fulfilled.*

F and g have a unique common coupled fixed point provided that the common coupled fixed points of F and g are comparable.

Proof 2.9 *Let (x, y) and (u, v) be two common coupled fixed points of F and g, i.e., $x = g(x) = F(x, y)$, $y = g(y) = F(y, x)$, and $u = g(u) = F(u, v)$, $v = g(v) = F(v, u)$.*

Suppose that (x, y) and (u, v) are comparable.

Since (u, v) is comparable with (x, y), we may assume that $(x, y) \preceq (u, v)$.

Now, applying 1 one obtains that

$$\begin{aligned} d(x,u) &= d(F(x,y), F(u,v)) \\ &\le \tfrac{1}{5}(d(gx,gu) + d(F(x,y),gx) + d(F(x,y),gu) \\ &+d(F(u,v),gx) + d(F(u,v),gu)) \\ &-\varphi(d(gx,gu), d(F(x,y),gx), d(F(x,y),gu) \\ &, d(F(u,v),gx), d(F(u,v),gu)) \\ &= \tfrac{1}{5}(d(gx,gu) + 0 + d(gx,gu) + d(gu,gx) + 0) \\ &-\varphi(d(gx,gu), 0, d(gx,gu), d(gu,gx), 0) \\ &= \tfrac{1}{5}(d(x,u) + 0 + d(x,u) + d(u,x) + 0) \\ &-\varphi(d(x,u), 0, d(x,u), d(u,x), 0). \end{aligned} \tag{20}$$

Therefore, $\varphi(d(x,u), 0, d(x,u), d(u,x), 0) \le -\frac{2}{5}d(x,u) \le 0$. Hence $x = u$. In a similar way, we have $y = v$.

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