Stability of Top-Points in Scale Space

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Abstract. This paper presents an algorithm for computing stability of top-points in scale-space. The potential usefulness of top-points in scale-space has already been shown for a number of applications, such as image reconstruction and image retrieval. In order to improve results only reliable top-points should be used. The algorithm is based on perturbation theory and noise propagation.

1 Introduction

Top-points have been shown to provide a sparse representation of an image that can potentially be used for image matching and image reconstruction [1]. To get rid of unstable top-points that may deteriorate performance, we derive a stability measure, which reflects the variance of top-point displacements induced by additive noise perturbation of given variance.

A top-point is an isolated point in scale-space where both gradient and Hessian determinant vanish. We consider only generic top-points [2]. Adding noise to the image leads to large displacements for some top-points and hardly noticeable displacements for others. In Sect. 2 we describe how to compute the dislocation of a top-point for each noise realization by using a perturbation approach. In order to obtain a realization-independent quantity, the variances of top-point displacement as a function of noise variances and image derivatives are derived in Sect. 3.

The variances of top-point displacement along coordinate directions are dependent on the coordinate system. In Sect. 4 invariants under Euclidean coordinate transformation are introduced.

We conclude the paper by experimental verification (Sect. 5). Experiments confirm our theoretical predictions. Thus we have obtained an operational criterion for distinguishing between stable and unstable top points.

2 Top-Points

Top-points of scale-space image representation $u(x, y, t)$ are defined by the following system of equations:

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$$\begin{cases} u_x = 0, \\ u_y = 0, \\ u_{xx}u_{yy} = u_{xy}^2. \end{cases} \tag{1}$$

Our scale parametrization convention is such that u satisfies the following heat equation:

$$u_t = u_{xx} + u_{yy}. \tag{2}$$

One idea of the "deep structure" rationale is to use information about top-points for different applications, for instance image matching and reconstruction. In order to get reliable results, the top-points, used by the algorithm, should be stable. Therefore the criterion of stability for top-points should be considered first.

Suppose (x_0, y_0, t_0) is a top-point for a fiducial scale-space image u. The stability of the top-point can be defined by measuring the distance over which the point moves after adding noise to the image.

Note that top-points are generic entities in scale space, thus cannot vanish or appear when the image is only slightly perturbed. Throughout we will assume that the noise variance is "sufficiently small" in the sense that the induced dislocation of the top-point can be investigated by means of a perturbation approach. For a given image u we consider its perturbations v under additive noise, i.e. $v = u + N$, in which N denotes the noise function. If (x_0, y_0, t_0) denotes a top-point in u, then due to noise perturbation it will move to some neighboring location $(x_0+\xi, y_0+\eta, t_0+\tau)$ in v. By using Taylor expansion, the displacement (ξ, η, τ) of the top-point (x_0, y_0, t_0) can be computed as

$$\begin{bmatrix} \xi, \\ \eta \\ \tau \end{bmatrix} = -\mathbf{M}^{-1} \begin{bmatrix} \mathbf{g} \\ \det\mathbf{H} \end{bmatrix}, \tag{3}$$

where

$$\mathbf{M} = \begin{bmatrix} \mathbf{H} & \mathbf{w} \\ \mathbf{z}^T & c \end{bmatrix}, \tag{4}$$

$$\mathbf{g} = \nabla v,\ \ \mathbf{H} = \nabla \mathbf{g},\ \ \mathbf{w} = \partial_t \mathbf{g},\ \ \mathbf{z} = \nabla \det\mathbf{H},\ \ c = \partial_t \det\mathbf{H}. \tag{5}$$

with all derivatives taken in the point (x_0, y_0, t_0). For a derivation we refer to [3].

Explicit expressions of ξ, η and τ in terms of image derivatives can be found in Appendix A.

3 Noise Propagation

In this section, the rules are discussed for the determination of the precision or reliability of a compound "measurement" f in terms of the precision of each constituent x_i. This subject is known as the propagation of errors [7].

Suppose that the derived property f is related to the measured properties x_1, ..., x_n by the functional relation

$$f = f(x_1, \ldots, x_n) \tag{6}$$

The function is assumed to be sufficiently regular.

Suppose that all $x_1, \ldots, x_n$ are random and possibly correlated between each other. The propagation of the variance of f can be approximated as

$$\langle (f(x_1, \ldots, x_n) - f(\bar{x}_1, \ldots, \ldots \bar{x}_n))^2 \rangle \approx \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \langle x_i x_j \rangle, \tag{7}$$

where all derivatives are calculated for the mean vector $(\bar{x}_1, \ldots, \ldots \bar{x}_n)$.

3.1 Noise Propagation for Top-Point Displacement

In our case the random variables $(x_1, \ldots, x_n)$ are the noise derivatives $(N_x, N_y, N_{xx}, \ldots, N_{yyyy})$. The computed "measurement" f is a vector of displacements $[\xi(N_x, \ldots, N_{yyyy}), \eta(N_x, \ldots, N_{yyyy}), \tau(N_x, \ldots, N_{yyyy})]^T$ in scale-space.

The mean vector $(\bar{N}_1, \ldots, \bar{N}_n)$ is zero, therefore the mean displacement is zero as well

$$\begin{bmatrix} \bar{\xi} \\ \bar{\eta} \\ \bar{\tau} \end{bmatrix} = \begin{bmatrix} \xi(\bar{N}_1, \ldots, \bar{N}_n) \\ \eta(\bar{N}_1, \ldots, \bar{N}_n) \\ \tau(\bar{N}_1, \ldots, \bar{N}_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{8}$$

Therefore the variance of the displacement vector equals the second order momentum of the displacement, $[\langle \xi^2 \rangle, \langle \eta^2 \rangle, \langle \tau^2 \rangle]^T$.

For simplicity, consider the variance in x direction $\langle \xi^2 \rangle$ only. Similar equations hold for $\langle \eta^2 \rangle$ and $\langle \tau^2 \rangle$.

Since the actual image v is obtained by adding noise N to the fiducial image u, i.e. $v = u + N$, for every i we have

$$\frac{\partial \xi}{\partial N_i} = \frac{\partial \xi}{\partial v_i}, \tag{9}$$

therefore (7) can be rewritten as

$$\langle \xi^2 \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \xi}{\partial v_i} \frac{\partial \xi}{\partial v_j} \langle N_i N_j \rangle. \tag{10}$$

N_i (v_i) is short notation for a partial derivative of the noise (image) function. More specifically the numerator of the expression for the displacement ξ (recall Appendix A) is a polynomial of $v_x, \ldots, v_{yyyy}$, which can be represented as

$$v_x F(v_x, \ldots, v_{yyyy}) + v_y G(v_x, \ldots, v_{yyyy}) + (v_{xy}^2 - v_{xx} v_{yy}) H(v_x, \ldots, v_{yyyy}). \tag{11}$$

From this representation it is easy to see that derivatives of (11) with respect to to third and higher order image derivatives taken in the mean point vanish since

$$v_x = u_x = 0,\ v_y = u_y = 0,\ v_{xy}^2 - v_{xx} v_{yy} = u_{xy}^2 - u_{xx} u_{yy} = 0, \tag{12}$$

in the respective top-point of u and v, recall (1).

Therefore, the sum (10) contains terms with derivatives with respect to $v_x, v_y, v_{xx}, v_{xy}, v_{yy}$ only. Hence in order to get the final expression for the variance we only need to compute the mutual correlations of noise derivatives $N_x, N_y, N_{xx}, N_{xy}, N_{yy}$. Higher order noise derivatives play no role.

Table 1. Some values of Q_n (Q_n=0 if n is odd)

n	0	2	4	6
Q_n	1	1	3	15

3.2 Noise Which Is Uncorrelated Between Neighboring Pixels

The momentum $M^2_{m_x,m_y,n_x,n_y} =< N_{m_x,m_y} N_{n_x,n_y} >$ of Gaussian derivatives of correlated noise in case the spatial noise correlation distance τ is much smaller than scale t is given by [10]

$$M^2_{m_x,m_y,n_x,n_y} \simeq < N^2 > \left(\frac{\tau}{2t}\right) \left(\frac{-1}{4t}\right)^{\frac{1}{2}(m_x+m_y+n_x+n_y)} Q_{m_x+n_x} Q_{m_y+n_y} \tag{13}$$

Let us take the correlation kernel with one pixel width, therefore $\tau = 1/2$. In this case Gaussian derivatives of the first and the second order have the following correlation matrix:

$$C = (\langle N_i N_j \rangle)_{ij} = \begin{pmatrix} 4t_0 & 0 & 0 & 0 & 0 \\ 0 & 4t_0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 3 \end{pmatrix} \frac{\langle N^2 \rangle}{(4t_0)^3}, \tag{14}$$

where $(N_1, \ldots, N_5) = (N_x, N_y, N_{xx}, N_{xy}, N_{yy})$.

4 Invariants

The variances $\langle \xi^2 \rangle$ and $\langle \eta^2 \rangle$ are not rotationally invariant, as they depend on the choice of Cartesian coordinate axes. By rotation we get variances as functions of angle φ, $\langle \xi^2 \rangle(\varphi)$ and $\langle \eta^2 \rangle(\varphi)$.

After some simplifications the rotated variances can be written as

$$\begin{aligned} \langle \xi^2 \rangle &= (A \sin^2 \varphi + B \sin \varphi \cos \varphi + C)/D, \\ \langle \eta^2 \rangle &= (A \cos^2 \varphi - B \sin \varphi \cos \varphi + C)/D, \end{aligned} \tag{15}$$

where A, B, C and D are functions of $u_{xx}, \ldots, u_{yyyy}$ (for sake of completeness the exact expressions are given in Appendix B). The variance of the total displacement $r = \sqrt{\xi^2 + \eta^2}$ can be easily computed from (15)

$$\langle r^2 \rangle = \langle \xi^2 \rangle + \langle \eta^2 \rangle = (A + 2C)/D. \tag{16}$$

Therefore $\langle r^2 \rangle$ is invariant under rotation, as expected

$$\langle \xi^2 \rangle' + \langle \eta^2 \rangle' = 0, \tag{17}$$

where prime denotes derivative with respect to angle of rotation. From (17) one can easy see, that if $\langle \xi^2 \rangle'$ is zero, then $\langle \eta^2 \rangle'$ is zero as well. This shows, that $\langle \xi^2 \rangle$

Fig. 1. Variances of top-point displacements for all top-points projected on the xy-plane

and $\langle\eta^2\rangle$ have an extremum under the same rotation of the axes. The extrema of $\langle\xi^2\rangle$ (and $\langle\eta^2\rangle$) can be reached by rotation, when

$$\chi = \tan\varphi = \frac{A}{B} + \sqrt{1 + \left(\frac{A}{B}\right)^2} \tag{18}$$

The extremal variances are

$$\begin{aligned} X &= \langle\xi^2\rangle = \tfrac{\chi B + 2C}{2D}, \\ Y &= \langle\eta^2\rangle = \tfrac{-B + 2\chi C}{2\chi D}. \end{aligned} \tag{19}$$

X and Y are obviously invariant under rotation and translation.

By rotating the coordinate system we find directions in which the variance is maximal, respectively minimal (these two directions are orthogonal) and we construct an ellipse[1] with principal directions and axes that reflect these extremal noise variances (Fig 1).

Note, that top-points, in the neighborhood of which there is a lot of structure, have ellipses with very small radiuses (stable), and top-points in rather flat

[1] Note, that (15) does not parameterize an ellips. An elliptical “gauge figure” however is merely used for simplicity.

locations tend to have large ellipses (unstable) (Fig. 1). Another invariant is the variance of τ (scale instability), the expression of which is given in Appendix B.

5 Experiments

In order to validate the theoretical results numerical experiments have been conducted. Adding noise to the image results in changing top-points coordinates. Some of them hardly move and others move quite a lot. It is practically impossible by comparing two top-point clouds to tell which top-point of the fiducial image corresponds to which top-point of the actual image, therefore it is impossible to investigate the stability in a pure experimental way. Instead, we choose a somewhat different approach, which combines theory and experiments.

For each noise realization N^i, where $i = 1 \ldots K$ labels the experiments, we use (3) as a refining algorithm in order to estimate the coordinates of the actual top-point $(x_0 + \xi_i, y_0 + \eta_i, t_0 + \tau_i)$, taking the coordinates of the original top-point (x_0, y_0, t_0) as an initial guess. The experiment consists of $K = 500$ noise realizations. Therefore, for original top-point (x_0, y_0, t_0) we compute an array $\{(\xi_i, \eta_i, \tau_i)\}_{1 \le i \le K}$ of 500 displacements.

The principal directions and maximum and minimum variances for the set of points, obtained by noise perturbation, have been calculated. In order to find principal directions, the extremum problem should be solved for the averages

$$\begin{aligned} \langle \xi^2 \rangle(\chi) &= \tfrac{1}{1+\chi^2} \textstyle\sum_{i=1}^{K} (\xi_i + \chi \eta_i)^2 / K, \\ \langle \eta^2 \rangle(\chi) &= \tfrac{1}{1+\chi^2} \textstyle\sum_{i=1}^{K} (-\chi \xi_i + \eta_i)^2 / K, \end{aligned} \tag{20}$$

where T is a tangent of the angle of rotation. The extreem for both variances are reached under identical rotations, since the sum $\langle \xi^2 \rangle(T) + \langle \eta^2 \rangle(T)$ does not depend on χ.

The extremum corresponds to the angle given by

$$\tilde{\chi} = -\frac{\sum_i (\xi_i^2 - \eta_i^2)}{2 \sum_i \xi_i \eta_i} + \sqrt{\left(\frac{\sum_i (\xi_i^2 - \eta_i^2)}{2 \sum_i \xi_i \eta_i} \right)^2 + 1}. \tag{21}$$

The variance in this direction is $\tilde{X} = \langle \eta^2 \rangle(\tilde{\chi})$

$$\tilde{X} = \frac{1}{1 + \tilde{\chi}^2} \sum_{i=1}^{K} (\xi_i + \tilde{\chi} \eta_i)^2 / K \tag{22}$$

and in the orthogonal direction

$$\tilde{Y} = \frac{1}{1 + \tilde{\chi}^2} \sum_{i=1}^{K} (-\tilde{\chi} \xi_i + \eta_i)^2 / K \tag{23}$$

The comparison of theory and the experiments is depicted in Fig. 2. Since both the theory and the experiments take into account derivatives up to fourth

Fig. 2. Examples of top-point movements projected on the xy-plane under noise realizations (crosses) and theoretical predictions (ellipses). Right column shows zooming in the neighborhood of the top-point

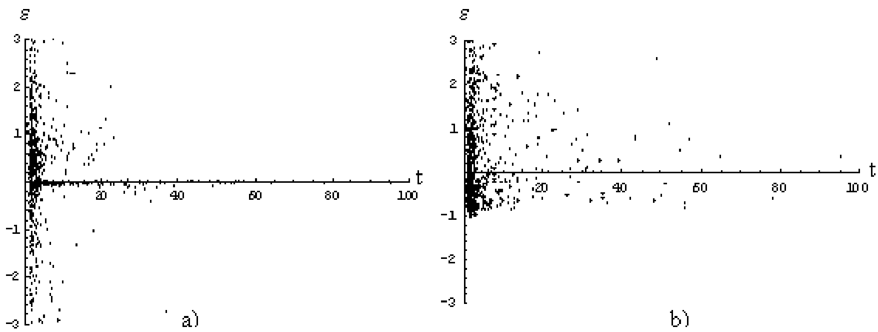

Fig. 3. Comparison of experimental and theoretical results. The value of ε denotes the ratio between theoretical and experimental variances, a) - for spatial displacement ε_X and b) - for scale-displacement ε_T

order, the scale of the top-point should be large enough to obtain reliable results. The value of ε denotes the relative difference between theoretical and experimental variances in space and scale

$$\varepsilon_X = \frac{X - \tilde{X}}{\tilde{X}} \tag{24}$$

$$\varepsilon_T = \frac{T - \tilde{T}}{\tilde{T}} \tag{25}$$

Figure 3 reveals that the relative difference between theoretical and experimental results is acceptably small for large scales and large for small scales due to computational errors in derivatives, as expected.

6 Results

In this paper we have described an algorithm for computing stability measures for top-points. The algorithm is based on a perturbation approach and uses properties of noise propagation in Gaussian scale-space.

Variances of top-point displacements can be computed on the basis of noise variance and fourth order differential structure at the top-point.

The advantage of this approach is that variances of displacements can be predicted theoretically on the basis of the local differential structure.

The experiments have shown correspondence between the analytical predictions and practice in cases where the scale of top-point is not too small for reliably computing fourth order derivatives.

Analytically computed variances can be used for several applications, such as stability measures and weight measures for top-point based image retrieval algorithm [1].

Applying the algorithm to problems listed above will be the next step in our research.

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Appendix A: Displacements Under Noise Perturbation

In this appendix we give expressions for displacements in spatial and scale directions. The refining equations (3) in terms of image derivatives are given by

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \Bigg((v_{xy}^2 - v_{xx}v_{yy}) \begin{bmatrix} v_{xy}(v_{xxy}+v_{yyy}) - v_{yy}(v_{xxx}+v_{xyy}) \\ v_{xy}(v_{xxx}+v_{xyy}) - v_{xx}(v_{xxy}+v_{yyy}) \end{bmatrix} -$$
$$(v_y(v_{xxx}+v_{xyy}) - v_x(v_{xxy}+v_{yyy})) \begin{bmatrix} -2v_{xy}v_{xyy} + v_{yy}v_{xxy} + v_{xx}v_{yyy} \\ 2v_{xy}v_{xxy} - v_{yy}v_{xxx} - v_{xx}v_{xyy} \end{bmatrix} + \tag{26}$$
$$(v_{yy}(v_{xxxx}+v_{xxyy}) + v_{xx}(v_{xxyy}+v_{yyyy}) - 2v_{xy}(v_{xxxy}+v_{xyyy}))$$
$$\begin{bmatrix} v_yv_{xy} - v_xv_{yy}v_xv_{xy} - v_yv_{xx} \end{bmatrix} \Bigg) /$$
$$\det M$$

The scale displacement equals

$$\tau = (-(v_{xy}^2 - v_{xx}v_{yy})^2 + v_y(2v_{xy}^2 v_{xxy} + u_{xx}(v_{yy}v_{xxy} + v_{xx}v_{yyy}) - v_{xy}(3v_{xx}v_{xyy} + v_{yy}v_{xxx})) + v_x(2v_{xy}^2 v_{xyy} + v_{yy}(v_{xx}v_{xyy} + v_{yy}v_{xxx}) - v_{xy}(3v_{yy}v_{xxy} + v_{xx}v_{yyy})))/\det M \tag{27}$$

In both formulas we have a denominator

$$\det M = (v_{yy}v_{xxy} + v_{xx}v_{yyy} - 2v_{xy}v_{xyy})(v_{xy}(v_{xxx} + v_{xyy}) - v_{xx}(v_{xxy} + v_{yyy})) + (v_{yy}v_{xxx} + v_{xx}v_{xyy} - 2v_{xy}v_{xxy})(v_{xy}(v_{xxy} + v_{yyy}) - v_{yy}(v_{xxx} + v_{xyy})) + (v_{xx}v_{yy} - v_{xy}^2)(v_{xx}(v_{xxyy} + v_{yyyy}) + v_{yy}(v_{xxxx} + v_{xxyy})) - 2v_{xy}(v_{xxxy} + v_{xyyy})) \tag{28}$$

Appendix B: Parameters for the Invariant Expressions

$$A = 3(u_{xx} - u_{yy})(u_{xx} + u_{yy})^2(u_{xx}(u_{xxy} + u_{yyy})^2 + (u_{xxx} + u_{xyy})((u_{xxx} + u_{xyy})u_{yy} - 2u_{xy}(u_{xxy} + u_{yyy}))) + 4t_0(((-2u_{xxy}u_{xy} + u_{xx}u_{xyy} + u_{xxx}u_{yy})(u_{xxy} + u_{yyy}) + (u_{xxx} + u_{xyy})(-2u_{xy}u_{xyy} + u_{xxy}u_{yy} + u_{xx}u_{yyy}) + 2u_{xy}(2u_{xy}(u_{xxxy} + u_{xyyy}) - (u_{xxxx} + u_{xxyy})u_{yy} - u_{xx}(u_{xxyy} + u_{yyyy})))^2 + 2(-2u_{xy}(u_{xxxy} + u_{xyyy})u_{yy} + (u_{xxxx} + u_{xxyy})u_{yy}^2 - (u_{xxy} + u_{yyy})(-2u_{xy}u_{xyy} + u_{xxy}u_{yy} + u_{xx}u_{yyy}) + u_{xy}^2(u_{xxyy} + u_{yyyy})) \times(-(u_{xxx} + u_{xyy})(u_{xx}u_{xyy} + u_{xxx}u_{yy}) + (u_{xxy} + u_{yyy})(u_{xxy}u_{yy} + u_{xx}u_{yyy}) + 2u_{xy}(u_{xxx}u_{xxy} - u_{xyy}u_{yyy}) + (-u_{xx} + u_{yy})(2u_{xy}(u_{xxxy} + u_{xyyy}) - (u_{xxxx} + u_{xxyy})u_{yy} - u_{xx}(u_{xxyy} + u_{yyyy}))) + (-(u_{xxx} + u_{xyy})(u_{xx}u_{xyy} + u_{xxx}u_{yy}) + (u_{xxy} + u_{yyy})(u_{xxy}u_{yy} + u_{xx}u_{yyy}) + 2u_{xy}(u_{xxx}u_{xxy} - u_{xyy}u_{yyy}) + (-u_{xx} + u_{yy})(2u_{xy}(u_{xxxy} + u_{xyyy}) - (u_{xxxx} + u_{xxyy})u_{yy} - u_{xx}(u_{xxyy} + u_{yyyy})))^2 - 2((-2u_{xxy}u_{xy} + u_{xx}u_{xyy} + u_{xxx}u_{yy})(u_{xxy} + u_{yyy}) + (u_{xxx} + u_{xyy})(-2u_{xy}u_{xyy} + u_{xxy}u_{yy} + u_{xx}u_{yyy}) + 2u_{xy}(2u_{xy}(u_{xxxy} + u_{xyyy}) - (u_{xxxx} + u_{xxyy})u_{yy} - u_{xx}(u_{xxyy} + u_{yyyy})))((u_{xxx} + u_{xyy}) \times(-2u_{xy}u_{xyy} + u_{xxy}u_{yy} + u_{xx}u_{yyy}) - u_{xy}(-2u_{xy}(u_{xxxy} + u_{xyyy}) + (u_{xxxx} + u_{xxyy})u_{yy} + u_{xx}(u_{xxyy} + u_{yyyy})))) \tag{29}$$

$$B = -6u_{xy}(u_{xx} + u_{yy})^2(u_{xx}(u_{xxy} + u_{yyy})^2 + (u_{xxx} + u_{xyy})((u_{xxx} + u_{xyy})u_{yy} - 2u_{xy}(u_{xxy} + u_{yyy}))) + 4t_0(2(-2u_{xy}(u_{xxxy} + u_{xyyy})u_{yy} + (u_{xxxx} + u_{xxyy})u_{yy}^2 - (u_{xxy} + u_{yyy})(-2u_{xy}u_{xyy} + u_{xxy}u_{yy} + u_{xx}u_{yyy}) + u_{xy}^2(u_{xxyy} + u_{yyyy}))((-2u_{xxy}u_{xy} + u_{xx}u_{xyy} + u_{xxx}u_{yy})(u_{xxy} + u_{yyy}) + (u_{xxx} + u_{xyy})(-2u_{xy}u_{xyy} + u_{xxy}u_{yy} + u_{xx}u_{yyy}) + 2u_{xy}(2u_{xy}(u_{xxxy} + u_{xyyy}) - (u_{xxxx} + u_{xxyy})u_{yy} - u_{xx}(u_{xxyy} + u_{yyyy}))) + 2(-(u_{xxx}+$$

$$u_{xyy}))(u_{xx}u_{xyy}+u_{xxx}u_{yy})+(u_{xxy}+u_{yyy})(u_{xxy}u_{yy}+u_{xx}u_{yyy})+ 2u_{xy}(u_{xxx}u_{xxy}-u_{xyy}u_{yyy})+(-u_{xx}+u_{yy})(2u_{xy}(u_{xxxy}+u_{xyyy})- (u_{xxxx}+u_{xxyy})u_{yy}-u_{xx}(u_{xxyy}+u_{yyyy})))((u_{xxx}+u_{xyy})(-2u_{xy}u_{xyy}+ u_{xxy}u_{yy}+u_{xx}u_{yyy})-u_{xy}(-2u_{xy}(u_{xxxy}+u_{xyyy})+(u_{xxxx}+u_{xxyy})u_{yy}+ u_{xx}(u_{xxyy}+u_{yyyy}))) \tag{30}$$

$$C=3(u_{xx}+u_{yy})^2(-(u_{xxx}+u_{xyy})u_{yy}+u_{xy}(u_{xxy}+u_{yyy}))^2+ 4t_0((-2u_{xy}(u_{xxxy}+u_{xyyy})u_{yy}+(u_{xxxx}+u_{xxyy})u_{yy}^2+(u_{xxy}+u_{yyy})\times (2u_{xy}u_{xyy}-u_{xxy}u_{yy}-u_{xx}u_{yyy})+u_{xy}^2(u_{xxyy}+u_{yyyy}))^2+((u_{xxx}+u_{xyy})\times (-2u_{xy}u_{xyy}+u_{xxy}u_{yy}+u_{xx}u_{yyy})-u_{xy}(-2u_{xy}(u_{xxxy}+u_{xyyy})+ (u_{xxxx}+u_{xxyy})u_{yy}+u_{xx}(u_{xxyy}+u_{yyyy})))^2) \tag{31}$$

$$D=8\sqrt{t_0^3}(v_{yy}v_{xxy}+v_{xx}v_{yyy}-2v_{xy}v_{xyy})(v_{xy}(v_{xxx}+v_{xyy})-v_{xx}(v_{xxy}+ v_{yyy}))+(v_{yy}v_{xxx}+v_{xx}v_{xyy}-2v_{xy}v_{xxy})(v_{xy}(v_{xxy}+v_{yyy})-v_{yy}(v_{xxx}+v_{xyy}))) \tag{32}$$

$$\langle\tau^2\rangle=4t_0((u_{xx}+u_{yy})(u_{yy}(3u_{xx}u_{xyy}+u_{xxx}u_{yy})^2-2u_{xy}(3u_{xx}u_{xyy}+u_{xxx}u_{yy})\times (3u_{xxy}u_{yy}+u_{xx}u_{yyy})+u_{xx}(3u_{xxy}u_{yy}+u_{xx}u_{yyy})^2))/D \tag{33}$$