E. S. Grood

W. J. Suntay

Giannestras Biomechanics Laboratory,
Department of Orthopaedic Surgery,
University of Cincinnati Medical Center,
Cincinnati, Ohio 45267

A Joint Coordinate System for the Clinical Description of Three-Dimensional Motions: Application to the Knee[1]

The experimental study of joint kinematics in three dimensions requires the description and measurement of six motion components. An important aspect of any method of description is the ease with which it is communicated to those who use the data. This paper presents a joint coordinate system that provides a simple geometric description of the three-dimensional rotational and translational motion between two rigid bodies. The coordinate system is applied to the knee and related to the commonly used clinical terms for knee joint motion. A convenient characteristic of the coordinate system shared by spatial linkages is that large joint displacements are independent of the order in which the component translations and rotations occur.

Introduction

A complete understanding of joint kinematics is important in the diagnosis of joint disorders resulting from injury or disease, in the quantitative assessment of treatment, in the design of better prosthetic devices, and in the general study of locomotion. The motions which occur in most anatomical joints involve three-dimensional movement which is described by six independent coordinates or degrees of freedom. Three are translations and three are rotations [1–4].

To date, most experimental studies of joint motion have considered only the relative rotational motion between the articulating bones of the knee [5–8], hip [9,10], and elbow [11, 12]. Typically, Euler angles are used as the rotational position coordinates.

Only a few investigators have considered both the translation and rotation which occurs between body segments such as vertebral bodies of the spine [13,14], the wrist [15], the knee [16–21], and the shoulder [22]. Most commonly, the displacement between two relative positions of the body segments is characterized by the method of screws [22, 23]. However, we have found that this description of motion is not readily understood by clinicians, who might otherwise use such data to improve diagnosis and treatment.

This paper is concerned with the description of three-dimensional joint motion in a way which facilitates the communication between biomechanician and physician. A convenient coordinate system for describing three-dimensional joint position is introduced along with its application to the human knee. The rotations about and translations along the defined coordinate axes form a set of independent generalized coordinates which can be described using commonly employed clinical terminology. A desirable characteristic of this coordinate system is that joint displacements within the system are independent of the order in which the component translations and rotations occur. This eliminates the requirement of specifying the order of the rotations, a precedure commonly believed to be necessary when Euler angles are used.

Description of Coordinate System

For the sake of clarity and generality, the coordinate system is first described in geometric terms. The corresponding equations are presented later, when the coordinate system is applied to the knee joint. Although basic, it is important to remember that the purpose of a coordinate system is to allow the relative position between two bodies to be specified. The description of motion is the characterization of how their relative position changes with time.

As shown in Fig. 1, the geometry of each body is specified by a Cartesian coordinate system with origins located at O_A and O_B, and a set of surfaces which describe its shape. For the case at hand, body A is a parallelepiped whose shape is defined by three sets of parallel planes. Body B is specified by its cylindrical surface and its two parallel end planes.

We start by considering the angular position and the corresponding rotational motion between these two arbitrary rigid bodies. Since angular position in three dimensions is specified by three independent angles [24,25], we specify the three spatial axes about which the corresponding rotational motions occur. Alternatively, we could specify the three planes which are perpendicular to the rotation axes.

The three rotation axes which comprise the joint coordinate system are shown in Fig. 1. The nonorthogonal unit base

[1] This work supported in part by Grant AM 21172 from the National Institute of Arthritis, Metabolic, Digestive and Kidney Diseases.

Contributed by the Bioengineering Division for publication in the JOURNAL OF BIOMECHANICAL ENGINEERING. Manuscript received by the Bioengineering Division, February 24, 1982; revised manuscript received January 3, 1983.

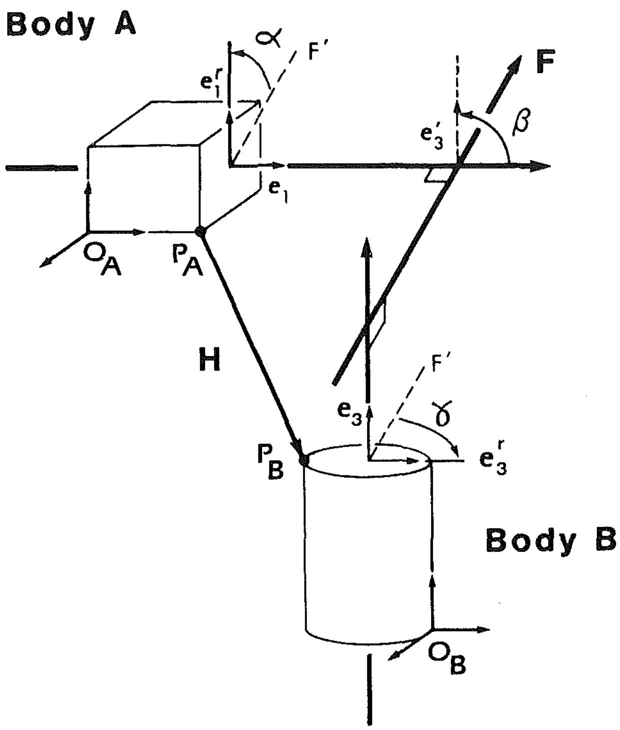

Fig. 1 The generalized joint coordinate system composed of three axes. Two axes are embedded in the two bodies whose relative motion is to be described. These axes which have unit base vectors e_1 and e_3, are called body fixed axes. The third axis, F, is the common perpendicular to both body fixed axes. Since the common perpendicular is not fixed to either body and moves in relation to both we call it the floating axis. The unit base vector for the floating axis is e_2.

vectors of the coordinate systems which define the axes are denoted as $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. Two of the axes, called body fixed axes, are embedded in the two bodies whose relative motion is to be described. Their direction is specified by unit base vectors $\mathbf{e}_1$ in body A, and $\mathbf{e}_3$ in body B. The fixed axes move with the bodies so the spatial relationship between them changes with the motion. The third axis, F, is the common perpendicular to the body fixed axes. Therefore, its orientation is given by the cross product of the unit base vectors which define the orientation of the fixed axes, $\mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1 / |\mathbf{e}_3 \times \mathbf{e}_1|$. We refer to the common perpendicular as the floating axis because it is not fixed in either body and moves in relation to both.

Two of the relative rotations between the bodies may be thought of as spin of each body about its own fixed axis while the other body remains stationary. The magnitude of these rotations are measured by the angles (α, γ) formed between the floating axis and a reference line embedded in each body, see Fig. 1. The direction and sense of the reference lines are described by unit vectors $\mathbf{e}_1{}^r$ and $\mathbf{e}_3{}^r$ in each body and are taken to be perpendicular to the fixed axis. Their orientation about the fixed axes is chosen based upon convenience in each application. The third relative rotation occurs about the floating axis and is measured by the angle, β, between the two body fixed axes, $\cos\beta = \mathbf{e}_1 \cdot \mathbf{e}_3$.

These three angular coordinates, (α, β, γ) provide a general geometric description of Euler angles. The floating axis, whose direction is defined by the unit vector $\mathbf{e}_2$, is always parallel to the line of nodes and is coincident with it when the three axes intersect at a common point.

Joint translations are described by the relative position of the two reference points, P_A and P_B, located in each body as shown in Fig. 1. The relative position of the reference points is characterized by the vector, $\mathbf{H}$, which connects them and is directed from body A to body B. The components of the translation vector are taken along the directions of the three coordinate axes. The magnitude of the translation vector, and its components, depends upon where the reference points are located in each body. This property is common to all methods of specifying translation and dictates the need for some rationale in the selection of the reference points. This will depend upon the specific application at hand and will be discussed later with reference to knee motions.

Application to the Knee

In constructing the coordinate system for the knee, or any other joint structure, it is necessary to specify: 1) the cartesian coordinate system fixed in each bone used to describe its shape; 2) the body fixed axes of the joint coordinate system and the reference axes of the joint coordinate system used to describe the relative motion between the two bones; and 3) the location of the translation reference point. It is convenient to establish the cartesian systems located in each bone so that two of their axes correspond to the body fixed and reference axes of the joint coordinate system and to locate the origin of the cartesian system so it is coincident with the translation reference point. While this approach is taken in the forthcoming, the reader should keep in mind the clear distinction between the Cartesian coordinate system located in each body and the joint coordinate system which is composed of the two body fixed axes, $\mathbf{e}_1$ and $\mathbf{e}_3$ and their mutual perpendicular, $\mathbf{e}_2$. The forthcoming discussion assumes that the coordinate system axes are established using bony landmarks identifiable on bi-planar X-rays.

For clarity, capitalized letters X, Y, Z will be used to denote the femoral cartesian coordinate system axes with $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ as the respective base vectors, and lower case letters x, y, z will be used with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as their respective base vectors for the tibial Cartesian coordinate system.

For the knee, we begin with the tibia as shown in Figs. 2 and 3. Table 1 summarizes for the reader the application of the joint coordinate system to the human knee. One clinical motion of interest is the internal-external rotation of the tibia about its mechanical axis. This axis, labelled as the z-axis in Fig. 2, is therefore selected as the tibial body fixed axis ($\mathbf{e}_3 = \mathbf{k}$). It is located so it passes midway between the two intercondylar eminences proximally and through the center of the ankle distally. The reference direction, $\mathbf{e}_3{}^r$, is taken oriented anteriorly and is identified as the tibial y-axis in Fig. 2 ($\mathbf{e}_3{}^r = \mathbf{j}$). Operationally we define the tibial anterior direction as the cross product of the fixed axis with a line connecting the approximate center of each plateau. The uncertainty, due to the error in defining the plateau centers from X-rays produces a fixed error in the measurement of absolute rotational position, but has no effect on quantifying rotational displacements. The third axis of the Cartesian coordinate system, the x direction, is obtained by completing a right-handed coordinate system. The x-axis is positive to the right, and therefore oriented laterally in the right knee and medially in the left knee.

In the femur, the body fixed axis is chosen so that rotations about it correspond to the clinical motion of flexion-extension. This is accomplished by choosing the fixed axis so it is perpendicular to the femoral sagittal plane, corresponding to the femoral X-axis in Fig. 2 ($\mathbf{e}_1 = \mathbf{I}$).

It is important that the flexion axis not be confused with the screw axis for functional knee flexion measured by Blacharski [16] and others. The screw axis is skewed with respect to the sagittal plane since it represents the combined effect of tibial axial rotation and adduction motions in addition to flexion.

To orient the flexion axis within the femur we start by defining the femoral mechanical axis, Z-axis, as shown in Fig. 2. Proximally this axis passes through the center of the

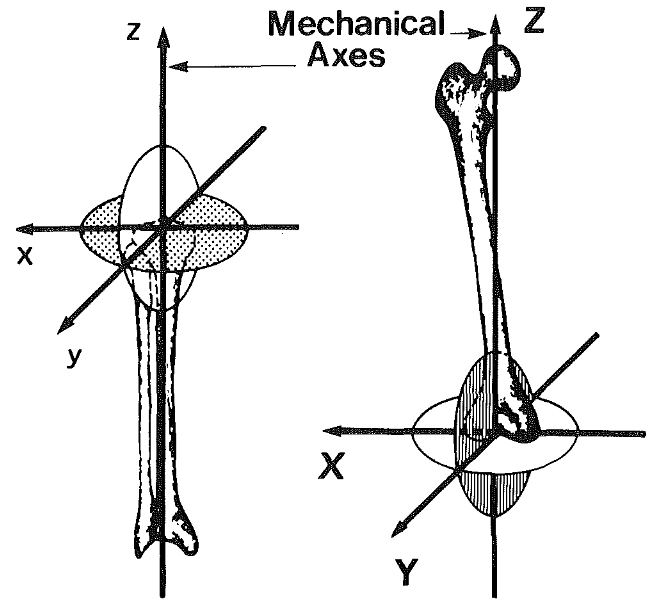

Fig. 2 Cartesian coordinate systems are defined in each bone. Capitalized letters X, Y, Z denote the femoral system axes while lower case letters, x, y, z, denote the tibial system axes. For both bones the z-axis is positive in the proximal direction, the y-axis is positive anteriorly, and the x-axis is positive to the right. The unit base vectors of these systems are I, J, K in the femur and i, j, k in the tibia.

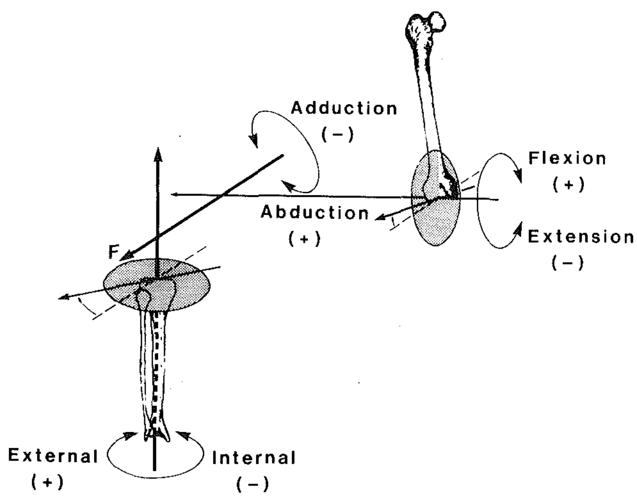

Fig. 3 Joint angles are defined by rotations occurring about the three joint coordinate axes. Flexion-extension is about the femoral body fixed axes. External-internal tibial rotation is about the tibial fixed axis and ab-adduction is about the floating axis (F).

femoral head. At the knee, it passes through the most distal point on the posterior surface of the femur midway between the medial and lateral condyles. The next step is to define the femoral frontal plane. The frontal plane contains the femoral mechanical axis, Z, and is oriented so that the most posterior points on the femoral condyles are equidistant from the plane. When using X-rays to define the bony landmarks, the points are taken at the level of the subchondral bones. Operationally, the normal to the frontal plane (femoral anterior or Y-axis) is obtained from the cross product of the mechanical axis and a line connecting the two points on the posterior surface of the femoral condyles. The direction of the flexion axis, which lies within the femoral frontal plane, is obtained as the cross product of the unit base vectors of the femoral anterior-posterior and mechanical axes. The femoral body fixed axes $\mathbf{e}_1$ corresponds to the X-axis whose base

Table 1 Joint coordinate system – application to the human knee

Femoral axis (flexion)	$\mathbf{e}_1 = \mathbf{I}$	$\mathbf{e}_1{}^r = \mathbf{J}$
Tibial axis (tibial rotation)	$\mathbf{e}_3 = \mathbf{k}$	$\mathbf{e}_3{}^r = \mathbf{j}$

i,**I** directed to the right
j,**J** directed anteriorly

Table 2 Clinical rotations and translations

Clinical rotations

- α = Flexion ($+ve$) — $\sin\alpha = -\mathbf{e}_2 \cdot \mathbf{K}$
- $\beta = \begin{cases} \pi/2 + \text{Adduction, right knee} \\ \pi/2 - \text{Adduction, left knee} \end{cases}$ — $\cos\beta = \mathbf{I} \cdot \mathbf{k}$
- γ = External rotation ($+ve$) — $\sin\gamma = \begin{cases} -\mathbf{e}_2 \cdot \mathbf{i} \text{ right knee} \\ \mathbf{e}_2 \cdot \mathbf{i} \text{ left knee} \end{cases}$

Clinical translations

- q_1 ($+ve$) lateral tibial displacement
- q_2 ($+ve$) anterior tibial drawer displacement
- q_3 ($+ve$) joint distraction

vector is **I**, and the reference axis is chosen in the anterior direction, $\mathbf{e}_1{}^r = \mathbf{J}$.

The relative joint rotations between the bones are shown in Fig. 3. Table 2 summarizes the sign convention used in defining the clinical rotations. Flexion-extension occurs about the femoral fixed axis and internal-external rotation is about the tibial fixed axis. Abduction-adduction motion occurs about the floating axis. Flexion and tibial rotation are the angles formed between the floating axis and the reference or anterior axis in each bone. Thus, we have the two relations:

$$\cos\alpha = \mathbf{e}_1{}^r \cdot \mathbf{e}_2 = \mathbf{J} \cdot \mathbf{e}_2 \qquad \alpha = \text{flexion} \qquad (1a)$$

$$\cos\gamma = \mathbf{e}_3{}^r \cdot \mathbf{e}_2 = \mathbf{j} \cdot \mathbf{e}_2 \qquad \gamma = \text{tibial rotation} \qquad (1b)$$

which can be used to obtain the magnitude of α and γ but not their sign, since $\cos(\alpha) = \cos(-\alpha)$. In order to determine the sign of the angles we use the following relations:

$$\mathbf{e}_2 \cdot \mathbf{K} = \cos(\pi/2 + \alpha) = -\sin\alpha \qquad (2a)$$

$$\mathbf{e}_2 \cdot \mathbf{i} = \begin{cases} \cos(\pi/2 + \gamma) = -\sin\gamma & \text{right knee} \quad (2b) \\ \cos(\pi/2 - \gamma) = \sin\gamma & \text{left knee} \quad (2c) \end{cases}$$

The signs in equation (2) are arbitrarily chosen so that flexion and external tibial rotation are both positive angles as summarized in Table 2.

The amount of joint adduction is obtained from the angle, β, between the tibial and femoral body fixed axes as follows:

$$\cos\beta = \mathbf{I} \cdot \mathbf{k}$$

$$\text{Adduction} = \begin{cases} \beta - \pi/2 & \text{right knee} \\ \pi/2 - \beta & \text{left knee} \end{cases} \qquad (3)$$

Translation between the femur and the tibia is denoted by the vector **H**, shown in Fig. 4, which is directed from the femoral origin to the tibial origin. The components of **H** with respect to the femoral system of axes are H_x, H_y, H_z, so:

$$\mathbf{H} = H_x\mathbf{I} + H_y\mathbf{J} + H_z\mathbf{k} \qquad (4a)$$

Similarly, with respect to the tibial system, we have

$$\mathbf{h} = -\mathbf{H} = h_x\mathbf{i} + h_y\mathbf{j} + h_z\mathbf{k} \qquad (4b)$$

where h_x, h_y and h_z are the components of the components of

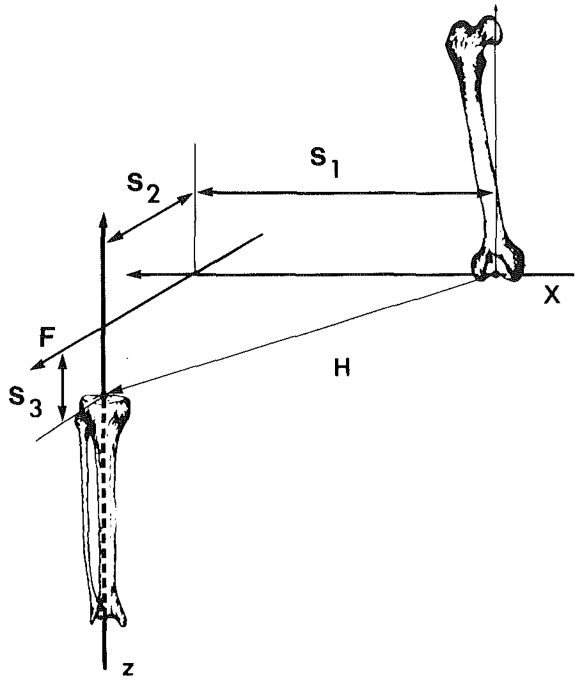

Fig. 4 Joint translations – when the femoral and tibial origins do not coincide, the floating axis, F, is located along the common perpendicular to the body fixed axes. Translation between the femoral and tibial origins is represented by the vector *H*.

the vector locating the femoral origin with respect to the tibial origin written in the tibial system of axes. The components of $\mathbf{H}$ with respect to the nonorthogonal base vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ of the joint coordinate system are the three joint translations and designated by S_1, S_2 and S_3, respectively, so that,

$$\mathbf{H} = S_1 \mathbf{e}_1 + S_2 \mathbf{e}_2 + S_3 \mathbf{e}_3 \qquad (4c)$$

Physically, S_1 is the distance from the femoral origin to the intersection of the $\mathbf{e}_1$ and $\mathbf{e}_2$-axes. S_2 is the distance between the $\mathbf{e}_1$ and $\mathbf{e}_3$-axes along their common perpendicular, $\mathbf{e}_2$, and S_3 the distance along the $\mathbf{e}_3$-axis from its intersection with the floating $\mathbf{e}_2$-axis to the tibial origin.

Clinical Translations

No widely accepted conventions currently exist for the clinical description of joint translation. Thus, it is necessary to adopt a set of mathematical definitions which correspond as closely as possible to existing clinical terminology. The clinical terms we adopt here are: medial-lateral tibial thrust or shift, designated by q_1, is a motion along the $\mathbf{e}_1$-axis; anterior-posterior tibial drawer, designated by q_2, is a motion along the floating $\mathbf{e}_2$-axis; and, joint distraction-compression, designated by q_3, is a motion along the $\mathbf{e}_3$-axis. Geometrically, medial-lateral thrust, q_1, is taken as the medial-lateral displacement of the tibial origin with respect to the femoral origin. Anterior drawer, q_2, is the displacement of the tibial origin along the floating, $\mathbf{e}_2$, axis; and joint distraction, q_3, is the height of the femoral origin above the tibial transverse plane.

Mathematically, the three clinical translations q_1, q_2 and q_3 are defined as the projections of the translation vector $\mathbf{H}$ along each of the axes of the joint coordinate system. One can write the following relations:

$$q_1 = \mathbf{H} \cdot \mathbf{e}_1 \qquad (5a)$$

$$q_2 = \mathbf{H} \cdot \mathbf{e}_2 \qquad (5b)$$

$$q_3 = -\mathbf{H} \cdot \mathbf{e}_3 \qquad (5c)$$

where the negative sign is introduced in equation (5*c*) to make joint distraction positive.

When the joint abduction angle is zero, $\beta = 90$ deg, the joint translations, S_i, introduced in the last section correspond to clinical descriptions of joint translation. Thus medial-lateral thrust is motion along the $\mathbf{e}_1$ axis and $q_1 = S_1$. Anterior drawer motion occurs along the $\mathbf{e}_2$, adduction, axis and $q_2 = S_2$. Finally, joint distraction is motion along the tibial mechanical axis, $\mathbf{e}_3$ and $q_3 = -S_3$.

When the body fixed axes are not perpendicular, $\beta \neq 90$ deg, the joint translations, S_i, are not equal to the clinical translations, q_i, since $\mathbf{e}_1$ and $\mathbf{e}_3$ are not perpendicular. Thus when $\beta \neq 90$ deg, distraction of the tibia along the $\mathbf{e}_3$-axis also affects medial-lateral thrust position, and medial-lateral shift along $\mathbf{e}_1$ also affects distraction and compression. This coupling of medial-laterial translation and distraction-compression translation in the skewed knee has presented no difficulty in communication with the physician since it is readily visualized and understood.

From the definitions of the clinical translations given in equations (5) and the selection of base vectors $\mathbf{e}_1$ and $\mathbf{e}_3$ given in Table 1, we see that $q_1 = H_x$ is the lateral displacement of the tibial origin along the femoral $X-$axis, $q_2 = S_2$ is the anterior drawer displacement between tibia and femur along the adduction axis and $q_3 = h_z$ is the proximal displacement of the femoral origin along the tibial z-axis. Note that when the joint is abducted, $\mathbf{H} \neq q_1 \mathbf{e}_1 + q_2 \mathbf{e}_2 + q_3 \mathbf{e}_3$ because $\mathbf{e}_1$ and $\mathbf{e}_3$ are not perpendicular.

For a right knee, one would express the clinical translations in terms of the joint translations as

$$q_1 = S_1 + S_3 \cos\beta \qquad (6)$$

$$q_3 = -S_3 - S_1 \cos\beta \qquad (7)$$

and $q_2 = S_2$ for all joint positions.

Coordinate Transformations

Although the joint coordinate system is convenient to use in describing joint motion, there is often a need to describe the location of specific points (i.e., ligament insertion sites, axes of rotation, etc.) relative to a coordinate system fixed to one of the bones. This dictates the need for describing the relation between the femoral and the tibial coordinate systems and the relation between the joint coordinate system and both the femoral and tibial coordinate systems.

If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ describes a point with respect to the tibial coordinate system, then the vector $\mathbf{R} = X\mathbf{I} + Y\mathbf{J} + Z\mathbf{K}$ describing the same point with respect to the femoral coordinate system is given by

$$\mathbf{R} = [R]\mathbf{r} + \mathbf{H} \qquad (8a)$$

where $\mathbf{H}$ is the vector which locates the tibial origin with respect to the femoral coordinate system and $[R]$ is a (3×3) rotation matrix. Equation (8*a*) can be rewritten as

$$\mathbf{r} = [R]^T \mathbf{R} + \mathbf{T} \qquad (8b)$$

where $\mathbf{T}$ locates the femoral origin with respect to the tibial coordinate system. The elements of the rotation matrix are derived in Appendix A in terms of the clinical rotations (α, β, γ).

Also, the components of the translation vector $\mathbf{H}$ with respect to the femoral system of axes, H_x, H_y, H_z are expressed in terms of the three joint translations S_1, S_2, S_3 in Appendix A as

$$\mathbf{H} = [U]\mathbf{S} \qquad (9)$$

where

$$[U] = \begin{bmatrix} 1 & 0 & \cos\beta \\ 0 & \cos\alpha & \sin\alpha\sin\beta \\ 0 & -\sin\alpha & \cos\alpha\sin\beta \end{bmatrix} \quad (10)$$

The three components of **S** are given in terms of **H** by inverting equation (9). Thus

$$\mathbf{S} = [U]^{-1}\mathbf{H} \quad (11a)$$

where

$$[U]^{-1} = \begin{bmatrix} 1 & -\sin\alpha\cos\beta/\sin\beta & -\cos\alpha\cos\beta/\sin\beta \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha/\sin\beta & \cos\alpha/\sin\beta \end{bmatrix} \quad (11b)$$

When β is zero, corresponding to 90 deg abduction, a singularity exists such that only the sum $(S_1 + S_3)$ is defined. Physically this occurs when both body fixed axes are parallel to each other but not coincident. In this situation the direction of the common perpendicular (floating axis) is well defined, but it has no single unique location. For this condition equation (11) reduces to two simultaneous equations:

$$S_2 = H_y\cos\alpha - H_z\sin\alpha \quad (12a)$$

$$S_1 = S_3 = H_x \quad (12b)$$

Note that although S_1 and S_3 are undefined for this condition, the clinical translations q_1 and q_3 are defined and $q_3 = \pm q_1$ depending upon whether $\beta = 0$ or 180 deg. This can be shown by expressing the clinical translations q_i in terms of the femoral components of the translation vector H_i, as given in Appendix A by:

$$\mathbf{q} = [V]\mathbf{H} \quad (13a)$$

where

$$[V] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ -\cos\beta & -\sin\alpha\sin\beta & -\cos\alpha\sin\beta \end{bmatrix} \quad (13b)$$

For the singularity where $\beta = 0$, this reduces to:

$$q_1 = -q_3 - H_x \quad (14a)$$

$$q_2 = H_y\cos\alpha - H_z\sin\alpha \quad (14b)$$

A second singularity exists when both β and S_2 are zero. Physically this occurs when both body fixed axes are not only parallel to each other but also co-linear with each other. This corresponds to the well-known gyroscopic singularity referred to as gimbal lock. Under these conditions, equation (8*a*) reduces to

$$\mathbf{R} = [R]\mathbf{r} + \mathbf{H}$$

where

$$[R] = \begin{bmatrix} 0 & 0 & 1 \\ -\sin(\alpha+\gamma) & \cos(\alpha+\gamma) & 0 \\ -\cos(\alpha+\gamma) & -\sin(\alpha+\gamma) & 0 \end{bmatrix} \quad (15a)$$

and

$$\mathbf{H} = [S_1 + S_3 \quad 0 \quad 0]^T \quad (15b)$$

Note that only the sums $(S_1 + S_3)$ and $(\alpha + \gamma)$ are defined for this singularity.

Although knowledge of the singularities is important, such conditions normally do not exist in human anatomical joints, with the possible exception of the glenoid-humeral joint. The dual singularity for the shoulder, however, can easily be prevented by selecting the body fixed axes so that they cannot become coincident when they are parallel.

Determination of the Clinical Parameters From the Position Matrix

When describing the transformation between the femur and the tibia, equation (8*a*) can be rewritten using equation (10) in the form

$$\mathbf{R} = [B]\mathbf{r} \quad (16a)$$

where

$$[B] = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline (S_1 + S_3\cos\beta) & & & \\ (S_2\cos\alpha + S_3\sin\alpha\sin\beta) & & [R] & \\ (-S_2\sin\alpha + S_3\cos\alpha\sin\beta) & & & \end{array}\right] \quad (16b)$$

Usually the transformation matrix is determined experimentally and it is necessary to calculate the clinical rotations and translations knowing the elements B_{ij}, $i,j = 1,2,3,4$. Using the rotation matrix calculated in Appendix A with equation (16) and the definitions of the clinical rotations given in equations (1)–(3), one can easily write the following relations:

$$\text{Adduction} = \cos^{-1}B_{24} - \pi/2 \quad (17a)$$

$$\text{Flexion angle} = \tan^{-1}(B_{34}/B_{44}) \quad (17b)$$

$$\text{Tibial rotation} = \tan^{-1}(B_{23}/B_{22}) \quad (17c)$$

The lateral translation of the tibia with respect to the femur is defined in equation (6) as being $q_1 = S_1 + S_3\cos\beta$. This is exactly equal to B_{12} in equation (16). Therefore

$$q_1 = B_{21} \quad (18)$$

The anterior drawer, q_2, is defined to be S_2. This is obtained from the following equations

$$q_2 = S_2 = B_{31}\cos\alpha - B_{14}\sin\alpha \quad (19)$$

The axial translation of the femur with respect to the tibia, q_3, is defined in equation (7) as $q_3 = -S_3 - S_1\cos\beta$. Using equations (9) and (16), q_3 may be obtained by the following equation:

$$q_3 = -(B_{42}B_{12} + B_{43}B_{13} + B_{44}B_{14}) \quad (20)$$

Selection of Translation Reference Points

The magnitude of the translation vector, and hence its components, depends upon where the reference points are located in each body. As described in the foregoing, the reference points are most often taken to be the origins of the femoral and tibial coordinate systems. These were initially chosen as identifiable bony landmarks observable on X-rays. The tibial origin was located on the tibial mechanical axis where the axis passes through subchondral bone between the two intercondylar eminences. The femoral origin was located on the femoral mechanical axis at the most distal bony point midway between the femoral condyles.

Shown as dashed lines in Fig. 5 are the three translational motions between these origins which occurred in one human cadaver knee during passive flexion with the tibia hanging vertically under the influence of gravity while the femur was flexed. The flexion was performed slowly so that the knee found its own equilibrium position at each flexion angle. The motion was measured using an instrumented spatial linkage with an absolute accuracy of ±0.5 mm and a resolution of 0.1 mm. As seen in Fig. 5, we found that large distraction

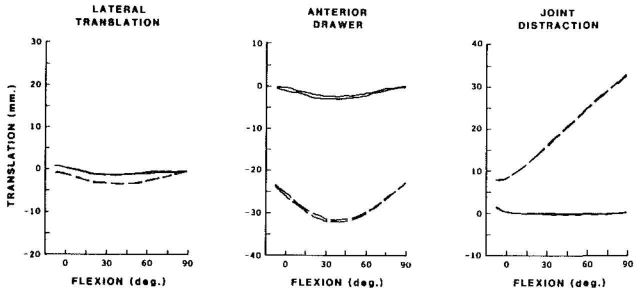

Fig. 5 Shown are the translational motions in millimeters versus flexion angle in degrees which occurred in one human cadaver knee during passive flexion. The dashed lines are the translations obtained when the origins of the femoral and tibial coordinate systems are used as the reference points. The solid lines represent the translations when the reference points are selected so as to minimize the magnitude of the translation vector for passive flexion.

translations resulted due to the position of the origins and the shape of the condyles. Also, a large anterior-posterior drawer motion occurred, again due to the position of the origin.

Since the curves shown in Fig. 5 correspond to a well-defined equilibrium position, in which only gravitational loading along the tibial axis is present, they represent one possible definition of the knee's translational neutral position. In most instances it is the deviation from this neutral position which is of interest. In order to more readily detect these deviations it is useful to minimize the translation associated with the neutral or reference condition. This may be accomplished by selecting translational reference points that are located on the screw displacement axis for the reference joint motion. To determine this, we calculated the screw displacement axis for passive flexion from 0 to 90 deg. The femoral origin was located at the intersection of the screw axis with the mid-femoral sagittal plane. The tibial origin was taken as the average of the two locations of the femoral origin corresponding to the beginning and the end of the motion. The translational motions computed using these definitions of the reference points is shown as solid lines in Fig. 5. A dramatic reduction in the anterior-posterior drawer and distraction compression motions are apparent.

The rotational and translational parameters described here (α, β, γ, S_i, $i=1, 2, 3$) or (α, β, γ, q_i, $i=1, 2, 3$) form a set of independent coordinates that completely define joint position. Further, for any motion the final joint position is a function only of the magnitude of the component rotations and translations, and not upon the order in which the individual motions occur. This is contrary to the general belief that finite rotational motions are sequence dependent. A proof of the sequence independency is given in the forthcoming.

Sequence Independency of Rotations

It is widely recognized that the net rotational displacement produced by a sequence of finite rotations performed about the axes of a cartesian coordinate system depends upon the order in which the individual rotations occur [28]. In contrast, we have noted [19, 21] that finite rotations performed about the axes of the joint coordinate system are not dependent upon sequence. Roth [29] has previously shown that sequences of screw rotations are independent of order when the screw axes combine in a chain structure such that the perpendicular distance and angle between adjacent screw axes is fixed. Recently, Chao [7] has applied Roth's proof to the gyroscopic mechanism which forms the motion sensing part of the triaxial goniometer [10].

It is easy to show that the property of sequence independence holds for any set of independent parameters which describe three-dimensional position. Since the parameters are independent, we can consider sequences of displacements where only one parameter changes at a time. Consider two sequences of differing order. If order is important then the final positions are different. In this case, the three parameters alone are not sufficient to describe the final position; we need the parameters plus their sequence. However, this contradicts the starting hypothesis, proved by Euler [24], that three independent parameters are sufficient to describe three-dimensional rotational position. Thus, we conclude that the final position produced by finite changes in independent rotational position parameters must be independent of their sequence.

The sequence dependency commonly referred to is for rotations performed about the three axes of a cartesian coordinate system. The reason for the sequence dependency is that these angles are not independent of each other [28].

Euler Angles Versus Joint Coordinate System Angles

Any review of the literature will show a wide variety of differing conventions for Euler angles. These different conventions have been classified into two axes and three axes types by several authors [7, 29]. The number of axes, two or three, refers to the axes in the moving body which the Euler rotations are thought to occur about. For example, a sequence like the z, x, z combination described by Goldstein [3] is of the two axes type, while the rotation sequence z, x, y sometimes referred to as Dexter angles would be of the three axes type. This description of Euler angles is sequence-dependent because the angles are defined by rotations performed about the axes of a cartesian system located in the moving body. As noted in the foregoing, such angles are not independent. Mathematically, the sequence dependency is observable from the coordinate transformation equation

$$\mathbf{r}_2 = [R_\psi][R_\theta][R_\varphi]\mathbf{r}_1 \tag{21}$$

where $\mathbf{r}_1$ is a vector in system 1, $\mathbf{r}_2$ is the same vector expressed in system 2 and $[R_\varphi]$, $[R_\theta]$, $[R_\psi]$ are an ordered set of rotations performed about the axes of the moving system 1. If the order of the ψ and θ rotations are reversed, the transformation equation becomes

$$\mathbf{r}_2{}' = [R_\theta][R_\psi][R_\varphi]\mathbf{r}_1 \tag{22}$$

The two sequences described by equations (21) and (22) result in different displacements because the inner product of two matrices is not commutative, $[R_\theta][R_\psi] \neq [R_\psi][R_\theta]$.

Our definition of Euler angles is sequence independent and eliminates much of the confusion relative to nomenclature. In our definition the rotations of equation (21) are thought of as occurring about the axes of the joint coordinate system. Thus the first and last rotations $[R_\varphi]$ and $[R_\psi]$ occur about the body fixed axes, while the second rotation, $[R_\theta]$, occurs about the floating axis. The first consequence is that all sets of Euler angles now have three axes making the prior distinction between two axes and three axes types inconsequential. For example, the two axes set of z, x, z described by Goldstein [3] is now thought of as Z, F, z. Where the first rotation occurs about the Z-axis of the fixed system, which is initially coincident with the z-axis of the moving system; the second rotation is about the floating axis, F, which corresponds to the line of nodes or x-axis; and the third rotation is about the z-axis of the moving system. Equation (21) can now be rewritten as

$$\mathbf{r}_2 = [R_z(\psi)][R_F(\theta)][R_Z(\varphi)]\mathbf{r}_1 \tag{23}$$

where $[R_Z(\varphi)]$ is thought of as a operator which produces a rotation of magnitude, φ, about the Z axis. The form of the operator is just the screw rotation matrix with the direction cosines of the screw axis expressed with respect to system 2. The direction cosines of the screw axis are, of course, those which describe its orientation when the rotation is performed.

When the sequence of rotation is changed

$$\mathbf{r}_2 = [R_F(\theta)][R_z(\psi)][R_Z(\varphi)]\mathbf{r}_1 \qquad (24)$$

two of the three axes will have a new orientation with respect to system 2 so that the individual elements of the operator are altered. Roth's proof of sequence independence [29] is based on the fact that the change in the elements of the operators is given by the similarity transformation so that different sequences all produce the same net displacement.

The relation between the conventional Euler angles and the angles in our joint coordinate system can be explained as follows. For a given set of Euler angles, there exists a unique set of axis corresponding to the joint coordinate system: a body fixed axis in the moving system, a body fixed axis in the stationary system and a common perpendicular to these two body fixed axes. When the conventional definition of Euler angles is used, a change in the order of rotations will produce a different joint coordinate system where the fixed axes have new orientations within their respective bodies. Thus, different sequences of the Euler rotations will produce different motions.

The body fixed axes which correspond to any set of Euler angles can be readily identified as follows: the first rotation performed when the coordinate system are coincident is about the fixed axis in the stationary body; the second rotation is about the floating axis, line of nodes; and the third rotation is about the fixed axis in the moving body. Thus, if the femur is assumed to be fixed and the tibia moving, the joint coordinate system presented here would correspond to a sequence of Euler rotation; first about the common x-axes, second about the rotated y in the tibia, and third about the twice rotated z-axis in the tibia.

Because the sequence of Euler rotations specifies the body fixed axes of the joint coordinate system, it can be concluded that the different sets of Euler angles have the axes of the joint coordinate system as their common geometric basis.

Discussion

Rigid body kinematics has been applied to the study of anatomical joints to determine the motions occurring during normal function and resulting from injury or disease. Several engineering descriptors of rigid body motion exist and have been applied to the description of joint motion. Often, difficulty exists in communicating these engineering descriptions to the clinician who must ultimately use them.

Townsend, et al. [18] measured knee motions in terms of flexion-extension, angulation (ab-adduction), and tibial axial rotation. Chao and Morrey [12] described elbow motion using Euler angles and related these to clinical descriptions for rotations. In this paper, they state that a particular sequence of rotation is necessary in order to define the unique orientation. However, Chao [7] applies a gyroscopic system as a special set of Euler angles and shows their sequence independence. Lewis and Lew [8] recognized the advantages of employing Euler angles and used them to describe knee rotations. To date, only the screw axis has been used to describe the translatory motions occurring in the knee [16, 17].

The joint coordinate system presented here provides a set of independent generalized coordinates for describing three-dimensional motion and permits a precise definition of the clinical descriptions of knee motion, both rotations and translations.

We have found that Euler angles, as described by the joint coordinate system, provide a precise mathematical description of clinical terminology for joint rotational motions. Further, we have shown that rotational motions about the axes of the proposed joint coordinate system commute. That is, the net displacement is independent of the sequence in which the individual rotations occur. This is not a new concept. Roth [29] has previously shown, using the similarity transformation, that screw displacements commute if two conditions are met. These are that the perpendicular distance and angle between adjacent screw axes are fixed. This is equivalent to saying the axes are the connecting joints in a series of rigid links forming a spatial linkage. Both of these conditions are met for the axes of the joint coordinate system we proposed.

The joint coordinate system presented here also overcomes the problems that exist if the joint translation vector $\mathbf{H}$ is described by its components along the cartesian systems fixed in the femur or in the tibia. For example, the direction of the clinical anterior-posterior drawer varies with respect to the femur as joint flexion changes while the direction corresponding to joint distraction motions varies as the knee is abducted. Similarly, the direction of anterior-posterior drawer varies with respect to the tibia as tibial axial rotation changes, while the direction corresponding to medial lateral thrust varies as the knee is abducted. These problems are eliminated when using the joint coordinate system, with the three nonorthogonal unit base vectors, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ to describe the joint translation vector $\mathbf{H}$. The two joint translations S_1 and S_3, locate the floating axis along the femoral and tibial fixed body axes, respectively. The clinical translations occur along the three base vectors of the joint coordinate system. When the joint abduction angle is zero, the joint translations correspond to the clinical ones.

It is important to emphasize that the magnitude of the translation vector depends upon where the reference points are located in each body. We have proposed a method for defining these reference points by minimizing the magnitude

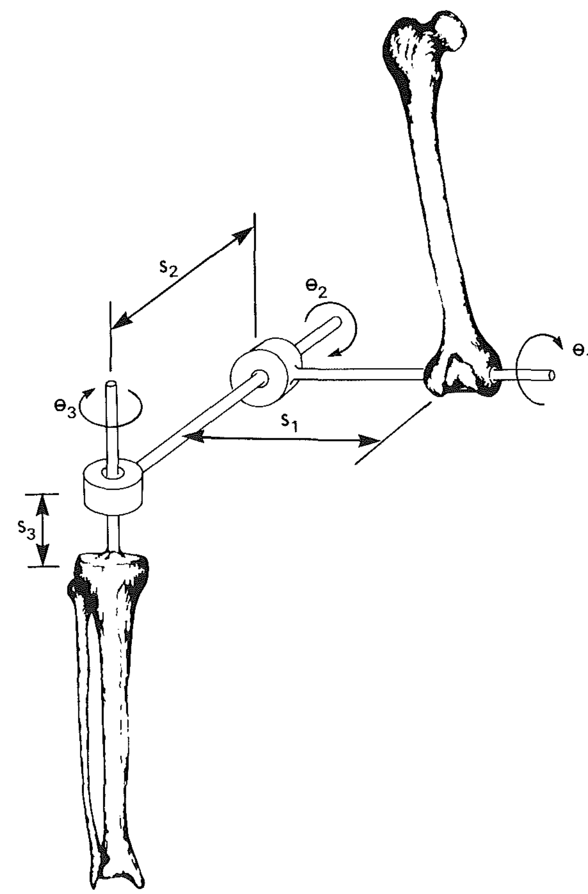

Fig. 6 The joint coordinate system is modeled as a four-link kinematic chain consisting of cylindrical joints. The first and last links are the tibia and femur, respectively. Two imaginary links exist between them. All three joints are cylindrical.

of the translation vector based on a selected reference motion. For the knee, we selected passive flexion from 0 to 90 deg. This permits translation motions that deviate from passive flexion to be detected more readily.

The joint coordinate system we have presented in the foregoing can be applied to any instrumented spatial linkage. For a typical triaxial goniometer with three degrees of freedom only the rotation can be measured after inherent crosstalk is accounted for. If a linkage with six degrees of freedom is used then the joint coordinate system provides the three-dimensional motion in terms of the clinical parameters for both rotations and translations. Errors in six degree of freedom systems result from mechanical and transducer tolerances alone, and not from crosstalk.

Interestingly, we have shown in [26] that the joint coordinate system introduced herein can be modeled as a four-link kinematic chain where the relative motion between links is described using Hartenberg and Denavit method [27]. For such a chain, the joints are cylindrical pairs which allow rotation and translation. The relative motion between links k and $k+1$ is then described by two variables: s_k for translation and θ_k for rotation. The geometry of the connecting links describing the relative position of the successive joints is determined by two more parameters: the common perpendicular between two successive joints, a_k, and the angle between joint axes at each end of the link , α_k. For the joint coordinate system of Fig. 4, the first and the last link are the tibia and femur, respectively, as shown in Fig. 6. Two imaginary links exist between the body fixed axes and the floating axis. Since the fixed and floating axes intersect, the lengths of the imaginary connecting links, a_k, are zero. Further, since the fixed and floating axes are perpendicular, the angles of twist, α_k, of the imaginary links are 90 deg.

Summary

1 We have presented a joint coordinate system having three nonorthogonal unit base vectors for describing the six degrees-of-freedom motion of the knee joint. In this system, joint position is independent of the order in which the translations and rotations are performed.

2 We have found that Euler angles, as described by the joint coordinate system, provide a mathematical description of clinical terminology for joint rotations.

3 We have introduced a set of mathematical definitions for the clinical translations which we believe make it easier to communicate translational motion to clinicians.

4 We proposed a method for the selection of the translation reference points which permits the magnitude of the translation reference vector to be minimized for a selected reference motion. For the knee we selected passive flexion as the reference motion. Thus, translational motions which deviate from those during passive flexion are easier to detect.

References

1 Routh, E. J., *Dynamics of a System of Rigid Bodies*, Macmillan and Co., Ltd., London, 1930.

2 Corben, H. C., and Stehle, P., *Classical Mechanics*, Wiley, New York, 1950.

3 Goldstein, H., *Classical Mechanics*, Addison-Wesley, New York, 1960, pp. 107–109.

4 Beggs, J. S., *Advanced Mechanisms*, The Macmillan Co., New York, 1966.

5 Levens, A. S., Inman, V. T., and Blosser, J. A., "Transverse Rotation of the Segments of the Lower Extremity in Location," *Journal of Bone and Joint Surgery*, Vol. 30-A, 1948, pp. 859–972.

6 Kettelkamp, D. B., Johnson, R. J., Smidt, G. L., Chao, E. Y. S., and Walker, M., "An Electrogoniometric Study of Knee Motion in Normal Gait," *Journal of Bone and Joint Surgery*, Vol. 52-A, 1970, pp. 775–790.

7 Chao, E. Y. S., "Justification of Triaxial Goniometer for the Measurement of Joint Motion," *Journal of Biomechanics*, Vol. 13, 1980, pp. 989–1006.

8 Lewis, J. L., and Lew, W. D., "A Note on the Description of Articulating Joint Motion," *Journal of Biomechanics*, Vol. 10, 1977, pp. 675–678.

9 Johnston, R. C., and Smidt, G. L., "Measurement of Hip Joint Motion During Walking," *Journal of Bone and Joint Surgery*, Vol. 51-A, 1969, pp. 1083–1094.

10 Chao, E. Y. S., Rim, K., Smidt, G. L., and Johnston, R. C., "The Application of 4 × 4 Matrix Method to the Correction of the Measurements of Hip Joint Rotations," *Journal of Biomechanics*, Vol. 3, 1970, pp. 459–471.

11 Morrey, B. F., and Chao, E. Y. S., "Passive Motion of the Elbow Joint," *Journal of Bone and Joint Surgery*, Vol. 58-A, 1976, pp. 501–508.

12 Chao, E. Y. S., and Morrey, B. F., "Three Dimensional Rotation of the Elbow," *Journal of Biomechanics*, Vol. 11, 1978, pp. 57–74.

13 Panjabi, M., and White, A. A., "A Mathematical Approach for Three-Dimensional Analysis of the Mechanics of the Spine," *Journal of Biomechanics*, Vol. 4, 1971, pp. 203–211.

14 Brown, R. H., Burstein, A. H., Nash, C. L., and Schock, C. C., "Spinal Analysis Using a Three-Dimensional Radiographic Technique," *Journal of Biomechanics*, Vol. 9, 1976, pp. 355–366.

15 Youm, Y., and Yoon, Y. S., "Analytical Development in Investigation of Wrist Kinematics," *Journal of Biomechanics*, Vol. 12, 1979, pp. 613–621.

16 Blacharski, P. A., and Somerset, J. H., "A Three Dimensional Study of the Kinematics of the Human Knee," *Journal of Biomechanics*, Vol. 8, 1975, pp. 375–384.

17 Thompson, C. T., "A System for Determining the Spatial Motions of Arbitrary Mechanisms—Demonstrated on a Human Knee," Ph.D. dissertation, Stanford University, 1972,

18 Townsend, M. A., Izak, M., and Jackson, R. W., "Total Motion Knee Goniometry," *Journal of Biomechanics*, Vol. 10, 1977, pp. 183–193.

19 Grood, E. S., Suntay, W. J., Noyes, F. R., Butler, D. L., Miller, E. H., and Malek, M., "Total Motion Measurement During Knee Laxity Tests," 25th Annual Orthopaedic Research Society, 1979, p. 80.

20 Suntay, W. J., "A Study on Spatial Linkages for Relative Motion Measurement With Application to the Knee Joint," M. S. thesis, University of Cincinnati, 1977.

21 Suntay, W. J., Grood, E. S., Noyes, F. R., and Butler, D. L., "A Coordinate System for Describing Joint Position," *Advances in Bioengineering*, 1978, pp. 59–62.

22 Kinzel, G. L., "On the Design of Instrumented Linkage for the Measurement of Relative Motion Between Two Rigid Bodies," Ph.D. thesis, Purdue University, 1973.

23 Suh, C. H., and Radcliff, C. W., *Kinematics and Mechanisms Design*, Wiley, 1978.

24 Euler, L., *Novi Comment*, Petrop XX, 1776.

25 Whittaker, E. T., *A Treatise of the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, London, 1904.

26 Suntay, W. J., "Three Dimensional Kinematics of the Human Cadaver Knee: Application to the Clinical Laxity Examination and the Knee Extension Exercise," Ph.D. dissertation, University of Cincinnati, Cincinnati, Ohio, 1982.

27 Hartenberg, R. S., and Denavit, J., *Kinematic Synthesis of Linkages*, McGraw-Hill Book Co., New York, 1964.

28 Shames, I. H., *Engineering Mechanics*, 2nd Edition, Prentice Hall, Englewood Cliffs, N.J., 1967, p. 653.

29 Roth, B., "Finite Position Theory Applied to Mechanism Synthesis," ASME *Journal of Applied Mechanics*, Sept. 1967, pp. 599–606.

APPENDIX A

1 Rotation Matrix. If **I**, **J**, **K** are unit base vectors in femoral system and **i**, **j**, **k**, are the base vectors in the tibial system and we let $\mathbf{R}=X\mathbf{I}+Y\mathbf{J}+Z\mathbf{K}$ describe a point with respect to the femoral coordinate system, then the vector $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ describing the same point with respect to the tibial coordinate system is given in accordance with equation (8*a*) as

$$\mathbf{r}=[R]^T\mathbf{R}+\mathbf{T} \qquad (25)$$

where **T** is the vector which locates the femoral origin with respect to the tibial coordinate system and $[R]^T$ is the transpose of a (3×3) rotation matrix. The rotation matrix is most commonly derived in the literature by considering a specific sequence of three rotations and forming the appropriate inner product of the individual rotation matrices. Since the inner product of two matrices is, in general, noncommutative, the reader is left with the impression that the resulting rotation matrix is only valid for a specific sequence of rotations. In the forthcoming we provide a derivation of the rotation matrix from the known geometry without any reference to a sequence of rotations.

The rotation matrix is defined as

$$[R]^T = \begin{bmatrix} \mathbf{I \cdot i} & \mathbf{J \cdot i} & \mathbf{K \cdot i} \\ \mathbf{I \cdot j} & \mathbf{J \cdot j} & \mathbf{K \cdot j} \\ \mathbf{I \cdot k} & \mathbf{J \cdot k} & \mathbf{K \cdot k} \end{bmatrix} \tag{26}$$

where its nine components are determined in terms of the clinical rotations (α, β, γ). Using the definitions of these clinical rotations, one can write the following relations between the unit base vectors of the femoral system, the tibial system and the joint coordinate system for a right knee:

$$\begin{aligned} &\mathbf{J \cdot e}_2 = \cos\alpha \\ &\mathbf{K \cdot e}_2 = \cos(\pi/2+\alpha) = -\sin\alpha \\ &\mathbf{j \cdot e}_2 = \cos\gamma \\ &\mathbf{i \cdot e}_2 = \cos(\pi/2+\gamma) = -\sin\gamma \\ &\mathbf{e}_2 \times \mathbf{J} = \mathbf{I}\sin\alpha \\ &\mathbf{e}_2 \times \mathbf{j} = -\mathbf{k}\sin\gamma \\ &\mathbf{I \cdot k} = \cos\beta \end{aligned} \tag{27}$$

At this point, one is able to evaluate the rotation matrix as follows:

First Column:

$$\begin{aligned} \mathbf{I \cdot i} &= \mathbf{I \cdot (j \times k)} = \mathbf{j \cdot (k \times I)} \\ &= \mathbf{j \cdot e}_2 \sin\beta \\ &= \cos\gamma \sin\beta \end{aligned} \tag{28}$$

$$\begin{aligned} \mathbf{I \cdot j} &= \mathbf{I \cdot (k \times i)} = \mathbf{i \cdot (I \times k)} \\ &= -\mathbf{i \cdot e}_2 \sin\beta \\ &= \sin\gamma \sin\beta \end{aligned} \tag{29}$$

$$\mathbf{I \cdot k} = \cos\beta \tag{30}$$

Second Column:

$$\begin{aligned} \mathbf{J \cdot i} &= \mathbf{J}\{(\mathbf{i \cdot e}_2)\mathbf{e}_2 + [\mathbf{i \cdot (e}_2 \times \mathbf{k})](\mathbf{e}_2 \times \mathbf{k})\} \\ &= (\mathbf{i \cdot e}_2)(\mathbf{J \cdot e}_2) + [\mathbf{i \cdot (e}_2 \times \mathbf{k})][\mathbf{J \cdot (e}_2 \times \mathbf{k})] \\ &= -\cos\alpha \sin\gamma + [\mathbf{e}_2 \mathbf{\cdot (k \times i)}][\mathbf{k \cdot (J \times e}_2)] \\ &= -\cos\alpha \sin\gamma + [\mathbf{e}_2 \mathbf{\cdot j}][\mathbf{k \cdot}(-\mathbf{I}\sin\alpha)] \\ &= -\cos\alpha \sin\gamma - \cos\gamma \sin\alpha \mathbf{k \cdot I} \\ &= -\cos\alpha \sin\gamma - \cos\gamma \sin\alpha \cos\beta \end{aligned} \tag{31}$$

$$\begin{aligned} \mathbf{J \cdot j} &= \mathbf{J \cdot (j \cdot e}_2)\mathbf{e}_2 + [\mathbf{j \cdot (e}_2 \times \mathbf{k})](\mathbf{e}_2 \times \mathbf{k}) \\ &= (\mathbf{j \cdot e}_2)(\mathbf{J \cdot e}_2) + [\mathbf{j \cdot (e}_2 \times \mathbf{k})][\mathbf{J \cdot (e}_2 \times \mathbf{k})] \\ &= \cos\alpha \cos\gamma + [\mathbf{e}_2 \mathbf{\cdot (k \times j)}][\mathbf{k \cdot (J \times e}_2)] \\ &= \cos\alpha \cos\gamma + [-(\mathbf{e}_2 \mathbf{\cdot i})][\mathbf{k \cdot}(-\mathbf{I}\sin\alpha)] \\ &= \cos\alpha \cos\gamma + (\mathbf{e}_2 \mathbf{\cdot i})(\mathbf{k \cdot I})\sin\alpha \\ &= \cos\alpha \cos\gamma + (-\sin\gamma)\cos\beta \sin\alpha \\ &= \cos\alpha \cos\gamma - \sin\gamma \sin\alpha \cos\beta \end{aligned} \tag{32}$$

$$\begin{aligned} \mathbf{J \cdot k} &= \mathbf{(K \cdot I) \cdot k} = \mathbf{k \cdot (K \times I)} = \mathbf{K \cdot (I \times k)} \\ &= -\sin\beta\, \mathbf{K \cdot e}_2 = -\sin\beta(-\sin\alpha) \\ &= \sin\beta \sin\alpha \end{aligned} \tag{33}$$

Third Column:

$$\begin{aligned} &\mathbf{K \cdot i K \cdot (i \cdot e}_2)\mathbf{e}_2 + [\mathbf{i \cdot (e}_2 \times \mathbf{k})](\mathbf{e}_2 \times \mathbf{k}) \\ &\quad = (\mathbf{i \cdot e}_2)(\mathbf{K \cdot e}_2) + [\mathbf{i \cdot (e}_2 \times \mathbf{k})][\mathbf{K \cdot (e}_2 \times \mathbf{k})] \\ &\quad = (\mathbf{i \cdot e}_2)(\mathbf{K \cdot e}_2) + (\mathbf{e}_2 \mathbf{\cdot j})[\mathbf{k \cdot (K \times e}_2)] \end{aligned} \tag{34}$$

$$\begin{aligned} &= \sin\alpha \sin\gamma + \cos\gamma\, [\mathbf{k \cdot} -\mathbf{I}\sin(\alpha + \pi/2)] \\ &= \sin\alpha \sin\gamma - \cos\gamma \cos\alpha\, (\mathbf{k \cdot I}) \\ &= \sin\alpha \sin\gamma - \cos\gamma \cos\alpha \cos\beta \end{aligned}$$

$$\begin{aligned} \mathbf{K \cdot j} &= \mathbf{K \cdot (j \cdot e}_2)\mathbf{e}_2 + [\mathbf{j \cdot (e}_2 \times \mathbf{k})](\mathbf{e}_2 \times \mathbf{k}) \\ &= (\mathbf{j \cdot e}_2)(\mathbf{K \cdot e}_2) + [\mathbf{j \cdot (e}_2 \times \mathbf{k})][\mathbf{K \cdot (e}_2 \times \mathbf{k})] \\ &= \cos\gamma\, (-\sin\alpha) + [\mathbf{e}_2 \mathbf{\cdot (k \times j)}][-\cos\alpha \cos\beta] \\ &= -\cos\gamma \sin\alpha - \cos\alpha \cos\beta \sin\gamma \end{aligned} \tag{35}$$

$$\begin{aligned} \mathbf{K \cdot k} &= \mathbf{(I \times J) \cdot k} = \mathbf{J \cdot (k \cdot I)} = \mathbf{J \cdot e}_2 \sin\beta \\ &= \cos\alpha \sin\beta \end{aligned} \tag{36}$$

2 Coordinate Transformations. The components of the translation vector **H** with respect to the femoral system of axes, H_x, H_y, H_z are expressed in terms of the three joint translations S_1, S_2, S_3 as

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 \mathbf{\cdot I} & \mathbf{e}_2 \mathbf{\cdot I} & \mathbf{e}_3 \mathbf{\cdot I} \\ \mathbf{e}_1 \mathbf{\cdot J} & \mathbf{e}_2 \mathbf{\cdot J} & \mathbf{e}_3 \mathbf{\cdot J} \\ \mathbf{e}_1 \mathbf{\cdot K} & \mathbf{e}_2 \mathbf{\cdot K} & \mathbf{e}_3 \mathbf{\cdot K} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} \tag{37}$$

while the clinical translations q_1, q_2, q_3 are expressed in terms of the femoral components of the translation vector as

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \mathbf{I \cdot I} & \mathbf{J \cdot I} & \mathbf{K \cdot I} \\ \mathbf{I \cdot e}_2 & \mathbf{J \cdot e}_2 & \mathbf{K \cdot e}_2 \\ -\mathbf{I \cdot k} & -\mathbf{J \cdot k} & -\mathbf{K \cdot k} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} \tag{38}$$

These equations readily reduce to equations (10) and (13) in the text when the relations $\mathbf{e}_1 = \mathbf{I}$ and $\mathbf{e}_3 = \mathbf{k}$ are substituted along with equations (28) through (36).