Journal of Mechanisms, Transmissions, and Automation in Design

Technical Briefs

A Generalization and Correction of the Welded Beam Optimal Design Problem Using Symbolic Computation

A. S. Almgren[1] and A. M. Agogino[1]

Nomenclature

(English units are used in order to facilitate comparison with previously published results.)

- a = bar cross-sectional area, in.^{2}
- b = bar width, in.
- f_i = constraints, $i = 1,6$
- f_0 = original objective function, \$'s
- F_0 = normalized parametric objective function, cu in.
- F = applied force, lbf
- h = weld height, in.
- K = cost ratio of weld (material and labor) to bar material costs
- l = weld length along the bar, in.
- L = bar length, taken to be 14 in.
- t = bar height, in.
- μ_i = Lagrange multipliers, $i = 1,6$

Introduction

A well-known weldment design optimization problem was formulated and solved by Wilde [1] in the September, 1986 issue of this journal under the title "A Maximal Activity Principle for Eliminating Overconstrained Optimization Cases." The Wilde paper describes a rigorous methodology for solving the weldment design problem by means of monotonicity analysis, the maximal activity principle, and constraint dominance. This Technical Note contributes two significant addenda to the solution of the weldment problem. First, it introduces a design methodology based entirely on automated symbolic computation and, in verifying the conclusions of Wilde, a minor algebraic mistake in the published solution has been detected. Second, the problem is resolved using a general parameter rather than a numerical value for the ratio of weld manufacturing costs to bar material costs, with relatively little additional effort. Three parametric design cases corresponding to an exhaustive range of cost ratios are generated rather than a single numerical solution to a particular numerical formulation.

The original problem was posed as follows: Minimize the cost of a weldment, with design variables: bar cross-sectional area a, bar height t, weld height h and weld length l. The bar area is defined as $a = bt$. Because the weldment represents one step in a mass production process, a small reduction in the cost of one component will have a significant impact on the total production costs, thus providing the motivation for a thorough analysis.

Using the numerical values in the original Ragsdell and Phillips formulation [2], the length L is 14 in. and the cost of manufacturing (weld material and labor) and the cost of the bar stock are \$1.1047/cu in. and \$0.04811/cu in., respectively, giving the cost function per weldment to be minimized over h, l, a, and t as

$$f_0 = 1.1047h^2l + 0.6735a + 0.04811al$$

The constraints governing this design problem are as follows:

$f_1 = h\,l - 1.5211 \geq 0$	(weld shear)
$f_2 = a\,t - 16.8 \geq 0$	(bar stress)
$f_3 = a - ht \geq 0$	(weld width)
$f_4 = at^2 - 9.08 \geq 0$	(deflection)
$f_5 = 1 - 0.02776t - 0.09428t^2/a^3 \geq 0$	(buckling)
$f_6 = h - 0.125 \geq 0$	(weld height)

SYMON and SYMFUNE: Optimization Programs Using Symbolic Computation

The rules of monotonicity analysis used by Wilde have been automated by Choy and Agogino [3] in the SYMON (SYmbolic MONotonicity analyzer) program, and extended by Agogino and Almgren [4, 5] in the SYMFUNE (SYMbolic FUNctional Evaluator, pronounced "symphony") program. SYMON and SYMFUNE are written in Vaxima/MACSYMA [6, 7], a symbolic math language written in FranzLISP [8]. They run on DEC Vax minicomputers under the Unix™ 4.3 BSD operating system. SYMON verified the constraint activity identification performed by Wilde, and SYMFUNE, in verifying the selection of the correct case for the numerically specific design, detected an algebraic mistake in the solution published by Wilde. This error did not invalidate Wilde's major conclusions, but led to incorrect values for some of the Lagrange multipliers in the optimal solution.

One major advantage in using symbolic computation, rather than numerical optimization, is that solutions can be obtained in parametric form. To illustrate its strength, consider a generalization of the weldment design problem, where the ratio of the cost of manufacturing (weld material and labor) to the cost of bar stock material is left as an unknown parameter in the objective function. Instead of a single optimal design, three potentially feasible and optimal design cases are found, the best design depending on the value of the

[1]Department of Mechanical Engineering, University of California at Berkeley, Berkeley, CA 94720.

Contributed by the Design Automation Committee for publication in the Journal of Mechanisms, Transmissions, and Automation in Design. Manuscript received at ASME Headquarters, August 10, 1988.

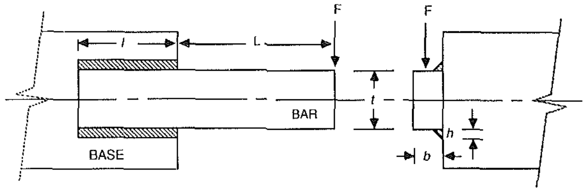

Fig. 1 Welded beam assembly

cost ratio parameter. SYMON uses symbolic computation to apply the principles of monotonicity analysis and the maximal activity principle in the same manner as Wilde and generates a list of cases, each identified by the combination of inactive constraints, which may yield potential solutions. These cases are then transferred to SYMFUNE, where they are evaluated for well-boundedness, feasibility and optimality. Constraint-bound cases are considered well-bounded, and cases with positive degrees of freedom (d.o.f.) are evaluated by substituting the solution into the objective function, finding the minimum value of the objective function with respect to the remaining d.o.f., and checking whether the minimum occurs at finite nonzero values of the design variables. Cases with detectable hidden monotonicities are eliminated at this point. The feasibility conditions require that the solution satisfies the remaining inactive inequalities, and the optimality conditions require that the Lagrange multipliers, computed in each case with known active constraints considered as equalities, be nonnegative. The set of inequalities comprised of the feasibility and optimality conditions is referred to as the *domain of optimality*. In the generalization of the welded beam optimization problem, we introduce the cost ratio parameter "K." The objective function normalized by the bar material cost then becomes

$$F_0(=f_0/0.0481)=Kh^2l+14a+al$$

where 14 in. is the value of the bar length used in the original Ragsdell and Phillips [2] formulation. The constraints remain unchanged. Since the monotonicities of the objective function are unchanged by the substitution of the parameter for the numerical value, the results of SYMON are identical to those obtained from a purely numerical formulation. SYMON generates 12 cases: four 1 d.o.f. cases ([2,3,6], [3,4,6], [4,5,6] and [2,5,6]) and eight zero d.o.f. cases ([5,6], [4,6], [4,5], [3,6], [3,4], [2,6], [2,5] and [2,3]). (Note the cases are identified by the subscript index of the inactive constraints.) The SYMON summary table for this problem is provided in Appendix A. Note that the weld shear constraint is active for *all* feasible cases generated by SYMON, indicating that constraint f_1 is unconditionally active for this problem. In the purely numerical problem (as presented by Wilde) SYMFUNE finds all cases to be bounded, but rejects cases [2,6], [2,5], [5,6], [3,4], and [2,3] as not optimal, and cases [4,5], [2,3,6], [3,4,6], [4,5,6], and [2,5,6] as potentially optimal but not feasible. Case [3,6] has an overconstrained subset of equations and is therefore not considered. The best design for the given value of the cost ratio, case [4,6], is the constraint-bound design with constraints f_4 and f_6 inactive and all other constraints set to strict equality. This case is determined by SYMFUNE to be the only one which is both feasible and optimal.

When the cost ratio is expressed by the single parameter K in the objective function, SYMFUNE rejects case [5,6] as not optimal, and cases [4,5], [2,6], [2,5], and [2,3] as potentially optimal but not feasible. SYMFUNE finds that case [4,6], which before was the sole solution, is now feasible for all values of K, but optimal if and only if $K \leq 34$. Case [3,4], previously rejected as not optimal for the numerical example, is feasible for all values of K but now optimal if and only if $K \geq 130$. Case [3,4,6] is again optimal, and now feasible if and only if

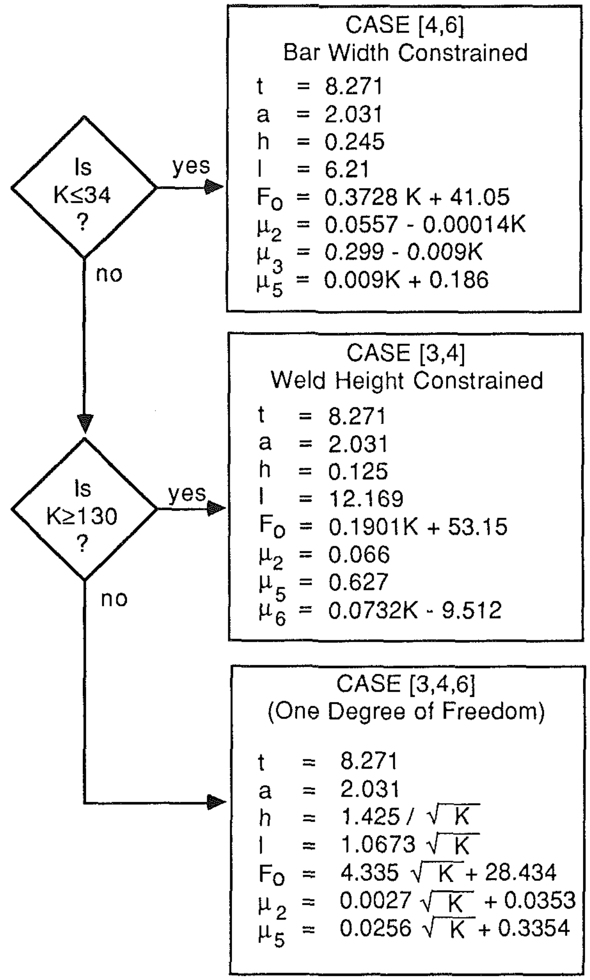

Fig. 2 Parametric design chart; all cases are constrained by weld shear, bar stress and buckling

$34 \leq K \leq 130$. SYMFUNE was not run on the remaining cases, because VAXIMA was unable to perform the calculations required to find the feasibility and optimality conditions (due to an overdetermined set of equations in case [3,6] and the complexity of the calculations involved in the others). However, since feasible optimal designs had been found for which the domains of optimality span all possible values of the parameter, and domains of optimality, by the manner in which they are constructed, must be mutually exclusive for convex problems, the remaining cases cannot generate solutions.

We now have, instead of a single design for a specific value of the cost ratio, a flow chart (Fig. 2) for any value of the cost ratio. If $K \leq 34$ then the best design is case [4,6]; if $34 \leq K \leq 130$ then the best design is case [3,4,6]; and if $K \geq 130$ then the best design is case [3,4]. *All three cases are limited by buckling, bar stress and weld shear.* Thus constraints f_1, f_2, and f_5 are active for all values of K, directly defining the optimal values of t and a. Note that in Wilde's formulation K is approximately 23, which from this chart would imply case [4,6], as Wilde found. A sample session of the SYMFUNE run is included in Appendix B. (Note: VAXIMA designates free variables with a "%" sign, e.g., %$r1$ in the first equation of Appendix B.)

Conclusion

In summary, this Technical Brief has introduced two significant addenda to the solution of the weldment constrained optimization problem. First, the problem is solved entirely by symbolic computation and, second, the problem is generalized to allow for any value of the cost ratio for use in parametric or multiobjective design [9]. In addition, an error in the published solution was detected and corrected, although the major conclusions reported were verified. The same techniques described can be further used to parameterize other critical variables in the design, such as the stress-concentration ratio or minimum weld height.

Acknowledgments

The SYMON/SYMFUNE programs were developed in part by funds from the National Science Foundation under Grant DMC-84521622. The authors are indebted to D. J. Wilde for many fruitful discussions on this problem.

References

1 Wilde, D. J., 1986, "A Maximal Activity Principle for Eliminating Overconstrained Optimization Cases," ASME Journal of Mechanisms, Transmissions, and Automation in Design, Vol. 108, No. 3, pp. 312-314.

2 Ragsdell, K. M., and Philips, D. T., 1976, "Optimal Design of a Class of Welded Structures Using Geometric Programming," ASME *Journal of Engineering for Industry*, pp. 1021-1025.

3 Choy, J. K., and Agogino, A. M., 1986, "SYMON: Automated SYmbolic MONotonicity Analysis System for Qualitative Design Optimization," in *Computers in Engineering 1986*, Vol. 1, (*Proceedings of the 1986 ASME International Computers in Engineering Conference and Exhibition*) pp. 207-212.

4 Agogino, A. M., and Almgren, A. S., 1987, "Symbolic Computation in Computer-Aided Optimal Design," *Expert Systems in Computer-Aided Design*, (*Proceedings of the IFIP Working Conference*) North-Holland, Amsterdam, pp. 267-284.

5 Agogino, A. M., and Almgren, A. S., 1987, "Techniques for Integrating Qualitative Reasoning and Symbolic Computation in Engineering Optimization," *Engineering Optimization*, Vol. 12, No. 2, Sept./Oct., pp. 117-135.

6 The Mathlab Group, 1982, *MACSYMA Reference Manual*, Laboratory for Computer Sciences, Massachusetts Institute of Technology, Cambridge, MA.

7 Rand, R. H., 1984, *Computer Algebra in Applied Mathematics: An Introduction to MACSYMA*, Pitman Advanced Publishing Program, Boston, MA.

8 Wilensky, R., 1984, *LISPcraft*, W. W. Norton & Company, New York, NY.

9 Michelena, N., and Agogino, A. M., 1988, "Multiobject Hydraulic Cylinder Design," ASME Journal of Mechanisms, Transmissions, and Automation in Design, Vol. 110, pp. 81-87.

APPENDIX A

SYMON Output

```
SYMON ANALYSIS SUMMARY :
************************************************************
*   Combinations of Assumed inactives which have been tried:
*   [5, 6]
*   [4, 6]
*   [4, 5, 6]  Result Obtained (A)
*   [4, 5]
*   [3, 6]
*   [3, 5]  Degenerate (B)
*   [3, 4, 6]  Result Obtained (A)
*   [3, 4]
*   [2, 6]
*   [2, 5, 6]  Result Obtained (A)
*   [2, 5]
*   [2, 4]  Degenerate (B)
*   [2, 3, 6]  Result Obtained (A)
*   [2, 3]
*   [6]
*   [5]
*   [4]
*   [3]
*   [2]
*   [1]  Degenerate (B)
*
*   Note(A) Combinations of constraints assumed inactive for which
*       results were obtained:
*   Note(B) Combinations of constraints assumed inactive which
*       led to unbounded or degenerate cases
************************************************************
```

```
* Table of combinations of active constraints which yield results:
*
* case  inactive constraints          active constraints        d.o.f
* ------ ---------------------------  ------------------------  --------
*   1        [4, 5, 6]                    [1, 2, 3]               1
*   2        [3, 4, 6]                    [1, 2, 5]               1
*   3        [2, 5, 6]                    [1, 3, 4]               1
*   4        [2, 3, 6]                    [1, 4, 5]               1
************************************************************
* Table of other combinations of active constraints:
* Subcases of Cases with 1 or more degrees of freedom
*
* case  inactive constraints          active constraints        d.o.f
* ------ ---------------------------  ------------------------  --------
*   5        [5, 6]                       [1, 2, 3, 4]            0
*   6        [4, 6]                       [1, 2, 3, 5]            0
*   7        [4, 5]                       [1, 2, 3, 6]            0
*   9        [3, 6]                       [1, 2, 4, 5]            0
*   10       [3, 4]                       [1, 2, 5, 6]            0
*   12       [2, 6]                       [1, 3, 4, 5]            0
*   13       [2, 5]                       [1, 3, 4, 6]            0
*   16       [2, 3]                       [1, 4, 5, 6]            0
************************************************************
```

APPENDIX B

Sample SYMFUNE Output

<u>CASE 8 :</u>
Inequality constraint(s) [3, 4, 6] are inactive.

The equation(s) which remove the remaining degree(s) of freedom:

$$-\frac{0.11131428\ K - 0.09771141\ \%r1^2}{\%r1^2} = 0$$

The solutions:

t = 8.271
a = 2.031

$$h = \frac{1.42513157}{K^{0.5}}$$

$$l = 1.06734005\ K^{0.5}$$

This case is BOUNDED.

<u>DOMAIN DEFINITIONS:</u>

Inequality constraints [3, 4, 6] are inactive.
The derivatives of the Lagrangian are:

$$-\frac{0.28284\ mu_5\ t^2}{a^4} - mu_2\ t + 0.04811\ \left(\frac{1.5211}{h} + 14\right) = 0$$

$$0.04811\ \left(1.5211\ K - \frac{1.5211\ a}{h^2}\right) = 0$$

$$mu_5\ \left(\frac{0.18856t}{a^3} + 0.02776\right) - mu_2\ a = 0$$

The Lagrange multipliers:

$$mu_2 = \left(\frac{0.03532397}{K^{0.5}} + 0.00269305\right) K^{0.5}$$

$$mu_5 = \left(\frac{0.33537873}{K^{0.5}} + 0.0255688\right) K^{0.5}$$

Setting $mu_2 > 0$ does not constrain the domain.

Setting $mu_5 > 0$ does not constrain the domain.

The feasibility conditions are:

$$2.031 >= \frac{11.78726324}{K^{0.5}}$$

$$138.93957467 >= 9.08$$

Note: the last equation does not restrict the domain.

$$\frac{1.42513157}{K^{0.5}} >= 0.125$$

Optimizing the Behavior of a Strain-Gaged Force Sensor

M. Reuber[1], C. Barratt[2], and J. Kornegay[1]

Introduction

Strain-gaged elastic members are the basis for a large variety of inexpensive and reliable sensors used to measure static and dynamic forces. Many strain-gaged force sensor designs have been documented (eg. [1]), but little research has been conducted in formalizing the rules which govern sensor design. Past studies of force-sensor design have focussed on related design issues such as cross-talk minimization [2], or the optimization of a specific sensor [3].

This brief examines the optimum design of single degree-of-freedom force sensors for maximum stiffness and sensitivity. Guidelines for force sensor design are derived and illustrated by analyzing the design of a sensor used to measure the forces produced by a robotic water-jet cutter.

General Form of Design Optimization Problem

The equations describing the stiffness and sensitivity of an elastic structure undergoing loading can be conveniently expressed in the following generic form:

$$k = \alpha_i(1/S), \quad i = 1, 4 \tag{1}$$

where

k = equivalent stiffness at point of load application
S = sensitivity of sensor (maximum strain per unit force)
α_i = parameter relating k and S for strain mode i (tension/compression, shear, bending, or torsion)

The parameter α_i is a function of a set of N design variables:

$$\alpha_i = \alpha_i(\underline{d}_i)$$

[1]Department of Mechanical Engineering and Applied Mechanics, University of Rhode Island, Kingston, RI.

[2]Department of Mechanical Engineering, University of New Haven, New Haven, CT.

Contributed by the Design Automation Committee for publication in the Journal of Mechanisms, Transmissions, and Automation in Design. Manuscript received October 1988.

where

$\underline{d}_i = [d_1 \ldots d_N]^{-1}{}_i$, the set of N design variables for strain mode i

To obtain maximum stiffness, k, for a given sensitivity, S, the value of α_i must be maximized. This problem can be expressed in standard minimization form:

Minimize $\psi_i(\underline{d}_i)$, where $\psi_i(\underline{d}_i) = \alpha_i^{-1}(\underline{d}_i)$
subject to

$E_j(\underline{d}_i) = 0, j = 1, J$, set of J equality constraints for strain mode i

$G_k(\underline{d}_i) \le 0$, $k = 1, K$, set of K inequality constraints for strain mode i

$\underline{d}_{i(L)} \le \underline{d}_i \le \underline{d}_{i(U)}$, upper and lower bounds on design variables for strain mode i

The design optimization problem is simplified further by posing the problem so that the equality and inequality constraints are not needed and formulating ψ_i as follows:

$$\psi_i = d_i \times d_2 \times \ldots \times d_N$$

The value of ψ_i is minimum and the value of α_i is maximum when all of the design variable assume their lower bounds for a given strain mode i:

$$\text{minimum } \psi_i = d_{i(L)} \times d_{2(L)} \times \ldots \times d_{N(L)}$$

This solution represents an optimum force-sensor design; a sensor which exhibits the maximum stiffness for a given sensitivity.

Form of Optimization for Specific Strain Modes

The equations which govern strain for each of the four strain modes can be cast into the general form described in equation (1). For example, for the bending mode ($i = 3$), a beam of length L is subjected to a moment of $f \times r$ through a rigid moment arm [Fig. 1(*c*)], yielding the following generic stiffness/sensitivity relation:

$$k = \alpha_3(1/S) \tag{2}$$

where

$$\alpha_3 = 1/(d_1 d_2 d_3) \tag{3}$$

$$S = d_2^3 d_3/(d_1 E\eta) \tag{4}$$

$$d_1 = L \tag{5}$$

$$d_2 = L/c \tag{6}$$

$$d_3 = r/L \tag{7}$$

The following variables are introduced to model bending:

E = modulus of elasticity
I = cross-section moment of inertia
c = distance from neutral axis to outermost fiber
$\eta = I/c^4$ (dimensionless cross-sectional shape factor)
L = length of member
r = moment arm

In a similar way, the strength of materials equations for tension/compression, shear, and torsion as described by Higdon et al. [4], are combined and reformulated in the generic form suitable for optimizing. Details of the derivations are given by Kornegay [5].

Optimum Force Sensor Design

The maximum value of α_i is found by substituting the lower bound values of the design variables, $\underline{d}_i$, into the expressions for α_i. The design variables are the length of the sensor, L (common to all strain modes), and several dimensionless shape factors (eg., L/c and r/L for bending); lower bounds are determined by reasonable assumptions about geometry and strain levels. Optimum values for α_i are

$\alpha_1 = 1.54 \text{ cm}^{-1}$ (tension/compression)

$\alpha_2 = 0.78 \text{ cm}^{-1}$ (shear)