A Unified Approach to Direct Kinematics of Some Reduced Motion Parallel Manipulators

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After discussing the Study point transformation operator, a unified way to formulate kinematic problems, using "points moving on planes or spheres" constraint equations, is introduced. Application to the direct kinematics problem solution of a number of different parallel Schönflies motion robots is then developed. Certain not widely used but useful tools of algebraic geometry are explained and applied for this purpose. These constraints and tools are also applied to some special parallel robots called "double triangular" to show that the approach is flexible and universally pertinent to manipulator kinematics in reducing the complexity of some previously achieved solutions. Finally a novel two-legged Schönflies architecture is revealed to emphasize that good design is not only essential to good performance but also to easily solve kinematic models. In this example architecture, with double basally actuated legs so as to minimize moving mass, the univariate polynomial solution turns out to be simplest, i.e., of degree 2.
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1 Introduction

This paper was originally intended only to revisit, with reformulation using Study parameters, the direct kinematic (DK) analysis of two special parallel mechanisms, so-called double triangular manipulators (DTMs). These parameters, eight homogeneous coordinates of kinematic image space, are also called the elements of a dual quaternion. Double triangular mechanisms include a planar, a spherical, and a full six degree of freedom (DOF) spatial type, all introduced by Daniali and co-workers [1,2]. This re-investigation of limited scope produced simplifications in solution and some insight that emboldened the authors to go farther afield and include a number of unrelated but possibly more practical parallel manipulators under the unifying umbrella of these analytical tools. The extended work reported herein concentrates on so-called Schönflies 4DOF manipulators, characterized by four distinctly different architectures and investigated by Nabat et al. [3], Angeles et al. [4], Gauthier [5], and Zsombor-Murray [6], respectively, that admit all three translational degrees and one rotation about a fixed axis. A treatment of spherical DTM DK analysis is included. Note that if one is given a 4DOF manipulator, like those confined to Schönflies motions, then the DK is completely specified with four constraint equations. Furthermore, in this paper, these equations describe points, transformed via kinematic mapping, to lie on planes or spheres. The main purpose is to investigate various parallel manipulator architectures and show how their DK is modeled with different combinations of constraints of this type. In every case the main result is a univariate polynomial of degree 2, 4, or 8, and a linear back substitution process to unambiguously evaluate all other unknown parameters. The relation between combinations and the degree of the univariate polynomial solution is explained.

The general Euclidean displacement β in 3-space can be described by

$$\mathbf{q} = \mathbf{M}\mathbf{p} \tag{1}$$

Here $\mathbf{M}$ is the 4×4 matrix

$$\mathbf{M} = \begin{bmatrix} t_0 & 0 & 0 & 0 \\ t_1 & x_0^2+x_1^2-x_2^2-x_3^2 & 2(x_1x_2-x_0x_3) & 2(x_1x_3+x_0x_2) \\ t_2 & 2(x_1x_2+x_0x_3) & x_0^2-x_1^2+x_2^2-x_3^2 & 2(x_2x_3-x_0x_1) \\ t_3 & 2(x_1x_3-x_0x_2) & 2(x_2x_3+x_0x_1) & x_0^2-x_1^2-x_2^2+x_3^2 \end{bmatrix} \tag{2}$$

where t_0 is the nonzero condition

$$t_0 = x_0^2+x_1^2+x_2^2+x_3^2 \neq 0 \tag{3}$$

and the rest of the first column are translational components in the respective x-, y-, and z-direction.

$$\begin{aligned} t_1 &= 2(x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2) \\ t_2 &= 2(x_0y_2 - x_2y_0 + x_3y_1 - x_1y_3) \\ t_3 &= 2(x_0y_3 - x_3y_0 + x_1y_2 - x_2y_1) \end{aligned} \tag{4}$$

The variables x_i, y_i, $i=0\ldots3$ are elements of the Study parameter vector, $\mathbf{s}$, in dual quaternion components

$$\mathbf{s} = [x_0,x_1,x_2,x_3,y_0,y_1,y_2,y_3]^\top$$

that must satisfy the so-called Study condition expressed as

$$x_0y_0+x_1y_1+x_2y_2+x_3y_3=0 \tag{5}$$

Finally $\mathbf{p}$ and $\mathbf{q}$ are homogeneous point coordinate vectors of a point P and its image Q under β.

$$\mathbf{p} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

The rest of the paper is organized as follows: Section 2 contains the general formulation of planar and spherical constraints in terms of Study parameters. In Secs. 3–5 we demonstrate the applicability of the method by treating some manipulator classes pertaining to the types mentioned above.

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2 Planar and Spherical Constraints

In general, a kinematic mapping approach to any problem involves the selection of a set of point, plane, and/or line elements, all on a chosen subassembly, called EE, because it often pertains to and is short for "end effector," of the mechanism in question, and displacing these according to some parameters, x_i and y_i, to be determined so that the selected elements fall on appropriate constraint surfaces on the remaining portion of the mechanism, called FF to indicate base or "fixed frame." In what follows only point elements and planar or spherical constraint surfaces will be used. Notwithstanding these restrictions it will be seen that a rich variety of mechanical situations can be dealt with.

2.1 Planar Constraints. Given the transformation relation, Eq. (1), consider a planar surface constraint equation. This can be written as

$$\mathbf{e}^{\top}\mathbf{q} = \mathbf{e}^{\top}\mathbf{M}\mathbf{p} = e_0q_0 + e_1q_1 + e_2q_2 + e_3q_3 = 0 \tag{6}$$

with

$$\mathbf{e} = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

denoting the homogeneous coordinate vector of the constraint plane ε in the fixed frame and

$$\mathbf{p} = \begin{bmatrix} 1 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

that of the point P in the moving frame whose image Q has to lie in ε.

Only *normalized homogeneous point coordinates* $(p_0=1)$ are used throughout to maintain points in Euclidean space. Then p_1, p_2, and p_3 are the Cartesian coordinates of P in the moving, end effector frame EE.

Equation (6) is a homogeneous quadratic constraint equation in terms of Study parameters x_0, x_1, x_2, x_3, y_0, y_1, y_2, and y_3. It can be compactly written as follows:

$$\mathbf{s}^{\top}\mathbf{C}\mathbf{s} = 0 \tag{7}$$

Here $\mathbf{C}$ is an 8×8 matrix of the form

$$\mathbf{C} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^{\top} & \mathbf{O} \end{bmatrix} \tag{8}$$

with 4×4 blocks $\mathbf{A}$, $\mathbf{O}$, and $\mathbf{B}$ where $\mathbf{O}$ is a zero block while $\mathbf{A}$ is symmetric and $\mathbf{B}$ is skew-symmetric. They can be written as follows:

$$\mathbf{A} = \begin{bmatrix} e_0+e_1p_1+e_2p_2+e_3p_3 & e_3p_2-e_2p_3 & e_1p_3-e_3p_1 & e_2p_1-e_1p_2 \\ e_3p_2-e_2p_3 & e_0+e_1p_1-e_2p_2-e_3p_3 & e_2p_1+e_1p_2 & e_1p_3+e_3p_1 \\ e_1p_3-e_3p_1 & e_2p_1+e_1p_2 & e_0-e_1p_1+e_2p_2-e_3p_3 & e_3p_2+e_2p_3 \\ e_2p_1-e_1p_2 & e_1p_3+e_3p_1 & e_3p_2+e_2p_3 & e_0-e_1p_1-e_2p_2+e_3p_3 \end{bmatrix} \tag{9}$$

$$\mathbf{B} = \begin{bmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & e_3 & -e_2 \\ -e_2 & -e_3 & 0 & e_1 \\ -e_3 & e_2 & -e_1 & 0 \end{bmatrix} \tag{10}$$

2.2 Spherical Constraints. A spherical constraint on the position of an image point $Q(q_0,q_1,q_2,q_3)$ is a condition of the form

$$q_1^2+q_2^2+q_3^2+e_1q_0q_1+e_2q_0q_2+e_3q_0q_3+e_0q_0^2=0 \tag{11}$$

where $e_i=-2m_i$, $i=1,2,3$, and $e_0=m_1^2+m_2^2+m_3^2-r^2$ with m_i being the center coordinates of the sphere κ under consideration and r denoting its radius.

Notice that symbols e_i, $i=0,1,2,3$ are used to denote *both* plane and sphere parameters to emphasize that these play the same role in formulating the constraint equation developed in either case.

Since condition (11) is quadratic in q_i and q_i themselves are quadratic in the Study parameters, an a priori quartic constraint on q_i is obtained. However, by applying a method due to Ref. [7],[1] this is thus reduced to a quadratic equation: Four times the square of Study condition (3) is added to implicit equation (11) to obtain a polynomial that is the product of

$$x_0^2+x_1^2+x_2^2+x_3^2$$

and a homogeneous quadratic factor f in the eight Study parameters

$$q_1^2+q_2^2+q_3^2+e_1q_0q_1+e_2q_0q_2+e_3q_0q_3+e_0q_0^2+4(x_0y_0+x_1y_1 + x_2y_2+x_3y_3)^2=(x_0^2+x_1^2+x_2^2+x_3^2)\cdot f(\mathbf{s}^{\top})$$

Since $x_0^2+x_1^2+x_2^2+x_3^2\neq 0$ and $x_0y_0+x_1y_1+x_2y_2+x_3y_3=0$ the constraint equation imposed by a sphere constraint is

$$f(\mathbf{s}^{\top})=0$$

Compressing coefficients, a compact matrix form is obtained as

$$f(\mathbf{s}^{\top})=\mathbf{s}^{\top}\mathbf{C}^{*}\mathbf{s}=0$$

The resulting 8×8 matrix $\mathbf{C}^{*}$ is abbreviated to block form as

$$\mathbf{C}^{*} = \begin{bmatrix} \mathbf{A}+(p_1^2+p_2^2+p_3^2)\mathbf{I} & \mathbf{B}^{*} \\ \mathbf{B}^{*\top} & 4\mathbf{I} \end{bmatrix} \tag{12}$$

where $\mathbf{A}$ is the 4×4 symmetric matrix (Eq. (9)), $\mathbf{B}^{*}$ is the 4×4 skew-symmetric matrix

$$\mathbf{B}^{*} = \begin{bmatrix} 0 & e_1+2p_1 & e_2+2p_2 & e_3+2p_3 \\ -e_1-2p_1 & 0 & e_3-2p_3 & -e_2+2p_2 \\ -e_2-2p_2 & -e_3+2p_3 & 0 & e_1-2p_1 \\ -e_3-2p_3 & e_2-2p_2 & -e_1+2p_1 & 0 \end{bmatrix} \tag{13}$$

and $\mathbf{I}$ is the 4×4 identity matrix.

[1]Husty [7] was first to apply this technique to formulate the DK algorithm for the general Stuart–Gough platform manipulator where six points in EE are displaced onto six spheres in FF.

2.3 Constraint Equation Structure. Comparing Eqs. (8)–(10), (12), and (13) one sees that matrices $\mathbf{C}$ and $\mathbf{C}^*$, which contain only given parameters, are quite similar in structure.

In the case of the point-on-plane (PoP) constraint the matrix $\mathbf{C}$ leads to an equation that is linear in y_i.

The point-on-sphere (PoS) constraint contains a term $4\Sigma_{i=0}^{3} y_i^2$ but there are no other quadratic terms in y_i.

In the case of more than one sphere constraint, only one constraint equation needs to remain quadratic in y_i because it can be subtracted from the others to remove all y_i^2.

For a full 6DOF manipulator problem, six constraint equations are required. The nonzero condition and the Study condition

$$\sum_{i=0}^{3} x_i^2 \neq 0, \quad \sum_{i=0}^{3} x_i y_i = 0$$

are added as additional constraints to handle eight unknown parameters.

3 Schönflies Manipulator DK With Plane and/or Sphere Constraints

The four parameter subgroup of Schönflies displacements contains the proper Euclidean transformations that confine rotation to a fixed axial direction. Here the common direction is taken parallel to the z- or x_3-axis of EE and FF. Analytic description of this group is obtained by substituting

$$x_1 = x_2 = 0$$

in the general displacement matrix, Eq. (2), so as to become the 4×4 matrix, Eq. (14).

$$\mathbf{M} = \begin{bmatrix} x_0^2 + x_3^2 & 0 & 0 & 0 \\ t_1 & x_0^2 - x_3^2 & -2x_0x_3 & 0 \\ t_2 & 2x_0x_3 & x_0^2 - x_3^2 & 0 \\ t_3 & 0 & 0 & x_0^2 + x_3^2 \end{bmatrix} \tag{14}$$

Simplified first column (translation) elements are shown above and are defined below.

$$\begin{aligned} t_1 &= 2(x_0y_1 - x_3y_2) \\ t_2 &= 2(x_0y_2 + x_3y_1) \\ t_3 &= 2(x_0y_3 - x_3y_0) \end{aligned} \tag{15}$$

The Study condition and the nonzero condition are similarly reduced.

$$x_0y_0 + x_3y_3 = 0 \tag{16}$$

$$x_0^2 + x_3^2 \neq 0 \tag{17}$$

A Schönflies manipulator is any mechanism that admits only Schönflies motions.

In case of Schönflies displacement, a plane constraint is represented by matrix $\mathbf{C}$ (Eq. (8)), with second and third rows and columns removed, that now reads as

$$a_1x_0^2 + 2a_2x_0x_3 + a_3x_3^2 + 2e_1(x_0y_1 - x_3y_2) + 2e_2(x_0y_2 + x_3y_1) + 2e_3(x_0y_3 - x_3y_0) = 0$$

or, using t_i, defined by Eq. (15), as

$$a_1x_0^2 + 2a_2x_0x_3 + a_3x_3^2 + e_1t_1 + e_2t_2 + e_3t_3 = 0 \tag{18}$$

where

$$a_1 = e_0 + e_1p_1 + e_2p_2 + e_3p_3$$

$$a_2 = e_2p_1 - e_1p_2$$

$$a_3 = e_0 - e_1p_1 - e_2p_2 + e_3p_3$$

Similarly a simplified[2] sphere constraint, in case of the Schönflies motion, is written as

$$a_1^*x_0^2 + 2a_2^*x_0x_3 + a_3^*x_3^2 + 2b_1^*x_0y_1 + 2b_2^*x_3y_2 + 2b_3^*x_0y_2 + 2b_4^*x_3y_1 + 2b_5^*(x_0y_3 - x_3y_0) + 4(y_0^2 + y_1^2 + y_2^2 + y_3^2) = 0 \tag{19}$$

where

$$a_1^* = e_0 + e_1p_1 + e_2p_2 + e_3p_3 + p_1^2 + p_2^2 + p_3^2$$

$$a_2^* = e_2p_1 - e_1p_2$$

$$a_3^* = e_0 - e_1p_1 - e_2p_2 + e_3p_3 + p_1^2 + p_2^2 + p_3^2$$

$$b_1^* = e_1 + 2p_1$$

$$b_2^* = -e_1 + 2p_1$$

$$b_3^* = e_2 + 2p_2$$

$$b_4^* = e_2 - 2p_2$$

$$b_5^* = e_3 + 2p_3$$

3.1 Schönflies Motion With Three PoP Constraints. To better understand geometric techniques used later in specific examples of parallel manipulator architectures it is useful to discuss *Darboux* motion, which is defined by the requirement that the path of each point is a planar curve. It turns out (cf. Ref. [8], pp. 304–310) that aside from trivially obvious cases, where all point paths lie in parallel planes, such a motion is one-parametric and the rigid body can rotate only about axes in some common, fixed direction. This means that Darboux motion is a subset of Schönflies motion. Moreover, it is well known that all point paths under a nontrivial Darboux motion are ellipses.

In the following we prove that a Schönflies motion with three PoP constraints is always a Darboux motion.[3] All we need to show is that, given three PoP constraints, the translational components t_1, t_2, and t_3 are homogeneous quadratic functions in x_0 or x_3; i.e., the resulting motion is *rational* of order 2.

Let

$$a_{i1}x_0^2 + 2a_{i2}x_0x_3 + a_{i3}x_3^2 + 2e_{i1}(x_0y_1 - x_3y_2) + 2e_{i2}(x_0y_2 + x_3y_1) + 2e_{i3}(x_0y_3 - x_3y_0) = 0 \tag{20}$$

be the three PoP constraints, $i = 1,2,3$ (compare with Eq. (18)). With some further symbolic compression, as noted afterward, the following four expressions, Eq. (21), generated with Eq. (20) via Cramer's rule, are offered, by way of proof, to show that one indeed obtains a Darboux motion.

$$\begin{aligned} y_0 &= -x_3 \cdot \frac{\gamma_2(x_0,x_3)}{2\Delta \cdot (x_0^2 + x_3^2)} \\ y_1 &= \frac{x_0 \cdot \alpha_2(x_0,x_3) + x_3 \cdot \beta_2(x_0,x_3)}{2\Delta \cdot (x_0^2 + x_3^2)} \\ y_2 &= \frac{x_0 \cdot \beta_2(x_0,x_3) - x_3 \cdot \alpha_2(x_0,x_3)}{2\Delta \cdot (x_0^2 + x_3^2)} \\ y_3 &= x_0 \cdot \frac{\gamma_2(x_0,x_3)}{2\Delta \cdot (x_0^2 + x_3^2)} \end{aligned} \tag{21}$$

where $\alpha_2(x_0,x_3)$, $\beta_2(x_0,x_3)$, and $\gamma_2(x_0,x_3)$ are the quadratic homogeneous polynomials

$$\alpha_2(x_0,x_3) = [|\mathbf{a}_1\mathbf{e}_2\mathbf{e}_3||\mathbf{a}_2\mathbf{e}_2\mathbf{e}_3||\mathbf{a}_3\mathbf{e}_2\mathbf{e}_3|][x_0^2 2x_0x_3 x_3^2]^\top$$

[2]Only six of the bilinear terms x_iy_j occur.

[3]Vogler [16] recently gave an alternative proof of this fact.

$$\beta_2(x_0,x_3)=[|\mathbf{e}_1\mathbf{a}_1\mathbf{e}_3||\mathbf{e}_1\mathbf{a}_2\mathbf{e}_3||\mathbf{e}_1\mathbf{a}_3\mathbf{e}_3|][x_0^2 2x_0x_3x_3^2]^\top$$

$$\gamma_2(x_0,x_3)=[|\mathbf{e}_1\mathbf{e}_2\mathbf{a}_1||\mathbf{e}_1\mathbf{e}_2\mathbf{a}_2||\mathbf{e}_1\mathbf{e}_2\mathbf{a}_3|][x_0^2 2x_0x_3x_3^2]^\top$$

and

$$\Delta=|\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3|,\quad \mathbf{a}_j=\begin{bmatrix}a_{1j}\\a_{2j}\\a_{3j}\end{bmatrix},\quad \mathbf{e}_j=\begin{bmatrix}e_{1j}\\e_{2j}\\e_{3j}\end{bmatrix}$$

By substitution of Eq. (21) into Eq. (15) we get

$$t_1=2(x_0y_1-x_3y_2)=\frac{\alpha_2(x_0,x_3)}{\Delta}$$
$$t_2=2(x_0y_2+x_3y_1)=\frac{\beta_2(x_0,x_3)}{\Delta}\tag{22}$$
$$t_3=2(x_0y_3-x_3y_0)=\frac{\gamma_2(x_0,x_3)}{\Delta}$$

This shows that the translational components t_1, t_2, and t_3 are indeed homogeneous quadratic functions in x_0,x_3, as stated.

3.2 DK of Schönflies Manipulators With Three PoP and a Fourth PoP or PoS Constraint. As shown in Sec. 3.1 the three given PoP constraints determine a Darboux motion. Hence, the path of the fourth given point undergoing this motion is an ellipse. On the other hand this point must lie on a plane or sphere according to the fourth given constraint. In conclusion the DK problem at hand can be reduced to finding the intersection of an ellipse with a plane or a sphere. The insight gained from this approach shows that such a problem must necessarily admit two or four DK solutions, at most.

Analytically, the solutions can be found as follows. From the three given PoP constraints we obtain expressions (21) and (22).

(a) If the fourth constraint surface is a plane represented by Eq. (18) then substitution of Eq. (22) produces a quadratic univariate in x_3 after dehomogenizing with $x_0=1$.

(b) If the fourth surface is a sphere represented by Eq. (19), then by substitution of Eq. (21) the quadratic term $4\Sigma_{i=0}^3 y_i^2$ becomes

$$4(y_0^2+y_1^2+y_2^2+y_3^2)=\frac{1}{\Delta^2(x_0^2+x_3^2)}[\alpha_2^2(x_0,x_3)+\beta_2^2(x_0,x_3)+\gamma_2^2(x_0,x_3)]$$

Thus, substitution of Eq. (21) in Eq. (19) clearly produces a quartic univariate in x_3 after multiplication with the denominator $x_0^2+x_3^2$ and dehomogenizing with $x_0=1$.

Once the values of x_3 are thus obtained, the three equations in Eq. (22) allow one to find the corresponding values of t_1, t_2, and t_3, thus completing the definition of the DK displacement implied by the problem.

3.3 DK of Schönflies Manipulators With Two or More PoS Constraints. Let at least two PoS constraints (Eq. (19)) be used to characterize the DK of a Schönflies motion. Then the difference between any two PoS equations removes the term $4\Sigma_{i=0}^3 y_i^2$ so as to always yield three equations (Eq. (23)) linear in y_0, y_1, y_2, and y_3.

$$m_{i1}x_0^2+m_{i2}x_0x_3+m_{i3}x_3^2-m_{i4}x_3y_0+(m_{i5}x_0+m_{i6}x_3)y_1+(m_{i7}x_0+m_{i8}x_3)y_2+m_{i4}x_0y_3=0\tag{23}$$

Each represents either a PoP constraint or the difference between two PoS constraints. The coefficients m_{ij}, $i=1,2,3$, $j=1,\ldots,8$ are formulated from appropriate combinations of given point, plane, or sphere parameters, p_{kl}, e_{kl}.

A fourth constraint contains the term $4\Sigma_{i=0}^3 y_i^2$. It has the form of Eq. (19).

A 4×5 matrix, whose rows are coefficients of 1, y_0, y_1, y_2, and y_3, as these appear in Study condition (16) and the three equations in Eq. (23), is set up. Taking determinants of all 4×4 minors with alternating $\pm$ sign and dividing all the rest by the first, i.e., Cramer's rule, yield $y_i=y_i(x_0,x_3)$.

$$\Delta=(x_0^2+x_3^2)[\mu_{457}x_0^2+(\mu_{458}+\mu_{467})x_0x_3+\mu_{468}x_3^2]=(x_0^2+x_3^2)\,\delta_2(x_0,x_3)$$

$$y_0=\frac{x_3}{\Delta}[\mu_{157}x_0^4+(\mu_{158}+\mu_{167}+\mu_{257})x_0^3x_3+(\mu_{168}+\mu_{258}+\mu_{267}+\mu_{357})x_0^2x_3^2+(\mu_{268}+\mu_{358}+\mu_{367})x_0x_3^3+\mu_{368}x_3^4]=\frac{x_3}{\Delta}\alpha_4(x_0,x_3)$$

$$y_1=\frac{x_0^2+x_3^2}{\Delta}[\mu_{147}x_0^3+(\mu_{148}+\mu_{247})x_0^2x_3+(\mu_{248}+\mu_{347})x_0x_3^2+\mu_{348}x_3^3]=\frac{x_0^2+x_3^2}{\Delta}\cdot\beta_3(x_0,x_3)\tag{24}$$

$$y_2=-\frac{x_0^2+x_3^2}{\Delta}[\mu_{145}x_0^3+(\mu_{146}+\mu_{245})x_0^2x_3+(\mu_{246}+\mu_{345})x_0x_3^2+\mu_{346}x_3^3]=-\frac{x_0^2+x_3^2}{\Delta}\gamma_3(x_0,x_3)$$

$$y_3=-\frac{x_0}{x_3}y_0=-\frac{x_0}{\Delta}\cdot\alpha_4(x_0,x_3)$$

where $\mu_{ijk}=|\mathbf{m}_i\mathbf{m}_j\mathbf{m}_k|$, $\mathbf{m}_j=[m_{1j},m_{2j},m_{3j}]^\top$.

Note that homogeneous polynomials $\delta_2(x_0,x_3)$, $\alpha_4(x_0,x_3)$, $\beta_3(x_0,x_3)$, and $\gamma_3(x_0,x_3)$ in x_0,x_3 are of degrees 2, 4, 3, and 3, respectively.

At this point, things do not look encouraging. The numerators in the expressions for y_i are of fifth order and the common denominator Δ is quartic. Improvement in prospects appears after substitution of these expressions into the quadratic term $\Sigma_{i=0}^3 y_i^2$ of the fourth constraint (Eq. (19))

$$\sum_{i=0}^{3} y_i^2=\frac{\alpha_4^2(x_0,x_3)+(x_0^2+x_3^2)\cdot(\beta_3^2(x_0,x_3)+\gamma_3^2(x_0,x_3))}{(x_0^2+x_3^2)\cdot\delta_2^2(x_0,x_3)}$$

Hence, substitution of Eq. (24) into the fourth equation yields, after multiplication with the denominator $(x_0^2+x_3^2)\delta_2^2(x_0,x_3)$, the following homogeneous octic equation in x_0,x_3:

$$(x_0^2+x_3^2)[\delta_2(x_0,x_3)((a_1^*x_0^2+2a_2^*x_0x_3+a_3^*x_3^2)\delta_2(x_0,x_3)+2(b_1^*x_0+b_4^*x_3)\cdot\beta_3(x_0,x_3)-2(b_3^*x_0+b_2^*x_3)\gamma_3(x_0,x_3)-2b_5^*\alpha_4(x_0,x_3))+4(\beta_3^2(x_0,x_3)+\gamma_3^2(x_0,x_3))]+4\alpha_4^2(x_0,x_3)=0\tag{25}$$

This establishes the upper bound of eight on the number of possible solutions for any Schönflies DK problem that is defined by PoP and PoS constraints and contains at least two of the latter. With solutions for x_3, and having set $x_0=1$, corresponding values of y_i are obtained explicitly with Eq. (24); so are elements of the transformation, Eq. (14). This essentially solves this DK problem. In Sec. 3.4 it will be shown that eight real DK solutions for such Schönflies architectures can occur.

3.4 Examples of Schönflies Manipulators

3.4.1 Fully Parallel Schönflies Manipulators. Figure 1 shows the Π or parallelogram joint, a feature common to many Schönflies manipulators because it provides a 1DOF circular translation to the distal link, with respect to the link at the opposite side of the parallelogram. Figure 1(*a*) shows two leg designs, with Π- and

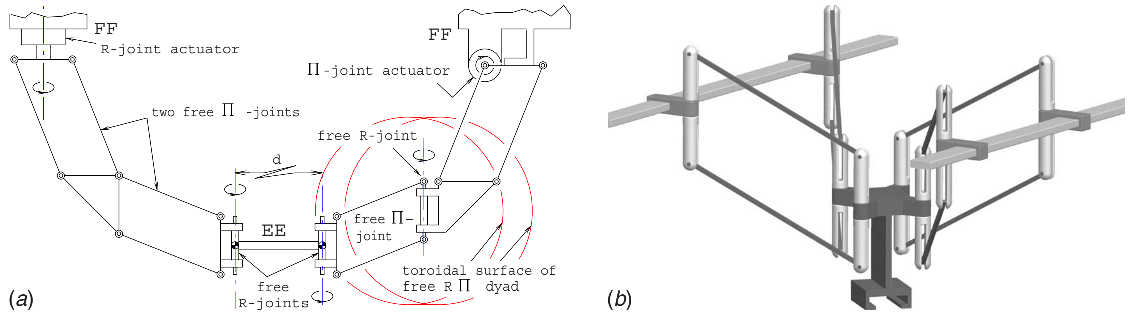

Fig. 1 Various leg architectures in a variety of Schönflies parallel manipulator contexts

R-joints, that may be used to build four-legged robots wherein EE executes Schönflies motion. Actuating a basal R-joint as shown on the left causes the remaining free joints to bind the EE attachment point, shown at the center of the terminal R-joint, to motion in the plane of the two Π-joints. Actuating a basal Π-joint as shown at the right causes that point to move, alas, on a *torus*. A four, similar legged manipulator of the first type is therefore seen to have four PoP constraints and a DK solution admitting two assembly modes as demonstrated in Sec. 3.2. In the case on the right, circular sections of the torus are shown. DK analysis of such models awaits a future treatment of toroidal constraint that promises to be more complicated. A glance at the manipulator of Zhou et al. [9], with $\underline{P}R\Pi R$ legs, shown in Fig. 1(b), fares—if one seeks solution simplicity—somewhat better. The EE attachment points cause the terminal R-joint centers to move on spheres and the DK problem solution admits an octic univariate polynomial. Referring to Ref. [10] the interested reader might try all leg architectures depicted therein to see which fit the "point-on-plane or -sphere" formulation paradigm. Certainly those that do not will afford fruitful avenue of future research.

3.4.2 Two-Legged Schönflies Manipulators. Shown in Fig. 2(a) is a novel design prototype revealed by Angeles et al. [4]. The idea was to achieve superior workspace and dexterity, which one might expect when the number of legs of a parallel robot is reduced from four to two, while retaining some advantages inherent in parallel architecture. Furthermore maintaining basal actuation is seen as an additional advantage of the design. This avoids placement of motors on moving links, as is done in many serial designs. Two joint actuation is achieved by means of an—also basally mounted—planetary gearbox that delivers torque to both the proximal R- and Π-joints. Only one would be actuated, in typical four-legged designs, like those depicted in Fig. 1.

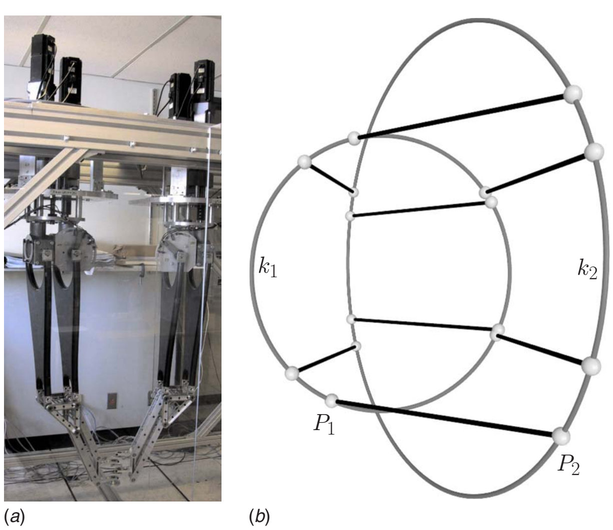

Fig. 2 Two-legged Schönflies manipulator

The DK of the two-legged Schönflies manipulator of Angeles et al. [4] is immediately seen to be modeled as the placement of each of the two EE attachment points P_i on a circle k_i represented by two surfaces: a sphere κ_i and a vertical plane ε_i, $i=1,2$. The solution paradigm is typical of all parallel Schönflies manipulators with PoP and more than one PoS constraint. Therefore the setup for the octic univariate, derived in Sec. 3.3, will be carried out here in some detail.

By appropriate choice of the coordinate system in EE one can assume that the two EE attachment points P_1 and P_2 are given by the vectors

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ d \\ 0 \\ 0 \end{bmatrix}$$

which means that P_1 is on the origin and P_2 on the x-axis of the EE coordinate system. The circles k_1 and k_2 are represented by the plane-sphere pairs $(\varepsilon_1, \kappa_1)$ and $(\varepsilon_2, \kappa_2)$ as follows:

$$\varepsilon_1 \cdots \mathbf{e}_1 = \begin{bmatrix} e_{10} \\ e_{11} \\ e_{12} \\ 0 \end{bmatrix}, \quad \kappa_1 \cdots \mathbf{e}_2 = \begin{bmatrix} e_{20} \\ e_{21} \\ e_{22} \\ e_{23} \end{bmatrix}$$

$$\varepsilon_2 \cdots \mathbf{e}_3 = \begin{bmatrix} e_{30} \\ e_{31} \\ e_{32} \\ 0 \end{bmatrix}, \quad \kappa_2 \cdots \mathbf{e}_4 = \begin{bmatrix} e_{40} \\ e_{41} \\ e_{42} \\ e_{43} \end{bmatrix}$$

From the constraints $P_1 \in \varepsilon_1, \kappa_1$ and $P_2 \in \varepsilon_2, \kappa_2$ we get the four equations

$$e_{10}(x_0^2 + x_3^2) + 2e_{11}(x_0y_1 - x_3y_2) + 2e_{12}(x_0y_2 + 2x_3y_1) = 0 \quad (26)$$

$$e_{20}(x_0^2 + x_3^2) + 2e_{21}(x_0y_1 - x_3y_2) + 2e_{22}(x_0y_2 + x_3y_1) + 2e_{23}(x_0y_3 - x_3y_0) + 4(y_0^2 + y_1^2 + y_2^2 + y_3^2) = 0 \quad (27)$$

$$(e_{30} + e_{31}d)x_0^2 + 2e_{32}dx_0x_3 + (e_{30} - e_{31}d)x_3^2 + 2e_{31}(x_0y_1 - x_3y_2) + 2e_{32}(x_0y_2 + x_3y_1) = 0 \quad (28)$$

$$(e_{40} + e_{41}d + d^2)x_0^2 + 2e_{42}dx_0x_3 + (e_{40} - e_{41}d + d^2)x_3^2 + 2(e_{41} + 2d)x_0y_1 - 2(e_{41} - 2d)x_3y_2 + 2e_{42}(x_0y_2 + x_3y_1) + 2e_{43}(x_0y_3 - x_3y_0) + 4(y_0^2 + y_1^2 + y_2^2 + y_3^2) = 0 \quad (29)$$

To obtain Eq. (23), the system of three equations linear in y_i, Eqs. (26) and (28) are selected along with the difference between Eqs. (29) and (27). The resulting coefficients m_{ij} are

$$m_{11}=m_{13}=e_{10}, \quad m_{12}=m_{14}=0$$

$$m_{15}=-m_{18}=2e_{11}, \quad m_{16}=m_{17}=2e_{12}$$

$$m_{21}=e_{30}+e_{31}d, \quad m_{22}=2e_{32}d, \quad m_{23}=e_{30}-e_{31}d$$

$$m_{24}=0, \quad m_{25}=-m_{28}=2e_{31}, \quad m_{26}=m_{27}=2e_{32}$$

$$m_{31}=e_{40}-e_{20}+e_{41}d+d^2, \quad m_{32}=2e_{42}d$$

$$m_{33}=e_{40}-e_{20}-e_{41}d+d^2, \quad m_{34}=2(e_{43}-e_{23})$$

$$m_{35}=2(e_{41}-e_{21}+2d), \quad m_{36}=m_{37}=2(e_{42}-e_{22})$$

$$m_{38}=2(-e_{41}+e_{21}+2d)$$

With the coefficients m_{ij} one defines the determinants μ_{ijk} and hence the polynomials $\alpha_4(x_0,x_3)$, $\beta_3(x_0,x_3)$, $\gamma_3(x_0,x_3)$, and $\delta_2(x_0,x_3)$ according to Eq. (24). Finally, one of the two given PoS constraints, say, Eq. (27), is used to produce univariate octic equation (25). In this case the resulting constants $a_1^*,\ldots,b_5^*$ are

$$a_1^*=a_3^*=e_{20}, \quad a_2^*=0$$

$$b_1^*=-b_2^*=e_{21}, \quad b_3^*=b_4^*=e_{22}, \quad b_5^*=e_{23}$$

Figure 2(b) shows an example with eight real solutions. The eight poses of EE are represented by eight horizontal—they do not appear so in the perspective image—bars whose end points P_1 and P_2 lie on the two given circles k_1 and k_2 representing EE anchor point free motion in FF. This example was solved using the following data:

$$d=5, \quad \mathbf{e}_1=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad \mathbf{e}_2=\begin{bmatrix}-9\\0\\0\\0\end{bmatrix}$$

$$\mathbf{e}_3=\begin{bmatrix}-0.98\\-0.1\\1\\0\end{bmatrix}, \quad \mathbf{e}_4=\begin{bmatrix}-23.87\\-0.4\\-2\\-0.6\end{bmatrix}$$

As a final example of two-legged Schönflies manipulators, the two view drawing in Fig. 3 shows for the first time how, after considerable further development of the basic design idea, to apply a very simple $\underline{P}\underline{P}PR$ leg architecture to achieve a DK model where two points S and T move on two lines $\mathcal{S}$ and $\mathcal{T}$, respectively. Each line is the intersection of a vertical plane and one normal to it. The two basal P-joints on each leg are actuated, possibly in the manner shown. With only PoP constraints, the DK admits two solutions, at most.

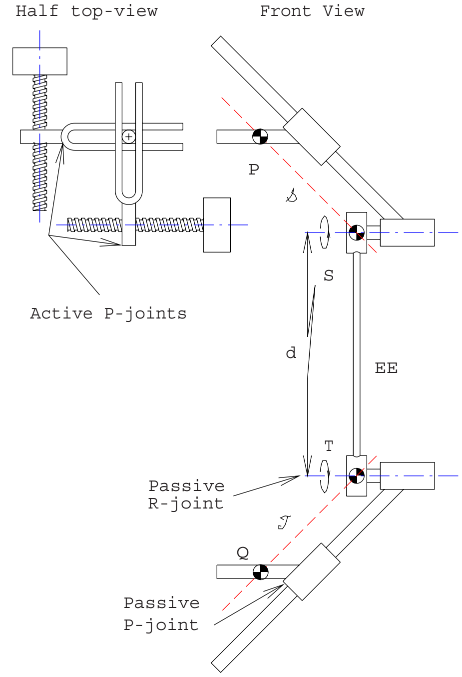

Fig. 3 Two screw actuators for double basal actuation

4 DK of the Spherical Double Triangular Manipulator

Figure 4(a) shows the mechanical layout of a regular spherical double triangular manipulator (spherical DTM). Keep in mind that the three short legs, each made up of curved sliders and intermediate R-joints, separating the curved rods of FF and EE, make this architecture kinematically equivalent to a classical three-legged $\underline{R}RR$ spherical parallel manipulator. Notwithstanding apparent similarity to the spatial DTM (see Sec. 5) this one, in contrast, is fully parallel; i.e., has one as opposed to more actuated joints per leg.

Under spherical displacement there are no terms containing y_i in point transformation (2) as follows:

$$\mathbf{M}=\begin{bmatrix} x_0^2+x_1^2+x_2^2+x_3^2 & 0 & 0 & 0\\ 0 & x_0^2+x_1^2-x_2^2-x_3^2 & 2(x_1x_2-x_0x_3) & 2(x_1x_3+x_0x_2)\\ 0 & 2(x_1x_2+x_0x_3) & x_0^2-x_1^2+x_2^2-x_3^2 & 2(x_2x_3-x_0x_1)\\ 0 & 2(x_1x_3-x_0x_2) & 2(x_2x_3+x_0x_1) & x_0^2-x_1^2-x_2^2+x_3^2 \end{bmatrix} \tag{30}$$

The DK of the spherical DTM can be reformulated as the following task:

Given a spherical triangle $P_1P_2P_3$ *on the unit sphere and three planes* $\varepsilon_1,\varepsilon_2,\varepsilon_3$ *on the origin (center of the unit sphere) find a congruent spherical triangle* $Q_1Q_2Q_3$ *with* $Q_i\in\varepsilon_i$.

In other words one has to find all spherical displacements that satisfy the three PoP conditions P_i,ε_i, $i=1,2,3$.

To solve this task one may simplify coefficients by choosing, without loss in generality, the three points P_i and the three planes ε_i as follows:

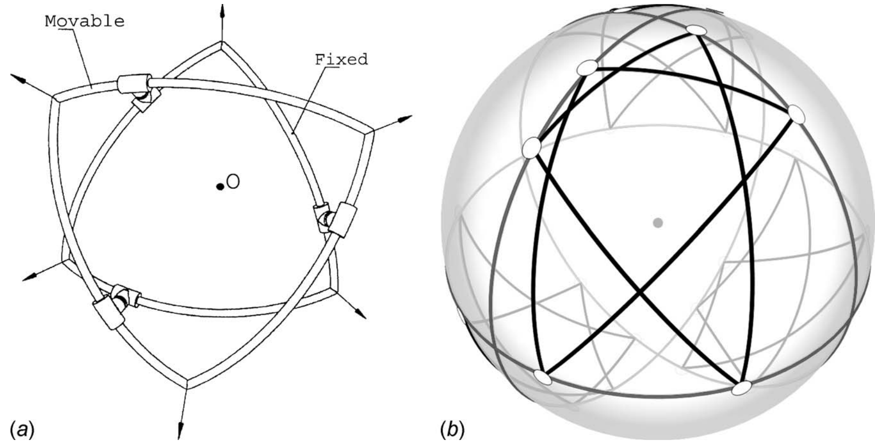

Fig. 4 Spherical double triangular manipulator

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ p_{2,1} \\ p_{2,2} \\ 0 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} 1 \\ p_{3,1} \\ p_{3,2} \\ p_{3,3} \end{bmatrix}$$

and

$$\mathbf{e}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ e_{2,1} \\ e_{2,2} \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ e_{3,1} \\ e_{3,2} \\ e_{3,3} \end{bmatrix}$$

Then the three planar constraints

$$\mathbf{e}_i^{\top}\mathbf{M}\mathbf{p}_i = 0, \quad i = 1,2,3$$

have the form

$$\begin{gathered} x_0^2 + x_1^2 - x_2^2 - x_3^2 = 0 \\ a_{00}x_0^2 + a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{03}x_0x_3 + 2a_{12}x_1x_2 = 0 \\ b_{00}x_0^2 + b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + 2b_{01}x_0x_1 + 2b_{02}x_0x_2 + 2b_{03}x_0x_3 \\ + 2b_{12}x_1x_2 + 2b_{13}x_1x_3 + 2b_{23}x_2x_3 = 0 \end{gathered} \tag{31}$$

where $\mathbf{M}$ is matrix equation (30) and

$$a_{00} = e_{21}p_{21} + e_{22}p_{22}, \quad a_{11} = e_{21}p_{21} - e_{22}p_{22}$$

$$a_{22} = -e_{21}p_{21} + e_{22}p_{22}, \quad a_{33} = -e_{21}p_{21} - e_{22}p_{22}$$

$$a_{03} = e_{22}p_{21} - e_{21}p_{22}, \quad a_{21} = e_{21}p_{22} + e_{22}p_{21}$$

$$b_{00} = e_{32}p_{32} + e_{31}p_{31} + e_{33}p_{33}$$

$$b_{11} = -e_{33}p_{33} + e_{31}p_{31} - e_{32}p_{32}$$

$$b_{22} = e_{32}p_{32} - e_{31}p_{31} - e_{33}p_{33}$$

$$b_{33} = -e_{31}p_{31} - e_{32}p_{32} + e_{33}p_{33}$$

$$b_{01} = e_{33}p_{32} - e_{32}p_{33}, \quad b_{02} = e_{31}p_{33} - e_{33}p_{31}$$

$$b_{03} = e_{32}p_{31} - e_{31}p_{32}, \quad b_{12} = e_{31}p_{32} + e_{32}p_{31}$$

$$b_{13} = e_{31}p_{33} + e_{33}p_{31}, \quad b_{23} = e_{32}p_{33} + e_{33}p_{32}$$

Remark. It is well known that each of the equations in Eq. (31) represents a Clifford-quadric in a homogeneous three dimensional vector space of Euler parameters x_0, x_1, x_2, and x_3. This is a Cayley–Klein space with an elliptic metric based on the absolute null-quadric $\mathcal{M}: x_0^2+x_1^2+x_2^2+x_3^2=0$. A Clifford-quadric is characterized by the property that its intersection with $\mathcal{M}$ is a skew quadrilateral consisting of two pairs of conjugate complex straight lines. See, for instance, Ref. [11].

In the following we will outline how the number of variables can be reduced from four to three, if dehomogenization is counted, to two by introducing a bilinear parametrization of the Clifford-quadric represented by the first of the three equations in Eq. (31); i.e.,

$$x_0^2 + x_1^2 - x_2^2 - x_3^2 = 0 \tag{32}$$

By means of the regular projective (coordinate) transformation

$$\begin{aligned} x_0 &= y_0 + y_3 \\ x_1 &= y_1 + y_2 \\ x_2 &= y_0 - y_3 \\ x_3 &= y_1 - y_2 \end{aligned} \tag{33}$$

the Clifford-quadric, Eq. (32), becomes the bilinear equation

$$4(y_0y_3 + y_1y_2) = 0$$

Now a parametrization can be easily tailored so as to null the expression above, viz.,

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ u \\ v \\ uv \end{bmatrix} \tag{34}$$

After application of the inverse projective transform (Eq. (33)) we obtain the mentioned bilinear parametrization of the original Clifford-quadric (Eq. (32))

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + uv \\ u + v \\ -1 - uv \\ u - v \end{bmatrix} \tag{35}$$

Hence, substitution of this parametrization nulls the left hand side of the first of the equations in Eq. (31). The other two, after a little rearrangement, assume the form of

$$\begin{aligned} \alpha_2(u)v^2 + \alpha_1(u)v + \alpha_0(u) &= 0 \\ \beta_2(u)v^2 + \beta_1(u)v + \beta_0(u) &= 0 \end{aligned} \tag{36}$$

where the coefficients are the following quadratics in u:

$$\alpha_2(u) = 2e_{22}(p_{22}u^2 - 2p_{21}u - p_{22})$$

$$\alpha_1(u) = -4e_{21}p_{22}(u^2 + 1)$$

$$\alpha_0(u) = -2e_{22}(p_{22}u^2 + 2p_{21}u - p_{22})$$

$$\beta_2(u) = -2(e_{31}p_{33} - e_{32}p_{32} - e_{33}p_{31})u^2 - 4(e_{32}p_{31} - e_{33}p_{32})u \\ - 2(e_{31}p_{33} + e_{32}p_{32} + e_{33}p_{31})$$

$$\beta_1(u) = -4(e_{31}p_{32} + e_{32}p_{33})u^2 - 8e_{33}p_{33}u - 4(e_{31}p_{32} - e_{32}p_{33})$$

$$\beta_0(u) = 2(e_{31}p_{33} - e_{32}p_{32} + e_{33}p_{31})u^2 - 4(e_{32}p_{31} + e_{33}p_{32})u \\ + 2(e_{31}p_{33} + e_{32}p_{32} - e_{33}p_{31})$$

An octic in u emerges when v is eliminated between equations in Eq. (36). A neat dialytic method to do this is given in

$$\begin{vmatrix} \alpha_2(u) & \alpha_1(u) & \alpha_0(u) & 0 \\ 0 & \alpha_2(u) & \alpha_1(u) & \alpha_0(u) \\ \beta_2(u) & \beta_1(u) & \beta_0(u) & 0 \\ 0 & \beta_2(u) & \beta_1(u) & \beta_0(u) \end{vmatrix} = 0 \tag{37}$$

To obtain values of v that correspond to the eight values of u obtained with Eq. (37) consider that a given $u=u_0$ numerically defines all coefficients in Eq. (36) so these two equations become redundant. Multiplying these, respectively, by $\alpha_2(u_0)$ and $\beta_2(u_0)$ and equating their difference to zero define $v=v_0$ as

$$v = v_0 = \frac{\alpha_0(u_0)\beta_2(u_0) - \alpha_2(u_0)\beta_0(u_0)}{\alpha_2(u_0)\beta_1(u_0) - \alpha_1(u_0)\beta_2(u_0)} \tag{38}$$

Using known pairs of $u=u_0$, $v=v_0$ in Eq. (35) yields all four x_i for up to eight poses of EE moved to FF via a spherical displacement constrained by three PoP equations.

The diagram in Fig. 4(*b*) displays an architecture with imposed joint parameters that generates a DK solution with eight assembly modes. So once again an octic univariate is minimal. This example uses three planes, $x=0$, $y=0$, and $z=0$, upon which three absolute EE points, initially with respective direction numbers (1,0,0), (0.5,0.48,0), and (0.27,0.71,1.64), are to be placed. The EE triangle is scalene. It was thus chosen to visually contrast, by its asymmetry, its double placement in each of the four octants of the sphere and, of course, to show a case with eight real assembly modes. The division of the FF sphere into eight congruent spherical triangles brings to ones attention that inscribing the EE triangle into any of the other (blank) octants would involve parity reversal of the EE triangle. I.e., exchanging concave and convex surface orientation, like flipping heads and tails in the planar case, is forbidden. Such "solutions" would thus not be valid ones.

5 Spatial Three-Legged Manipulator DK With Three Line Constraints

This is a full mobility, i.e., 6DOF manipulator. It fits into the category of reduced mobility—or rather reduced complexity—because it is not fully parallel. Its three legs require two actuators each and thus its DK is much easier to solve than, say, Husty's general six-points-on-six-spheres problem [7]. Equation (1) and six PoP constraints may be used in this case if each given point must satisfy a pair of these; i.e., each pair of planes intersects on one of the given lines. The spatial DTM can be modeled in this way. The following three points P_i, $i=1,2,3$ and six planes ε_i, $i=1,\ldots,6$, the latter to be taken in successive pairs to represent lines, l_i, are without loss in generality chosen to simplify equation coefficients and, more important, to obtain a system that admits a reparametrization approach to solution quite similar to that used, in Sec. 4, for the DK of three-legged spherical robots.

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ p_{21} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} 1 \\ p_{31} \\ p_{32} \\ 0 \end{bmatrix}$$

$$l_1 \cdots \mathbf{e}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$l_2 \cdots \mathbf{e}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ e_{33} \end{bmatrix}, \quad \mathbf{e}_4 = \begin{bmatrix} 0 \\ e_{41} \\ e_{42} \\ 0 \end{bmatrix}$$

$$l_3 \cdots \mathbf{e}_5 = \begin{bmatrix} 1 \\ e_{51} \\ e_{52} \\ 0 \end{bmatrix}, \quad \mathbf{e}_6 = \begin{bmatrix} 1 \\ 0 \\ e_{62} \\ e_{63} \end{bmatrix}$$

This means that the first line l_1 is the x-axis of the coordinate frame in FF, that the z-axis of that coordinate system is the common perpendicular of l_1, l_2, and that one of the two planes fixing the third line l_3 is parallel to z and the other one is parallel to x.

Now the three terms, which contain y_i, $i=0,1,2,3$ in the first column of the matrix in Eq. (2), are replaced with the translational components t_i, $i=1,2,3$, according to Eq. (5) and, after carrying out the six transformations with Eq. (1) to get $\mathbf{q}_i$, the products $\mathbf{e}_{1,2}^\top \mathbf{q}_1$, $\mathbf{e}_{3,4}^\top \mathbf{q}_2$, and $\mathbf{e}_{5,6}^\top \mathbf{q}_3$ provide six constraint equations. Notice that the original eight homogeneous Study parameters have been reduced to seven by the replacement of all terms containing y_0, y_1, y_2, and y_3 with t_1, t_2, and t_3 so these six equations are sufficient when the new system is dehomogenized by setting $x_0=1$. The first two, which express $P_1 \in l_1$, yield $t_2=0$ and $t_3=0$. Substituting this result into the rest leaves only t_1, in two of the remaining four equations

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 - 2e_{33}p_{21}(x_0x_2 - x_1x_3) = 0 \tag{39}$$

$$p_{21}[e_{41}(x_0^2 + x_1^2 - x_2^2 - x_3^2) + 2e_{42}(x_0x_3 + x_1x_2)] + e_{41}t_1 = 0 \tag{40}$$

$$(1 + e_{51}p_{31} + e_{52}p_{32})x_0^2 + (1 + e_{51}p_{31} - e_{52}p_{32})x_1^2 + (1 - e_{51}p_{31} \\ + e_{52}p_{32})x_2^2 + (1 - e_{51}p_{31} - e_{52}p_{32})x_3^2 - 2(e_{51}p_{32} - e_{52}p_{31})x_0x_3 \\ + 2(e_{51}p_{32} + e_{52}p_{31})x_1x_2 + e_{51}t_1 = 0 \tag{41}$$

$$(1 + e_{62}p_{32})x_0^2 + (1 - e_{62}p_{32})x_1^2 + (1 + e_{62}p_{32})x_2^2 + (1 - e_{62}p_{32})x_3^2 \\ + 2e_{63}p_{32}(x_0x_1 + x_2x_3) - 2e_{63}p_{31}(x_0x_2 - x_1x_3) + 2e_{62}p_{31}(x_0x_3 \\ + x_1x_2) = 0 \tag{42}$$

Next, t_1 is eliminated from Eqs. (40) and (41) as follows:

$$e_{41}[1 - e_{51}(p_{21} - p_{31}) + e_{52}p_{32}]x_0^2 + e_{41}[1 - e_{51}(p_{21} - p_{31}) \\ - e_{52}p_{32}]x_1^2 + e_{41}[1 + e_{51}(p_{21} - p_{31}) + e_{52}p_{32}]x_2^2 \\ + e_{41}[1 + e_{51}(p_{21} - p_{31}) - e_{52}p_{32}]x_3^2 - 2(e_{41}e_{51}p_{32} - e_{41}e_{52}p_{31} \\ + e_{42}e_{51}p_{21})x_0x_3 + 2(e_{41}e_{51}p_{32} + e_{41}e_{52}p_{31} - e_{42}e_{51}p_{21})x_1x_2 = 0 \tag{43}$$

With Eqs. (39), (43), and (42) we have obtained a system of three homogeneous quadratic equations in x_0, x_1, x_2, and x_3. The coefficients of this system are shown compressed in the following equation to make the final steps easier to follow:

$$x_0^2+x_1^2+x_2^2+x_3^2-2k(x_0x_2-x_1x_3)=0$$

$$a_0x_0^2+a_1x_1^2+a_2x_2^2+a_3x_3^2-2a_4x_0x_3+2a_5x_1x_2=0 \tag{44}$$

$$b_0(x_0^2+x_2^2)+b_1(x_1^2+x_3^2)+2b_2(x_0x_1+x_2x_3)-2b_3(x_0x_2-x_1x_3) \\ +2b_4(x_0x_3+x_1x_2)=0$$

In the following we adapt the parametrization technique, introduced in Sec. 4, to the case at hand. The left hand side of the first of the equations in Eq. (44) can be written as a sum of two products whose factors are linear in x_0, x_1, x_2, and x_3 as follows:

$$x_0^2+x_1^2+x_2^2+x_3^2-2k(x_0x_2-x_1x_3)=[x_0-(k+l)x_2][x_0-(k-l)x_2] \\ +[x_1+(k+l)x_3][x_1+(k-l)x_3]$$

where[4]

$$l=\sqrt{k^2-1}$$

Hence, if we apply the regular projective (coordinate) transformation

$$\begin{aligned} y_0 &= x_0-(k+l)x_2 \\ y_1 &= x_1+(k+l)x_3 \\ y_2 &= x_1+(k-l)x_3 \\ y_3 &= x_0-(k-l)x_2 \end{aligned} \tag{45}$$

the quadric represented by the first of the equations in Eq. (44) becomes a simple bilinear expression

$$y_0y_3+y_1y_2=0$$

whose left hand side is again nulled by parametrization equation (34). Substitution of this parametrization into the inverse transform

$$x_0=\frac{1}{2l}[-(k-l)y_0+(k+l)y_3]$$

$$x_1=\frac{1}{2l}[-(k-l)y_1+(k+l)y_2]$$

$$x_2=\frac{1}{2l}[-y_0+y_3]$$

$$x_3=\frac{1}{2l}[y_1-y_2]$$

of (Eq. (45)) yields

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} k-l+(k+l)uv \\ (l-k)u+(l+k)v \\ 1+uv \\ u-v \end{bmatrix} \tag{46}$$

i.e., a parametrization of the original quadric.[5]

Equations (37) and (38) are applied exactly as before, except for the definition of the quadratic polynomials $\alpha_2(u),\ldots,\beta_0(u)$, described as follows:

$$\begin{aligned} \alpha_2(u) &= [2a_0k(k+l)-a_0+a_2]u^2+2(a_4+a_5)(k+l)u+2a_1k(k+l) \\ &\quad -a_1+a_3 \\ \alpha_1(u) &= -2[a_4(k+l)+a_5(k-l)]u^2+2(a_0-a_1+a_2-a_3)u \\ &\quad +2[a_5(k+l)+a_4(k-l)] \\ \alpha_0(u) &= [2a_1k(k-l)-a_1+a_3]u^2-2(a_4+a_5)(k-l)u+2a_0k(k-l) \\ &\quad -a_0+a_2 \\ \beta_2(u) &= 2(b_0k-b_3)(k+l)u^2-4b_2[1-k(k+l)]u \\ &\quad +2(b_1k-b_3)(k+l) \\ \beta_1(u) &= 4b_4lu^2+4(b_0-b_1)u+4b_4l \\ \beta_0(u) &= 2(b_1k-b_3)(k-l)u^2+4b_2[1-k(k-l)]u \\ &\quad +2(b_0k-b_3)(k-l) \end{aligned} \tag{47}$$

Again the solutions of an octic univariate in u, produced with the determinant of Eq. (37), are back substituted into Eq. (38) and the corresponding v is solved linearly.

5.1 The Spatial Double Triangular Manipulator. A possible mechanical realization of the three-points-on-three-lines paradigm is the so-called spatial double triangular manipulator (spatial DTM) as introduced in Ref. [2].

Figure 5(*a*) shows two frames, each consisting of three skew lines. These are connected by three short $\underline{C}CC$ legs where C is a cylindrical joint. Both DOFs of the ones on FF are actuated. This was the design envisaged by Daniali [2] who carried out no DK analysis. Figure 5(*b*), on the right, shows such a leg. This design, though theoretically feasible, embodies a three-intersecting-line-pairs paradigm, which is fraught with singularities and even 2DOF self-motion as described by Zsombor-Murray and Hyder [12]. Adopting $\underline{C}RRC$ legs as shown on the left of Fig. 5(*b*) solves the problem. The centers of the three unactuated C-joints become the three points in FF upon which the three pairs of planes, which intersect on the three lines in EE, are to be placed. These six planes can be transformed by the procedure outlined above.

The sample solution in Fig. 5(*c*), revealing eight real assembly modes, is an inversion; i.e., the three points in EE, (0,0,0), (5,0,0), and $\left(\frac{5}{2},\frac{5\sqrt{3}}{2},0\right)$, were placed on the respective plane pairs $y=0\cap z=0$, $x=0\cap z=1$, and $x=1\cap y=1$. Thus the octic polynomial (see above) is demonstrated to be minimal.

6 Conclusion

Direct kinematic problems for a wide variety of parallel manipulators have been solved in a unified fashion using *point* kinematic mapping. All cases involved the writing of constraint equations that place a number of points on corresponding surfaces, not always in the same number. However, once one begins to look at problems in this way, the writing of a sufficient set of such equations is made a lot easier. These equation sets were then solved by introducing, or rather resurrecting in a more general engineering context, some not so widely known algebraic techniques, found in Refs. [7,8,11,13–16], and thereby obtaining some new results.

(a) reducing the PoP constraint to a quadric in Study parameters and similarly reducing the PoS constraint by intersecting the original quartic with the Study quadric and confining the transformed point P to Euclidean space[6]
(b) reduction in a partial set of constraints to a one parameter motion trajectory of the last point that is then intersected with the remaining surface
(c) reparametrization to reduce the number of variables and constraint equations

[4]As one can easily check $k^2-1\geq 0$ is equivalent with $\mathrm{dist}(P_1,P_2)\geq \mathrm{dist}(l_1,l_2)$. Clearly a solution to the DK problem exists only if the latter condition holds.

[5]The factor $1/2l$ can be omitted since we deal with homogeneous equations.

[6]Strictly speaking, Husty [7] introduced the technique in his notable DK solution of the general Stewart–Gough platform. Here we have reintroduced the technique in the context of parallel Schönflies robots.

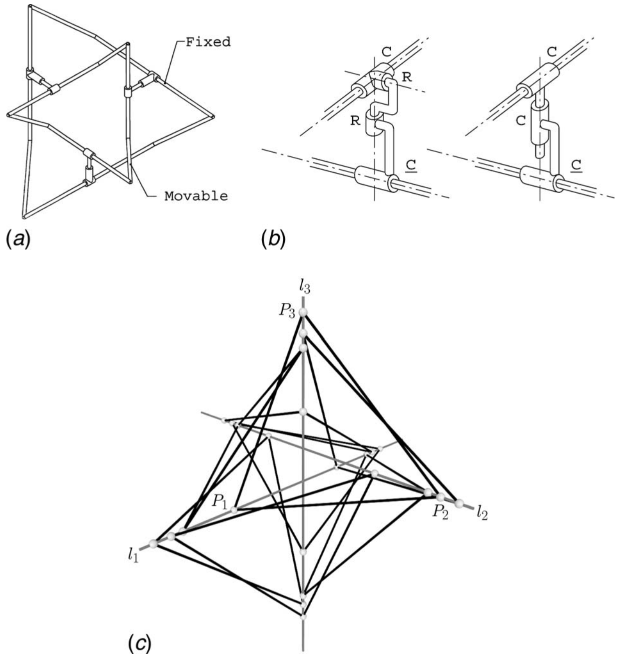

Fig. 5 Spatial double triangular manipulator

(d) neatly extracting an octic univariate from a pair of simultaneous bivariate quartics, like system (36) wherein there are no cubic or quartic variable terms, as a simple 4×4 determinant, Eq. (37), and exposing a linear back substitution, Eq. (38), to obtain corresponding values of the other variable

(e) revealing for the first time an octic univariate polynomial and eight real DK solutions for the spherical DTM

(f) revealing for the first time an octic univariate polynomial and eight real DK solutions for the spatial DTM

Almost all cases examined pertained to manipulators of less than 6DOF though there was one fully mobile example, the spatial DTM, albeit a simplified problem because it was not fully parallel, i.e., had more than one actuated joint per leg.

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References

[1] Daniali, H. R. M., Zsombor-Murray, P. J., and Angeles, J., 1993, "The Kinematics of 3-DOF Planar and Spherical Double-Triangular Manipulators," *Computational Kinematics*, Kluwer, Dordrecht, pp. 153–164.

[2] Daniali, H. R. M., 1995, "Contribution to the Synthesis of Parallel Manipulators," Ph.D. thesis, McGill University, Montreal, QC, Canada.

[3] Nabat, V., Rodriguez, M., Company, O., Pierrot, F., and Dauchez, P., 2005, "Very Fast Schoenflies Motion Generator," IEEE Publication No. 0-7803-9484-4/05, pp. 365–370.

[4] Angeles, J., Caro, S., Khan, W., and Morozov, A., 2008, "Kinetostatic Design of an Innovative Schönflies Motion Generator," J. Mech. Eng. Sci., **220**, pp. 935–943.

[5] Gauthier, J.-F., 2008, "Contributions to the Optimum Design of Schönflies Motion Generators," M.Eng. thesis, McGill University, Montreal, QC, Canada.

[6] Zsombor-Murray, P. J., 2008, "Kinematic Mapping and Kinematics of Some Parallel Manipulators," Proceedings of the Second International Workshop on Fundamental Issues and Future Research Directions for Parallel Mechanisms and Manipulators, Montpellier, France, pp. 41–48.

[7] Husty, M., 1996, "An Algorithm for Solving the Direct Kinematics of General Stewart-Gough Platforms," Mech. Mach. Theory, **31**(4), pp. 365–379.

[8] Bottema, O., and Roth, B., 1990, *Theoretical Kinematics*, Dover, New York.

[9] Zhou, K., Mao, D., and Tao, Z., 2002, "Kinematic Analysis and Application Research on a High-Speed Travelling Double Four-Rod Spatial Parallel Mechanism," Int. J. Adv. Manuf. Technol., **19**(12), pp. 873–878.

[10] Lee, C.-C., and Hervé, J. M., 2009, "On Some Applications of Primitive Schönflies-Motion Generators," Mech. Mach. Theory, **44**, pp. 2153–2163.

[11] Müller, H. R., 1962, *Sphärische Kinematik*, Deutscher Verlag der Wissenschaften, Berlin.

[12] Zsombor-Murray, P. J., and Hyder, A., 1992, "Design, Mobility Analysis and Animation of a Double Equilateral Tetrahedral Mechanism," Robot. Auton. Syst., **9**, pp. 227–236.

[13] Blaschke, W., 1960, *Kinematik und Quaternionen*, Deutscher Verlag der Wissenschaften, Berlin.

[14] Husty, M., Pfurner, M., and Schröcker, H.-P., 2007, "A New and Efficient Algorithm for the Inverse Kinematics of a General Serial 6R Manipulator," Mech. Mach. Theory, **42**(1), pp. 66–81.

[15] Study, E., 1903, *Geometrie der Dynamen*, B.G. Teubner, Leipzig.

[16] Vogler, H., 2008, "Vorwort des Herausgebers—Bemerkungen zu einem Satz vom W. Blaschke und zur Methode von Borel-Bricard," Grazer Mathematische Berichte, **352**, pp. 1–16.