

HEREDITARITY OF CLOSURE OPERATORS AND INJECTIVITY

G. Castellini¹ E. Giuli²

ABSTRACT: A notion of heredity of a closure operator with respect to a class of monomorphisms is introduced. Let C be a regular closure operator induced by a subcategory \mathcal{A} . It is shown that, if every object of \mathcal{A} is a subobject of an \mathcal{A} -object which is injective with respect to a given class of monomorphisms, then the closure operator C is hereditary with respect to that class of monomorphisms.

KEY WORDS: Closure operator, hereditary closure operator, injective object, factorization pair.

CLASSIFICATION: 18A32, 18G05, 18A20.

INTRODUCTION

Let C be a closure operator on a category \mathcal{X} with respect to a class \mathcal{M} of \mathcal{X} -monomorphisms. In this paper we introduce the notion of heredity of C with respect to a subclass \mathcal{M}' of \mathcal{M} . We show that if \mathcal{M}' and \mathcal{M}'' are two subclasses of \mathcal{M} which form a factorization pair for \mathcal{M} (cf. Definition 7) then, the heredity of C with respect to both \mathcal{M}' and \mathcal{M}'' implies the heredity of C with respect to \mathcal{M} .

The main purpose of this paper is to show that heredity of a regular closure operator is strongly related to the notion of injectivity. As a matter of fact, let C be a regular closure operator induced by a subcategory \mathcal{A} and let $\mathcal{M}' \subseteq \mathcal{M}$. If \mathcal{A} satisfies the condition that every object of \mathcal{A} is a subobject of an \mathcal{M}' -injective object of \mathcal{A} , then C is \mathcal{M}' -hereditary. Some examples show that in general if C is \mathcal{M}' -hereditary, \mathcal{A} need not satisfy the above condition.

We conclude the paper with an example which shows that neither heredity nor C -dense heredity is preserved under the construction of idempotent hulls.

We use the terminology of [HS] throughout.

¹ Research partially supported by the Research Office of the Faculty of Arts and Sciences – University of Puerto Rico – Mayagüez Campus.

² The second author acknowledges support from the Italian Ministry of Public Education and from the University of Puerto Rico at Mayagüez. He also would like to thank the Mayagüez Campus for its hospitality while working on this paper.

PRELIMINARIES

Throughout we consider a category \mathcal{X} and a fixed class \mathcal{M} of \mathcal{X} -monomorphisms, which contains all \mathcal{X} -isomorphisms. It is assumed that:

- (1) \mathcal{M} is closed under composition
- (2) Pullbacks of \mathcal{M} -morphisms exist and belong to \mathcal{M} , and multiple pullbacks of (possibly large) families of \mathcal{M} -morphisms with common codomain exist and belong to \mathcal{M} .

In addition, we require \mathcal{X} to have equalizers and \mathcal{M} to contain all regular monomorphisms.

One of the consequences of the above assumptions is that there is a uniquely determined class \mathcal{E} of morphisms in \mathcal{X} such that $(\mathcal{E}, \mathcal{M})$ is a factorization structure on \mathcal{X} , i.e., each morphism f in \mathcal{X} has a factorization $f = m \circ e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, and if $A \xrightarrow{e} B$, $B \xrightarrow{h} D$, $A \xrightarrow{g} C$ and $C \xrightarrow{m} D$ are \mathcal{X} -morphisms with $m \in \mathcal{M}$, and $e \in \mathcal{E}$ such that $m \circ g = h \circ e$, then there exists a unique diagonal, i.e., a morphism $B \xrightarrow{d} C$ such that for each $i \in I$ the both triangles of the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{e} & B \\
 g \downarrow & \swarrow d & \downarrow h \\
 C & \xrightarrow[m]{} & D
 \end{array}$$

commute (cf. [DG₁]).

We regard \mathcal{M} as a full subcategory of the arrow category of \mathcal{X} , with the codomain functor from \mathcal{M} to \mathcal{X} denoted by U . Since U is faithful, \mathcal{M} is concrete over \mathcal{X} .

DEFINITION 1

A *closure operator* on \mathcal{X} (with respect to \mathcal{M}) is a pair $C = (\gamma, F)$, where F is an endofunctor on \mathcal{M} that satisfies $UF = U$, and γ is a natural transformation from $id_{\mathcal{M}}$ to F that satisfies $(id_U)\gamma = id_U$.

Thus, given a closure operator $C = (\gamma, F)$, every member m of \mathcal{M} has a canonical factorization

$$\begin{array}{ccc}
 \bullet & \xrightarrow{]m[_C} & \bullet \\
 m \searrow & & \downarrow [m]_C \\
 & & \bullet
 \end{array}$$

where $[m]_C = F(m)$ is called the C -closure of m , and $]m[_C$ is the domain of the m -component

of γ . The class of all \mathcal{M} -morphisms of the form $]m[_C$ ($[m]_C$) will be denoted by $\Delta(C)$ ($\nabla(C)$). In particular, $]]_C$ induces an order-preserving increasing function on the \mathcal{M} -subobject lattice of every \mathcal{X} -object. Also, these functions are related in the following sense: if p is the pullback of a morphism $m \in \mathcal{M}$ along some \mathcal{X} -morphism f , and q is the pullback of $[m]_C$ along f , then $]p]_C \leq q$. Conversely, every family of functions on the \mathcal{M} -subobject lattices that has the above properties uniquely determines a closure operator.

DEFINITION 2

Given a closure operator C , we say that $m \in \mathcal{M}$ is *C-closed* if $]m[_C$ is an isomorphism. An \mathcal{X} -morphism f is called *C-dense* if for every $(\mathcal{E}, \mathcal{M})$ -factorization (e, m) of f we have that $[m]_C$ is an isomorphism. We call C *idempotent* provided that $]]_C \circ []_C \simeq []_C$, i.e., provided that $[m]_C$ is C -closed for every $m \in \mathcal{M}$. C is called *weakly hereditary* if $]m[_C$ is C -dense for every $m \in \mathcal{M}$.

For more background on closure operators see, e.g., [DG₁], [DG₂], [C], [K] and [DGT].

A special case of an idempotent closure operator arises in the following way. Given any class \mathcal{A} of \mathcal{X} -objects and $M \xrightarrow{m} X$ in \mathcal{M} , define $[m]_{\mathcal{A}}$ to be the intersection of all equalizers of pairs of \mathcal{X} -morphisms r, s from X to some \mathcal{A} -object A that satisfy $r \circ m = s \circ m$, and let $]m[_{\mathcal{A}} \in \mathcal{M}$ be the unique \mathcal{X} -morphism by which m factors through $[m]_{\mathcal{A}}$. It is easy to see that $(]]_{\mathcal{A}}, []_{\mathcal{A}})$ forms an idempotent closure operator. This generalizes the Salbany construction of closure operators induced by classes of topological spaces, cf. [S]. Such a closure operator was called *regular* in [DG₂]. To simplify the notation, instead of “[]_{mathcal{A}}-dense” we usually write “ \mathcal{A} -dense”.

We denote the collection of all closure operators on \mathcal{M} by $\mathbf{CL}(\mathcal{X}, \mathcal{M})$ pre-ordered as follows: $C \sqsubseteq D$ if $]m[_C \leq]m[_D$ for all $m \in \mathcal{M}$ (where \leq is the usual order on subobjects).

DEFINITION 3

An \mathcal{X} -object I is said to be *injective* with respect to the class of \mathcal{X} -morphisms \mathcal{U} (in short *\mathcal{U} -injective*) if for each $X \xrightarrow{m} Y$ in \mathcal{U} and $X \xrightarrow{f} I$, there exists $Y \xrightarrow{g} I$ such that $g \circ m = f$. Then g is called an *extension of f along m* . $\text{Inj}(\mathcal{U})$ will denote the class of all \mathcal{U} -injective \mathcal{X} -objects.

Let C be a closure operator on \mathcal{X} and let $\mathcal{M}' \subseteq \mathcal{M}$. If \mathcal{M}' is the class of all C -dense \mathcal{M} -morphisms (C -closed \mathcal{M} -morphisms), then the class of all \mathcal{M}' -injective \mathcal{X} -objects will be denoted by $\text{Inj}_d(C)$ ($\text{Inj}_c(C)$).

MAIN RESULTS

In what follows \hat{C} (\check{C}) will denote the idempotent hull (weakly hereditary core) of the closure operator C (cf. [DG₂]).

PROPOSITION 4

- (a) $Inj_c(C) = Inj_c(\hat{C})$
- (b) If \mathcal{X} is \mathcal{M} -well powered and C is weakly hereditary then $Inj_d(C) = Inj_d(\hat{C})$.
- (c) $Inj_d(C) = Inj_d(\check{C})$

Proof:

(a). It follows from the fact that an \mathcal{M} -subobject is C -closed iff it is \hat{C} -closed.

(b). Since C -dense always implies \hat{C} -dense, we have that $Inj_d(\hat{C}) \subseteq Inj_d(C)$. Now, let $Z \in Inj_d(C)$ and let $M \xrightarrow{m} X$ be a \hat{C} -dense \mathcal{M} -subobject. Since C is weakly hereditary, $]m[_C^X$ is C -dense. Consequently, for any \mathcal{X} -morphism $M \xrightarrow{f} Z$ there exists an \mathcal{X} -morphism $]m[_C^X \xrightarrow{g} Z$ such that $g \circ]m[_C^X = f$. Since \mathcal{X} is \mathcal{M} -well powered, using transfinite induction we obtain that there exists an \mathcal{X} -morphism $]m[_C^X \xrightarrow{h} Z$ such that $h \circ]m[_C^X = f$. Since m is \hat{C} -dense, $]m[_C^X$ is an isomorphism and $k = h \circ ([m]_C^X)^{-1}$ is an extension of f along m . Therefore $Z \in Inj_d(\hat{C})$ (cf. [DG₂] with $\hat{C} = C^\infty$).

(c). It follows from the fact that an \mathcal{M} -subobject is C -dense iff it is \check{C} -dense. □

The question of whether item (b) of the above proposition might hold without C being weakly hereditary and without \mathcal{X} being \mathcal{M} -well powered, remains open.

Since C -closed always implies \check{C} -closed, $Inj_c(\check{C}) \subseteq Inj_c(C)$.

DEFINITION 5

Let $\mathcal{M}' \subseteq \mathcal{M}$ and let C be a closure operator on \mathcal{X} with respect to \mathcal{M} . C is called \mathcal{M}' -hereditary if given two \mathcal{M} -subobjects of X , (M, m) and (N, n) , with $(M, m) \leq (N, n)$ and $(N, n) \in \mathcal{M}'$, we have that $]M[_C^X \cap N \simeq]M[_C^N$.

Three particularly important cases are $(C$ -dense)-hereditary, $(C$ -closed)-hereditary and hereditary that occur exactly when \mathcal{M}' equals the class of C -dense \mathcal{M} -subobjects, the class of C -closed \mathcal{M} -subobjects and all of \mathcal{M} , respectively.

Notice that $]M[_C^X \cap N$ is isomorphic to the pullback of $(]M[_C^X,]m[_C^X)$ along n .

LEMMA 6 ([DG₂])

An idempotent closure operator C is weakly hereditary iff it is C -closed-hereditary. □

DEFINITION 7

Let \mathcal{M}' and \mathcal{M}'' be two subclasses of \mathcal{M} . We say that \mathcal{M} factors through the pair $(\mathcal{M}', \mathcal{M}'')$ iff every $m \in \mathcal{M}$ can be written as $m = m'' \circ m'$ with $m' \in \mathcal{M}'$ and $m'' \in \mathcal{M}''$. $(\mathcal{M}', \mathcal{M}'')$ will be called a *factorization pair* for \mathcal{M} .

PROPOSITION 8

Let C be a closure operator on \mathcal{X} and let $(\mathcal{M}', \mathcal{M}'')$ be a factorization pair for \mathcal{M} . Then, C is hereditary iff C is \mathcal{M}' -hereditary and \mathcal{M}'' -hereditary.

Proof:

(\Rightarrow). It is obvious.

(\Leftarrow). Let us consider the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{m} & X \\ t \downarrow & \nearrow n & \uparrow n'' \\ N & \xrightarrow[n']{} & N' \end{array}$$

with $n' \in \mathcal{M}'$ and $n'' \in \mathcal{M}''$. From the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{n' \circ t} & N' \\ t \downarrow & \nearrow n' & \\ N & & \end{array}$$

and the fact that C is \mathcal{M}' -hereditary, we obtain that $[M]_c^N \simeq N \cap [M]_c^{N'}$. From the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{m} & X \\ n' \circ t \downarrow & \nearrow n'' & \\ N' & & \end{array}$$

and the fact that C is \mathcal{M}'' -hereditary, we obtain that $[M]_c^{N'} \simeq N' \cap [M]_c^X$. Therefore, $[M]_c^N \simeq N \cap [M]_c^{N'} \simeq N \cap N' \cap [M]_c^X \simeq N \cap [M]_c^X$. Therefore C is hereditary. \square

COROLLARY 9

- (a) Let C be a closure operator on \mathcal{X} . C is hereditary iff it is $\Delta(C)$ -hereditary and $\nabla(C)$ -hereditary
- (b) Let C be an idempotent closure operator on \mathcal{X} . C is hereditary iff it is $(C$ -dense)-hereditary and $(C$ -closed)-hereditary.

Proof:

- (a). Clearly because $(\Delta(C), \nabla(C))$ always forms a factorization pair for \mathcal{M} .

(b). It follows immediately from the fact that if C is idempotent and (C -closed)-hereditary, then (C -dense \mathcal{M} -morphisms, C -closed \mathcal{M} -morphisms) forms a factorization pair for \mathcal{M} (cf. Lemma 6 and [DG₂]). \square

PROPOSITION 10

Let $(\mathcal{M}', \mathcal{M}'')$ be a factorization pair for \mathcal{M} . Then we have: $Inj(\mathcal{M}') \cap Inj(\mathcal{M}'') = Inj(\mathcal{M})$.

Proof:

We need to prove only one inclusion. Let $X \xrightarrow{m} Y$ be a morphism in \mathcal{M} and let $X \xrightarrow{f} I$ be an \mathcal{X} -morphism with $I \in Inj(\mathcal{M}') \cap Inj(\mathcal{M}'')$. By hypothesis, $m = m'' \circ m'$ with $m' \in \mathcal{M}'$ and $m'' \in \mathcal{M}''$. So, there exists an \mathcal{X} -morphism g such that $g \circ m' = f$ as well as an \mathcal{X} -morphism h such that $h \circ m'' = g$. Therefore $h \circ m = h \circ m'' \circ m' = g \circ m' = f$. Thus, $I \in Inj(\mathcal{M})$. \square

COROLLARY 11

(a) Let C be a closure operator on \mathcal{X} . Then $Inj(\Delta(C)) \cap Inj(\nabla(C)) = Inj(\mathcal{M})$

(b) Let C be a weakly hereditary and idempotent closure operator on \mathcal{X} . Then $Inj_d(C) \cap Inj_c(C) = Inj(\mathcal{M})$.

Proof:

(a). Just notice that $(\Delta(C), \nabla(C))$ always forms a factorization pair for \mathcal{M} .

(b). If C is weakly hereditary and idempotent, then (C -dense \mathcal{M} -morphisms, C -closed \mathcal{M} -morphisms) forms a factorization pair for \mathcal{M} (cf. [DG₂]). \square

For the next few results we assume the additional condition that \mathcal{X} is a regular well-powered category with products.

The following result is well known.

LEMMA 12

Let $\mathcal{M}' \subseteq \mathcal{M}$. $Inj(\mathcal{M}')$ is closed under products. \square

THEOREM 13

Let \mathcal{A} be a class of \mathcal{X} -objects and let $\mathcal{M}' \subseteq \mathcal{M}$. Suppose that for each $A \in \mathcal{A}$, there is an \mathcal{X} -monomorphism $A \xrightarrow{k} A'$ with $A' \in \mathcal{A}$ being \mathcal{M}' -injective. Then the \mathcal{A} -closure is \mathcal{M}' -hereditary.

Proof:

Let $\Pi(Inj(\mathcal{M}') \cap \mathcal{A})$ denote the family of all possible products of the objects of $Inj(\mathcal{M}') \cap \mathcal{A}$. Let $M \xrightarrow{m} X$ be an \mathcal{M} -subobject of X and let $X \xrightarrow[f]{g} A$ be two \mathcal{X} -morphisms with $A \in \mathcal{A}$ and $f \circ m = g \circ m$. If $A \xrightarrow{k} A'$ is an \mathcal{X} -monomorphism with $A' \in Inj(\mathcal{M}') \cap \mathcal{A}$, then it is easy to

see that $equ(f, g) \simeq equ(k \circ f, k \circ g)$. Therefore, the \mathcal{A} -closure agrees with the regular closure operator induced by the family $Inj(\mathcal{M}') \cap \mathcal{A}$ as well as with the one induced by $\Pi(Inj(\mathcal{M}') \cap \mathcal{A})$ (cf. [C, Proposition 1.4] and [G]).

Let us consider the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{m} & X \\ p \downarrow & \searrow t & \uparrow n \\ [M]_{\mathcal{A}}^N & \xrightarrow{[t]_{\mathcal{A}}^N} & N \end{array}$$

with $m \in \mathcal{M}$ and $n \in \mathcal{M}'$. Consider two morphisms r and s with domain N and codomain in $\Pi(Inj(\mathcal{M}') \cap \mathcal{A})$, such that $[t]_{\mathcal{A}}^N = equ(r, s)$ (cf. [C, Proposition 1.6]). Since every $Y \in \Pi(Inj(\mathcal{M}') \cap \mathcal{A})$ is \mathcal{M}' -injective (cf. Lemma 12), we get that there exist two morphisms h and k such that $h \circ n = r$ and $k \circ n = s$. Now, $r \circ t = s \circ t$ implies that $h \circ m = h \circ n \circ t = r \circ t = s \circ t = k \circ n \circ t = k \circ m$. Therefore $h \circ [m]_{\mathcal{A}}^X = k \circ [m]_{\mathcal{A}}^X$.

Let us consider the diagram

$$\begin{array}{ccc} [M]_{\mathcal{A}}^X \cap N & \xrightarrow{\beta} & [M]_{\mathcal{A}}^X \\ \alpha \downarrow & \searrow \alpha & \mu \nearrow \\ N & \xleftarrow{[t]_{\mathcal{A}}^N} [M]_{\mathcal{A}}^N & \searrow m \circ [t]_{\mathcal{A}}^N \\ \downarrow id & & \downarrow [m]_{\mathcal{A}}^X \\ N & \xrightarrow{n} & X \end{array}$$

$h \circ [m]_{\mathcal{A}}^X = k \circ [m]_{\mathcal{A}}^X$ implies that $h \circ [m]_{\mathcal{A}}^X \circ \beta = k \circ [m]_{\mathcal{A}}^X \circ \beta$. From $[m]_{\mathcal{A}}^X \circ \beta = n \circ \alpha$, we get that $r \circ \alpha = h \circ n \circ \alpha = h \circ [m]_{\mathcal{A}}^X \circ \beta = k \circ [m]_{\mathcal{A}}^X \circ \beta = k \circ n \circ \alpha = s \circ \alpha$. Since $[t]_{\mathcal{A}}^N = equ(r, s)$, there exists a morphism $[M]_{\mathcal{A}}^X \cap N \xrightarrow{\gamma} [M]_{\mathcal{A}}^N$ such that $[t]_{\mathcal{A}}^N \circ \gamma = \alpha$. $[M]_{\mathcal{A}}^N$ is an \mathcal{M} -subobject of N and by functoriality of $[\]_{\mathcal{A}}$, it is also an \mathcal{M} -subobject of $[M]_{\mathcal{A}}^X$. So, there exists a morphism $[M]_{\mathcal{A}}^N \xrightarrow{c} [M]_{\mathcal{A}}^X \cap N$ such that $\alpha \circ c = [t]_{\mathcal{A}}^N$. Now $\alpha \circ c \circ \gamma = \alpha$ implies that $c \circ \gamma = id$, since α is a monomorphism. Thus, c is an isomorphism, since it is a monomorphism and a retraction. \square

COROLLARY 14

- (a) If \mathcal{A} has enough \mathcal{M}' -injectives, (i.e., for every $A \in \mathcal{A}$, there is a monomorphism $A \xrightarrow{k} A'$ with $k \in \mathcal{M}'$ and with $A' \in \mathcal{A}$ being \mathcal{M}' -injective), then the \mathcal{A} -closure is \mathcal{M}' -hereditary.
- (b) If \mathcal{A} is epireflective in \mathcal{X} and admits a system of \mathcal{M}' -injective cogenerators, then the \mathcal{A} -closure is \mathcal{M}' -hereditary.

Proof:

- (a). We just observe that $\mathcal{M}' \subseteq \mathcal{M} \subseteq \mathcal{X}$ -monomorphisms.
- (b). It just follows from the fact that every $A \in \mathcal{A}$ is an extremal subobject of a product of \mathcal{M}' -injective objects of \mathcal{A} . \square

Notice that Lemma 6, Corollary 9 and the above corollary yield the following interesting special cases.

- (a) Hereditariness of the \mathcal{A} -closure is implied by \mathcal{A} having enough \mathcal{M} -injectives.
- (b) Weakly hereditariness of the \mathcal{A} -closure ($=$ (\mathcal{A} -closed)-hereditariness) is implied by \mathcal{A} having enough (\mathcal{A} -closed)-injectives.
- (c) (\mathcal{A} -dense)-hereditariness of the \mathcal{A} -closure is implied by \mathcal{A} having enough (\mathcal{A} -dense)-injectives.
- (d) The \mathcal{A} -closure is hereditary iff \mathcal{A} has enough (\mathcal{A} -closed)-injectives and enough (\mathcal{A} -dense)-injectives.

Example 17 and 18 below show that the implications in items (a)–(c) cannot be reversed in general. Example 19 provides a case in which item (a) becomes a characterization.

REMARK 15

For any idempotent closure operator C , its weakly hereditary core \check{C} is hereditary iff C is C -dense hereditary. As a matter of fact, since every closure operator C and its weakly hereditary core, \check{C} , determine the same dense morphisms (i.e., C -dense $=$ \check{C} -dense), if C is C -dense hereditary, so is \check{C} and if C is idempotent, so is \check{C} (cf. [DG₂, Theorem 4.2(3)]). Therefore from Corollary 9 and Lemma 6, we get that \check{C} is hereditary iff C is C -dense hereditary.

In all of the following examples \mathcal{M} will be the class of embeddings.

EXAMPLE 16

If $\mathcal{X} = \mathbf{TOP}$ and $\mathcal{A} = \mathbf{TOP}_0$, then the Sierpinski space S , which is a cogenerator for \mathbf{TOP}_0 is trivially injective. Thus the \mathbf{TOP}_0 -closure (b -closure, [Sk]) is a hereditary operator.

EXAMPLE 17

(a) Let $\mathcal{X} = \mathcal{A}$ be any epireflective non bireflective subcategory of \mathbf{TOP} different from \mathbf{TOP}_0 and from \mathbf{Sgl} (spaces with at most one point). Then, $\mathcal{A} \subseteq \mathbf{TOP}_1$ (cf. [G]) and the injective objects with respect to embeddings are the spaces with exactly one point. As a matter of fact, by assumption \mathcal{A} contains a discrete two-point space, so it also contains any 0-dimensional Hausdorff space. In particular it contains the one-point compactification of the discrete space of natural numbers, \mathcal{N}_∞ . Now, suppose that $I \in \mathcal{A}$ has at least two points, say $I = \{0, 1\}$ and let

$\mathcal{N} \xrightarrow{f} I$ be the continuous map defined by $f(n) = 0$ for n odd and $f(n) = 1$ for n even. Now, if we take the embedding $\mathcal{N} \xrightarrow{e} \mathcal{N}_\infty$, there is no extension of f along e .

(b) If \mathcal{A} is one of the categories **Haus**, **Tych** or **0-Dim**, the morphism f of item (a) is \mathcal{A} -dense (= dense cf. [DG₁]). So, in these cases, the injective objects with respect to the dense embeddings are the spaces with exactly one point.

EXAMPLE 18

For $\mathcal{A} = \mathbf{Tych}$, the cogenerator $[0, 1]$ is not closed injective. In fact, if X is a Tychonoff not normal space, we know from Tietze's Theorem that there exist a closed subset F of X and a continuous function $F \xrightarrow{f} [0, 1]$ that cannot be extended to all of X . Since every cogenerator of **Tych** must contain a copy of the unit interval $[0, 1]$, it is easy to conclude that **Tych** does not have a **Tych**-closed-injective cogenerator. This proves that the implications in Corollary 14 cannot be reversed in general. As a matter of fact if $\mathcal{A} = \mathbf{Tych}$, then the \mathcal{A} -closure in **Tych** is the ordinary closure (cf. [DG₁]), which is hereditary.

EXAMPLE 19

For a fixed ring R with unity, let \mathcal{X} be the category $R\mathbf{Mod}$ of left R -modules, let \mathcal{M} be the class of monomorphisms in $R\mathbf{Mod}$ and let $(\mathcal{T}, \mathcal{F})$ be a torsion theory. $(\mathcal{T}, \mathcal{F})$ is hereditary iff \mathcal{F} is simply cogenerated by an injective module (cf. [DG₃] and [L]). Thus $[\]_{\mathcal{F}}$ is hereditary iff \mathcal{F} is simply cogenerated by an injective object. This shows that in the category $R\mathbf{Mod}$, item (a) of Corollary 14 can be reversed.

Neither hereditariness nor dense-hereditariness is preserved under the construction of idempotent hulls, as the following example shows.

EXAMPLE 20

Let us consider the sets: $M = \{(m, n) : m, n \in \mathcal{N}\}$, $X = M \cup \{\infty_1, \infty_2, \dots\} \cup \{\infty\}$ and $N = M \cup \{\infty\}$. We consider in X the pretopological structure in which every point of the form (m, n) is isolated, a basic nbhd of ∞_i is of the form $\{(i, m) : \bar{m} \leq m \text{ for some } \bar{m} \in \mathcal{N}\} \cup \{\infty_i\}$ and a basic nbhd of ∞ is of the form $\{\infty_j, \infty_{j+1}, \dots\} \cup \{\infty\}$ for some $j \in \mathcal{N}$. Let \hat{K} be the idempotent hull of the closure operator K induced by the pretopology in **PrTOP** (cf. [DG₄]). Clearly $\hat{K}_X(N) = X$, i.e., N is \hat{K} -dense. Now, $\hat{K}_X(M) = X$, so $\hat{K}_X(M) \cap N = N$, but $\hat{K}_N(M) = M$, since N is discrete as a pretopological subspace. Thus \hat{K} is \hat{K} -closed-hereditary but not \hat{K} -dense-hereditary and therefore is not hereditary, although K is.

REFERENCES

- [C] G. Castellini, "Closure operators, monomorphisms and epimorphisms in categories of groups," *Cahiers Topologie Geom. Differentielle Categoriqes* **27(2)** (1986), 151-167.
- [CS] G. Castellini, G. E. Strecker, "Global closure operators vs. subcategories," *Quaestiones Mathematicae* **13** (1990), 417-424.
- [DG₁] D. Dikranjan, E. Giuli, "Closure operators induced by topological epireflections," *Coll. Math. Soc. J. Bolyai* **41** (1983), 233-246.
- [DG₂] D. Dikranjan, E. Giuli, "Closure operators I," *Topology Appl.* **27** (1987), 129-143.
- [DG₃] D. Dikranjan, E. Giuli, "Factorizations, injectivity and compactness in categories of modules," *Comm. Algebra* **19(1)** (1991), 45-83.
- [DG₄] D. Dikranjan, E. Giuli, "C-perfect morphisms and C-compactness," preprint.
- [DGT] D. Dikranjan, E. Giuli, W. Tholen, "Closure operators II," Proceedings of the Conference in Categorical Topology (Prague, 1988) World Scientific (1989), 297-335.
- [G] E. Giuli, "Bases of topological epireflections," *Topology Appl.* **11** (1980), 265-273.
- [HS] H. Herrlich, G. E. Strecker, *Category Theory*, 2nd ed., Helderman Verlag, Berlin, 1979.
- [HSS] H. Herrlich, G. Salicrup, G. E. Strecker, "Factorizations, denseness, separation, and relatively compact objects," *Topology Appl.* **27** (1987), 157-169.
- [K] J. Koslowski, "Closure operators with prescribed properties," *Category Theory and its Applications* (Louvain-la-Neuve, 1987) Springer L.N.M. **1248** (1988), 208-220.
- [L] J. Lambek, *Torsion theories, additive semantics and rings of quotients*, Springer L.N.M. **177**, 1971.
- [S] S. Salbany, "Reflective subcategories and closure operators," *Proceedings of the Conference in Categorical Topology* (Mannheim, 1975), Springer L.N.M. **540** (1976), 548-565.
- [Sk] L. Skula, "On a reflective subcategory of the category of all topological spaces," *Trans. Amer. Mat. Soc.* **142** (1969), 37-41.

Gabriele Castellini
Department of Mathematics
University of Puerto Rico
P.O. Box 5000
Mayagüez, PR 00709-5000
U.S.A.

Eraldo Giuli
Dipartimento di Matematica
Università di L'Aquila
67100 L'Aquila
ITALY