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# A Strong Triangle Inequality in Hyperbolic Geometry

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**Abstract**. For a triangle in the hyperbolic plane, let $\alpha, \beta, \gamma$ denote the angles opposite the sides $a, b, c$, respectively. Also, let $h$ be the height of the altitude to side $c$. Under the assumption that $\alpha, \beta, \gamma$ can be chosen uniformly in the interval $(0, \pi)$ and it is given that $\alpha + \beta + \gamma < \pi$, we show that the strong triangle inequality $a + b > c + h$ holds approximately 79% of the time. To accomplish this, we prove a number of theoretical results to make sure that the probability can be computed to an arbitrary precision, and the error can be bounded.

## 1. Introduction

It is well known that the Euclidean and hyperbolic planes satisfy the triangle inequality. What is less known is that in many cases a stronger triangle inequality holds. Specifically,

$$a + b > c + h \tag{1}$$

where $a, b, c$ are the lengths of the three sides of the triangle and $h$ is the height of the altitude to side $c$. We refer to inequality (1) as the *strong triangle inequality* and note that this inequality depends on which side of the triangle is labeled $c$.

The strong triangle inequality was first introduced for the Euclidean plane by Bailey and Bannister in [1]. They proved, see also Klamkin [4], that inequality (1) holds for all Euclidean triangles if $\gamma < \arctan\left(\frac{24}{7}\right)$ where $\gamma$ is the angle opposite side $c$. Bailey and Bannister also showed that $a + b = c + h$ for any Euclidean isosceles triangle such that $\gamma = \arctan\left(\frac{24}{7}\right)$ and $\gamma$ is the unique largest angle of the triangle. We let $B = \arctan\left(\frac{24}{7}\right)$ and refer to $B$ as the Bailey-Bannister bound.

In 2007, Baker and Powers [2] showed that the strong triangle inequality holds for any hyperbolic triangle if $\gamma \leq \Gamma$ where $\Gamma$ is the unique root of the function

$$f(\gamma) = -1 - \cos\gamma + \sin\gamma + \sin\frac{\gamma}{2}\sin\gamma$$

in the interval $[0, \frac{\pi}{2}]$. It turns out that $B \approx 74^\circ$ and $\Gamma \approx 66^\circ$ leading to roughly an $8^\circ$ difference between the Euclidean and hyperbolic bounds. It appears that the strong triangle inequality holds more often in the Euclidean plane than in the hyperbolic plane.

Let $\alpha$ and $\beta$ denote the angles opposite the sides $a$ and $b$, respectively. Under the assumption that the angles $\alpha$ and $\beta$ can be chosen uniformly in the interval $(0, \pi)$ and $\alpha + \beta < \pi$, Faĭziev et al. [3] showed the strong triangle inequality holds in the

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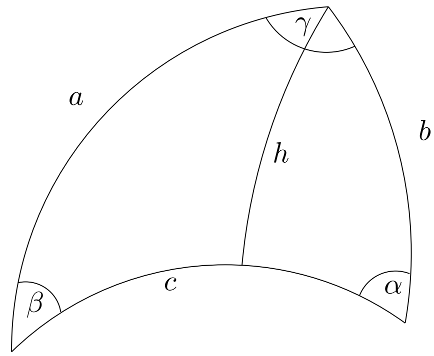


Figure 1. A hyperbolic triangle

Euclidean plane approximately 69% of the time. In addition, they asked how this percentage will change when working with triangles in the hyperbolic plane. In this paper, we answer this question by showing that the strong triangle inequality holds approximately 79% of the time. Moreover, we show that the stated probability can be computed to an arbitrary precision and that the error can be bounded.

Unless otherwise noted, all geometric notions in this paper are on the hyperbolic plane. Since our problem is invariant under scaling, we will assume that the Gaussian curvature of the plane is $-1$. We will use the notations $a$, $b$, $c$, $h$, $\alpha$, $\beta$, $\gamma$ for sides, height, and angles of a given triangle. (See Figure 1.) We will extensively use hyperbolic trigonometric formulas such as the law of sines and the two versions of the law of cosines. We refer the reader to Chapter 8 in [5] for a list of these various formulas.

## 2. Simple observations

In this section we mention a few simple, but important observations about the main question.

**Proposition 1.** *If $\gamma$ is not the unique greatest angle in a triangle, then the strong triangle inequality holds.*

*Proof.* Suppose that $\gamma$ is not the greatest angle, say, $\alpha \geq \gamma$. Then $a \geq c$, so $a + b \geq c + b \geq c + h$. Equality could only hold, if $\alpha = \gamma = \pi/2$, which is impossible. $\square$

**Proposition 2.** *If $\gamma \geq \pi/2$, then the strong triangle inequality does not hold.*

We will start with a lemma that is interesting in its own right.

**Lemma 3.** *In every triangle the following equation holds.*

$$\sinh c \sinh h = \sinh a \sinh b \sin \gamma$$

Note that in Euclidean geometry the analogous theorem would be the statement that $ch = ab \sin \gamma$, which is true by the fact that both sides of the equation represent twice the area of the triangle. Interestingly, in hyperbolic geometry, the sides of the corresponding equation do not represent the area of the triangle.

*Proof.* By the law of sines,

$$\frac{\sinh b}{\sin\beta} = \frac{\sinh c}{\sin\gamma},$$

so

$$\sinh c = \frac{\sinh b \sin\gamma}{\sin\beta}.$$

By right triangle trigonometry, $\sinh h = \sinh a \sin\beta$. Multiplying these equations, the result follows. □

*Proof of Proposition 2.* Note that $a+b > c+h$ if and only if $\cosh(a+b) > \cosh(c+h)$. Using the addition formula for $\cosh$, then the fact that $\cosh h \geq 1$ and $\cosh c \geq 0$, and then the law of cosines, in this order, we get

$$\begin{aligned}
\cos(c+h) &= \cosh c\cosh h + \sinh c \sinh h \geq \cosh c + \sinh c \sinh h \\
&= \cosh a \cosh b - \sinh a \sinh b \cos\gamma + \sinh a \sinh b \sin\gamma \\
&= \cosh a \cosh b + \sinh a \sinh b(\sin\gamma - \cos\gamma)
\end{aligned}$$

Notice that $\sin\gamma - \cos\gamma \geq 1$ if $\pi/2 \leq \gamma \leq \pi$. So

$$\begin{aligned}
&\cosh a \cosh b + \sinh a \sinh b(\sin\gamma - \cos\gamma) \\
\geq\ &\cosh a \cosh b + \sinh a \sinh b \\
=\ &\cosh(a+b).
\end{aligned}$$

□

## 3. Converting angles to lengths

Since the angles of a hyperbolic triangle uniquely determine the triangle, it is possible to rephrase the condition $a+b > c+h$ with $\alpha, \beta, \gamma$. In what follows, our goal is find a function $f(\alpha, \beta, \gamma)$, as simple as possible, such that $a+b > c+h$ if and only if $f(\alpha, \beta, \gamma) > 0$. Following Proposition 1 and Proposition 2, in the rest of the section we will assume that $\gamma < \pi/2$ is the greatest angle of the triangle.

The following lemma is implicit in [2]. We include the proof for completeness.

**Lemma 4.** *A triangle satisfies the strong triangle inequality if and only if*

$$\frac{\cos\alpha\cos\beta + \cos\gamma}{\cos\gamma + 1 - \sin\gamma} - 1 < \cosh h.$$

*Furthermore, the formula holds with equality if and only if* $a+b = c+h$.

*Proof.* Recall that $a+b > c+h$ if and only if $\cosh(a+b) - \cosh(c+h) > 0$. Using the $\cosh$ addition formula and the law of cosines on $\cosh c$, we have

$$\begin{aligned}
&\cosh(a+b) - \cosh(c+h) \\
=\ &\cosh a \cosh b + \sinh a \sinh b - \cosh c \cosh h - \sinh c \sinh h \\
=\ &\cosh c + \sinh a \sinh b \cos\gamma + \sinh a \sinh b - \cosh c \cosh h - \sinh c \sinh h \\
=\ &\cosh c(1 - \cosh h) + \sinh a \sinh b(\cos\gamma + 1) - \sinh c \sinh h
\end{aligned}$$

By Lemma 3,

$$\begin{aligned}
&\cosh c(1-\cosh h)+\sinh a\sinh b(\cos\gamma+1)-\sinh c\sinh h\\
=\ &\cosh c(1-\cosh h)+\sinh a\sinh b(\cos\gamma+1)-\sinh a\sinh b\sin\gamma\\
=\ &\frac{\sinh a\sinh b}{1+\cosh h}\left[\cosh c\frac{1-\cosh^2 h}{\sinh a\sinh b}+(1+\cosh h)(\cos\gamma+1-\sin\gamma)\right]
\end{aligned}$$

By the fact that $\sinh h=\sinh b\sin\alpha=\sinh a\sin\beta$, we have

$$\frac{1-\cosh^2 h}{\sinh a\sinh b}=\frac{-\sinh^2 h}{\sinh a\sinh b}=\frac{-\sinh b\sin\alpha\cdot\sinh a\sin\beta}{\sinh a\sinh b}=-\sin\alpha\sin\beta,$$

so, using the dual form of the law of cosines,

$$\begin{aligned}
&\cos(a+b)-\cosh(c+h)\\
=&\frac{\sinh a\sinh b}{1+\cosh h}\left[-\cosh c\sin\alpha\sin\beta+(1+\cosh h)(\cos\gamma+1-\sin\gamma)\right]\\
=&\frac{\sinh a\sinh b}{1+\cosh h}\left[-(\cos\alpha\cos\beta+\cos\gamma)+(1+\cosh h)(\cos\gamma+1-\sin\gamma)\right].
\end{aligned}$$

Since $\frac{\sinh a\sinh b}{1+\cosh h}>0$, we have the strong triangle inequality holds, if and only if

$$\cos\alpha\cos\beta+\cos\gamma<(1+\cosh h)(\cos\gamma+1-\sin\gamma),$$

and the result follows.

A minor variation of the proof shows the case of equality. □

**Lemma 5.** *For all triangles with* $\gamma>\max\{\alpha,\beta\}$,

$$\frac{\cos\alpha\cos\beta+\cos\gamma}{\cos\gamma+1-\sin\gamma}>1.$$

*Proof.* Without loss of generality, $0<\alpha\le\beta<\gamma<\pi/2$. Then

$$\begin{gathered}
0>\sin\gamma(\sin\gamma-1)=\sin^2\gamma-\sin\gamma>\sin^2\beta-\sin\gamma\\
\cos^2\beta>\cos^2\beta+\sin^2\beta-\sin\gamma=1-\sin\gamma\\
\cos^2\beta+\cos\gamma>1-\sin\gamma+\cos\gamma,
\end{gathered}$$

so

$$\frac{\cos^2\beta+\cos\gamma}{1-\sin\gamma+\cos\gamma}>1.$$

Since $0<\cos\beta\le\cos\alpha$, we have

$$\frac{\cos^2\beta+\cos\gamma}{1-\sin\gamma+\cos\gamma}\le\frac{\cos\alpha\cos\beta+\cos\gamma}{1-\sin\gamma+\cos\gamma},$$

and the results follows. □

By Lemma 4 and Lemma 5, we can conclude that the strong triangle inequality holds if and only if

$$\left(\frac{\cos\alpha\cos\beta+\cos\gamma}{\cos\gamma+1-\sin\gamma}-1\right)^2<\cosh^2 h. \tag{2}$$

Using the law of cosines,

$$\begin{aligned}\cosh^2 h = &= \sinh^2 h + 1 = \sin^2\beta \sinh^2 a + 1 = \sin^2\beta(\cosh^2 a - 1) + 1\\ &= \sin^2\beta\left(\frac{\cos\beta\cos\gamma + \cos\alpha}{\sin\beta\sin\gamma}\right)^2 - \sin^2\beta + 1\\ &= \cos^2\beta + \left(\frac{\cos\beta\cos\gamma + \cos\alpha}{\sin\gamma}\right)^2. \qquad (3)\end{aligned}$$

Equations (2) and (3) together imply the following statement.

**Lemma 6.** *The strong triangle inequality holds if and only if*

$$f(\alpha,\beta,\gamma) = \cos^2\beta + \left(\frac{\cos\beta\cos\gamma + \cos\alpha}{\sin\gamma}\right)^2 - \left(\frac{\cos\alpha\cos\beta + \cos\gamma}{\cos\gamma + 1 - \sin\gamma} - 1\right)^2 > 0$$

Notes:

(1) $f(\alpha,\beta,\gamma) = 0$ if and only if $a + b = c + h$. The proof of this is a minor variation of that of Lemma 6.
(2) $f(\alpha,\beta,\gamma)$ is symmetric in $\alpha$ and $\beta$. This is obvious from the geometry, but it is also not hard to prove directly.
(3) $f(\alpha,\beta,\gamma)$ is quadratic in $\cos\alpha$ and $\cos\beta$.
(4) $f(\alpha,\beta,\gamma)$ is not monotone in either $a+b-c-h$ or in $\cosh(a+b) - \cosh(c+h)$. Therefore it is not directly useful for studying the difference of the two sides in the strong triangle inequality.

Also note that it is fairly trivial to write down the condition $a+b > c+h$ with an inequality involving only $\alpha$, $\beta$, and $\gamma$. Indeed, one can just use the law of cosines to compute $a$, $b$, and $c$ from the angles, and some right triangle trigonometry to compute $h$. But just doing this simple approach will result in a formidable formula with inverse trigonometric functions and square roots. Even if one uses the fact that the condition is equivalent to $\cosh(a+b) > \cosh(c+h)$, the resulting naive formula is hopelessly complicated, and certainly not trivial to solve for $\alpha$ and $\beta$. Therefore, the importance and depth of Lemma 6 should not be underestimated.

## 4. Computing probabilities

Motivated by the original goal of computing the probability that the strong triangle inequality holds in hyperbolic geometry, we need to clarify first under what model we compute this probability.

In hyperbolic geometry there exists a triangle for arbitrarily chosen angles, provided that their sum is less than $\pi$. So it is natural to choose the three angles independently uniformly at random in $(0,\pi)$, and then aim to compute the probability that the strong triangle inequality holds, given that the sum of the chosen angles is less than $\pi$.

Of course, the computation can be reduced to a computation of volumes. Let

$$F = \{(\alpha,\beta,\gamma) \in (0,\pi)^3 : \alpha+\beta+\gamma < \pi,\ \max\{\alpha,\beta\} < \gamma < \pi/2\},$$

and let

$$S = \{(\alpha,\beta,\gamma) \in F : f(\alpha,\beta,\gamma) > 0\}.$$

Since $f$ is continuous when $0 < \gamma < \pi/2$, and $S$ is the level set of $f$ (within $F$), $S$ is measurable, so its volume is well-defined. The desired probability is then

$$\frac{\mathrm{Vol}(S)}{\pi^3/6},$$

where the denominator is the volume of the tetrahedron for which $\alpha + \beta + \gamma < \pi$.

So it remains to compute the volume of $S$. Fix $0 < \gamma < \pi/2$, and let

$$P_\gamma = \{(\alpha, \beta) : (\alpha, \beta, \gamma) \in F \text{ and } f(\alpha, \beta, \gamma) > 0\},$$
$$N_\gamma = \{(\alpha, \beta) : (\alpha, \beta, \gamma) \in F \text{ and } f(\alpha, \beta, \gamma) < 0\},$$
$$Z_\gamma = \{(\alpha, \beta) : (\alpha, \beta, \gamma) \in F \text{ and } f(\alpha, \beta, \gamma) = 0\}.$$

(See Figures 2 and 3 for illustration for $\gamma = 1.2$ and $\gamma = 1.3$ respectively.) It is clear that

$$\mathrm{Vol}(S) = \int_0^{\pi/2} \mu(P_\gamma)\, d\gamma,$$

where $\mu$ is the 2-dimensional Lebesgue measure.

It is not hard to see why it will be useful for us to solve the equation $f(\alpha, \beta, \gamma) = 0$: it will provide a description of the set $Z_\gamma$, which will help us analyze the sets $P_\gamma$, and $N_\gamma$. This is easy, because $f$ is quadratic in $\cos\beta$. The following extremely useful lemma shows that at most one of the quadratic solutions will lie in $F$.

**Lemma 7.** *Let* $(\alpha, \beta, \gamma) \in F$ *such that* $f(\alpha, \beta, \gamma) = 0$. *Let*

$$a = \csc^2\gamma - \left(\frac{\cos\alpha}{\cos\gamma + 1 - \sin\gamma}\right)^2,$$
$$b = \frac{\cos\alpha(\cos\gamma + 1)}{\sin^2\gamma},$$
$$c = \left(\frac{\cos\alpha}{\sin\gamma}\right)^2 - \left(\frac{1 - \sin\gamma}{\cos\gamma + 1 - \sin\gamma}\right)^2.$$

*Then*

$$\cos\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

*Proof.* By tedious, but simple algebra one can see that $f(\alpha, \beta, \gamma) = 0$ if and only if $a\cos^2\beta + b\cos\beta + c = 0$. To see the result, we will show that if $(\alpha, \beta, \gamma) \in F$, then $(-b + \sqrt{b^2 - 4ac})/(2a) < 0$. We will proceed by showing that for $(\alpha, \beta, \gamma) \in F$, we have $b > 0$, and $c > 0$. The former is trivial. For the latter, here follows the sequence of implied inequalities.

$$1 + \cos\gamma > \sin\gamma(1 + \cos\gamma) = \sin\gamma + \sin\gamma\cos\gamma$$
$$\cos^2\gamma + \sin^2\gamma + \cos\gamma > \sin\gamma + \sin\gamma\cos\gamma$$
$$\cos\gamma(\cos\gamma + 1 - \sin\gamma) > \sin\gamma(1 - \sin\gamma)$$
$$\frac{\cos\alpha}{\sin\gamma} \geq \frac{\cos\gamma}{\sin\gamma} > \frac{1 - \sin\gamma}{\cos\gamma + 1 - \sin\gamma}$$

Squaring both sides will give $c > 0$.

We have shown that $b, c > 0$. If $a = 0$, then $\cos\beta = -c/b < 0$, and that is inadmissible. If $a > 0$, then $b^2 - 4ac < b^2$, so $(-b + \sqrt{b^2 - 4ac})/(2a) < 0$, and similarly, if $a < 0$, then $b^2 - 4ac > b^2$, so $(-b + \sqrt{b^2 - 4ac})/(2a) < 0$ again. □

Recall that a set $R \subseteq \mathbb{R} \times \mathbb{R}$ is called a *function*, if for all $x \in \mathbb{R}$ there is at most one $y \in \mathbb{R}$ such that $(x, y) \in R$. Also $R^{-1} = \{(y, x) : (x, y) \in R\}$. $R$ is *symmetric*, if $R = R^{-1}$. The domain of a function $R$ is the set $\{x : \exists y, (x, y) \in R\}$.

So far we have learned the following about $Z_\gamma$.

- $Z_\gamma$ is a function (by Lemma 7).
- $Z_\gamma$ is symmetric.
- $Z_\gamma$ is injective (that is $Z_\gamma^{-1}$ is function).
- $\mu(Z_\gamma) = 0$.

The last fact follows, because $Z_\gamma$ is closed, and hence, it is measurable.

Therefore the computation may be reduced to that of $\mu(N_\gamma)$, which will turn out to be more convenient.

Let $\gamma$ be fixed, and let $0 < \alpha < \gamma$. We will say that $\alpha$ is *all-positive*, if for all $\beta$ we have $f(\alpha, \beta, \gamma) \geq 0$. Similarly, $\alpha$ is *all-negative*, if for all $\beta$, $f(\alpha, \beta, \gamma) \leq 0$. If there is a $\beta'$ such that $f(\alpha, \beta', \gamma) = 0$, then there are two possibilities: if for all $\beta < \beta'$, we have $f(\alpha, \beta, \gamma) < 0$, and for all $\beta > \beta'$, we have $f(\alpha, \beta, \gamma) > 0$, then we will say $\alpha$ is *negative-positive*. If it's the other way around, we will say $\alpha$ is *positive-negative*.

We will use the function notation $z(\alpha) = \beta$, when $(\alpha, \beta) \in Z_\gamma$; $z(\alpha)$ is undefined if $\alpha$ is not in the domain of $Z_\gamma$. When we want to emphasize the dependence on $\gamma$, we may write $z_\gamma(\alpha)$ for $z(\alpha)$.

**Lemma 8.** *If $z_\gamma$ is defined at $\alpha$, then*

$$z_\gamma(\alpha) = \cos^{-1}\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right),$$

*where $a, b, c$ are as in Lemma 7.*

*Proof.* This is direct consequence of Lemma 7. □

Our next goal is to extend the set $F$ as follows:

$$\overline{F} = \{(\alpha, \beta, \gamma) \in (0, \pi)^3 : \alpha + \beta + \gamma \leq \pi,\ \max\{\alpha, \beta\} \leq \gamma < \pi/2\}.$$

So we are extending $F$ by considering cases where $\alpha + \beta + \gamma = \pi$ and where $\max\{\alpha, \beta\} = \gamma$. For fixed $\gamma$ such that $0 < \gamma < \pi/2$, the collection $\{P_\gamma, N_\gamma, Z_\gamma\}$ is extended by letting

$$\overline{P}_\gamma = P_\gamma \cup \{(\alpha, \beta) : (\alpha, \beta, \gamma) \in \overline{F} \setminus F \text{ and } a + b > c + h\},$$
$$\overline{N}_\gamma = N_\gamma \cup \{(\alpha, \beta) : (\alpha, \beta, \gamma) \in \overline{F} \setminus F \text{ and } a + b < c + h\},$$
$$\overline{Z}_\gamma = Z_\gamma \cup \{(\alpha, \beta) : (\alpha, \beta, \gamma) \in \overline{F} \setminus F \text{ and } a + b = c + h\}.$$

We will show that with this extension, sequences of points entirely outside of $P_\gamma$ can not converge to a point in $\overline{P}_\gamma$, and similarly for $N_\gamma$. But first, we need a lemma that, in a way, formalizes the well-known intuition that infinitesimally small hyperbolic triangles are becoming arbitrarily similar to Euclidean triangles.

**Lemma 9.** *Let* $\{(\alpha_i, \beta_i, \gamma_i)\}_{i=1}^{\infty}$ *be a sequence in* $\mathbb{R}^3$*, such that* $(\alpha_i, \beta_i, \gamma_i) \to (\alpha, \beta, \gamma)$ *with* $\alpha_i + \beta_i + \gamma_i < \pi$ *for all* $i$*, and* $\alpha + \beta + \gamma = \pi$*. Let* $a_i, b_i, c_i$ *be the sides of the hyperbolic triangle determined by* $\alpha_i, \beta_i, \gamma_i$*, and let* $h_i$ *be the height corresponding to* $c_i$*. Furthermore, consider the class of similar Euclidean triangles with angles* $\alpha, \beta, \gamma$*, and let* $a, b, c$ *be the sides of an element of this class, and let* $h$ *be the height corresponding to* $c$*. Then*

$$\lim_{i\to\infty} \frac{a_i + b_i - c_i}{h_i} = \frac{a + b - c}{h}.$$

*Proof.* First we will prove that $a_i/b_i \to a/b$. By the law of sines for both the hyperbolic and Euclidean planes,

$$\frac{\sinh a_i}{\sinh b_i} = \frac{\sin \alpha_i}{\sin \beta_i} \to \frac{\sin \alpha}{\sin \beta} = \frac{a}{b}.$$

Since $a_i, b_i \to 0$, $\lim(\sinh a_i/\sinh b_i) = \lim(a_i/b_i)$, and the claim follows.

Note that applying this to various triangles formed by the height and the sides, this also implies $a_i/h_i \to a/h$, and $b_i/h_i \to b/h$. To see that $c_i/h_i \to c/h$, just observe that $c_i/h_i = (c_i/b_i)(b_i/h_i)$. □

**Corollary 10.** *Let* $\gamma \in (0, \pi/2)$*. Let* $\{(\alpha_i, \beta_i)\}_{i=1}^{\infty}$ *be a sequence in* $\mathbb{R}^2$ *such that* $(\alpha_i, \beta_i) \to (\alpha, \beta)$ *with* $(\alpha_i, \beta_i, \gamma) \in F$ *and* $(\alpha, \beta, \gamma) \in \overline{F}$*. Then* $(\alpha_i, \beta_i) \notin P_\gamma$ *implies* $(\alpha, \beta) \notin \overline{P}_\gamma$*, and* $(\alpha_i, \beta_i) \notin N_\gamma$ *implies* $(\alpha, \beta) \notin \overline{N}_\gamma$*.*

*Proof.* If $(\alpha, \beta, \gamma) \in F$, then this is a direct consequence of the continuity of $f$. If $(\alpha, \beta, \gamma)$ belongs to $\overline{F} \setminus F$, then $\alpha = \gamma$ or $\beta = \gamma$ or $\alpha + \beta + \gamma = \pi$. In the first two cases, all distances in the triangles determined by the angles $\alpha_i, \beta_i, \gamma_i$ converge to the corresponding distances in the limiting isosceles hyperbolic triangle determined by the angles $\alpha, \beta, \gamma$. Finally, if $\alpha + \beta + \gamma = \pi$, then this is a consequence of Lemma 9 with the observation that the strong triangle inequality holds if and only if $\frac{a+b-c}{h} > 1$. □

Recall the notations $\Gamma$ for the Baker–Powers constant, and $B$ for the Bailey–Bannister constant. We will use the following lemma which was proven by Baker and Powers [2].

**Lemma 11.** *If* $\gamma > \Gamma$*, then there exists* $\alpha' > 0$ *such that for all* $0 < \alpha < \alpha'$*, the strong triangle inequality fails for the triangle with angles* $\alpha$*,* $\alpha$*, and* $\gamma$*.*

**Lemma 12.** *Let* $\gamma \in (\Gamma, B)$*. Then the set of values of* $\alpha$ *for which* $\alpha$ *is negative-positive is an open interval* $(0, i_\gamma)$*, and* $z$ *is defined, continuous, and decreasing on this interval. Furthermore, the region* $N_\gamma$ *is exactly the region under the function* $z$ *on the interval* $(0, i_\gamma)$*.*

See Figure 2 for illustration.

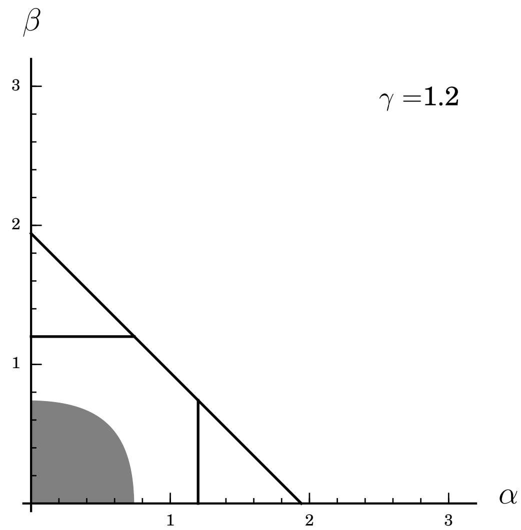


Figure 2. For $\gamma = 1.2$ here, the shaded region is $N_\gamma$, and the unshaded is $P_\gamma$. The boundary is $Z_\gamma$. The horizontal and vertical line segments represent $\beta = \gamma$ and $\alpha = \gamma$, beyond which it is guaranteed that the strong triangle inequality holds. The diagonal $\beta = \pi - \gamma - \alpha$ represents Euclidean triangles.

*Proof.* First note that the condition on $\gamma$ implies that every $\alpha$ is either all-positive or negative-positive. Indeed, any other type of $\alpha$ would give rise to a sequence of points in $N_\gamma$ converging to a point in $\overline{P}_\gamma$. Let $C$ be the set of values of $\alpha$, for which $\alpha$ is negative-positive. Clearly $z$ is defined on $C$.

By Lemma 11, there exists $\alpha' > 0$ such that for all $0 < \alpha < \alpha'$, $f(\alpha, \alpha, \gamma) < 0$. This also means that for all $0 < \alpha < \alpha'$, $\alpha \in C$.

Since $\gamma < B$ it follows that the diagonal $\alpha+\beta+\gamma = \pi$ lies in $\overline{P}_\gamma$. Moreover, by Proposition 1, the appropriate vertical line segment $\alpha = \gamma$ belongs to $\overline{P}_\gamma$. Consider the open segment going right from the point $(\alpha'/2, \alpha'/2)$ and ending at the point $(x, \alpha'/2)$ such that $(x, \alpha'/2, \gamma) \in \overline{F} \setminus F$. By Corollary 10, we can not have the open segment entirely in $N_\gamma$. So there exists $\alpha'/2 < \alpha_0 < \gamma$ with $(\alpha_0, \alpha'/2) \in Z_\gamma$. We also have that $(0, \alpha_0) \subseteq C$.

By Lemma 8, $z$ is continuous on $(0, \alpha_0)$. Injective continuous functions are monotone, and by symmetry again, $z$ must be monotone decreasing on $(0, \alpha_0)$. The portion of $z$ on $(0, \alpha'/2)$ is "copied" to the portion after $\alpha_0$, so there exists $\alpha_1 > \alpha_0$ such that $(0, \alpha_1) \subseteq C$, and $z$ is continuous, monotone decreasing on $(0, \alpha_1)$, and $\lim_{\alpha \to \alpha_1^-} z(\alpha) = 0$.

We claim that in fact $(0, \alpha_1) = C$. Suppose not, and there exists $\alpha_2 > \alpha_1$ with $\alpha_2 \in C$. For all $0 < \beta_2 < \min\{z(\alpha_2), \alpha'/2\}$, the horizontal line $\beta = \beta_2$ contains only one point from $Z_\gamma$. That implies that in fact the entire open line segment between $(0, \beta_2)$ to $(\min\{\pi - \gamma - \beta_2, \gamma\}, \beta_2)$ lies in $N_\gamma \cup Z_\gamma$. Thereby, we could construct a sequence in $N_\gamma \cup Z_\gamma$ converging to a point in $\overline{P}_\gamma$ contrary to Corollary 10. So the first part of the statement holds with $i_\gamma = \alpha_1$.

The second part of the statement is obvious after the first part, which is necessary to show that there is a well-defined region under the function on the interval $(0, i_\gamma)$. □

For the actual computations, we will need to numerically compute the value of $i_\gamma$. Since $i_\gamma = \lim_{\alpha\to 0^+} z(\alpha)$, and since $z$ remains continuous even if we extend the function by its formula for $\alpha = 0$, it is easy to compute its value. In fact it turns out that it has a relatively simple formal expression:

$$i_\gamma = \cos^{-1}\left(\frac{(\sin\gamma - 1)^2 + \cos\gamma}{2\sin\gamma - \cos\gamma - 1}\right).$$

Now we will start to work on the more difficult case when $\gamma \in (B, \pi/2)$. First we need two technical lemmas.

**Lemma 13.** *$f(\alpha, \beta, \gamma)$ is monotone decreasing in $\gamma$.*

*Proof.* Let

$$f_1(\alpha,\beta,\gamma) = \frac{\cos\beta\cos\gamma + \cos\alpha}{\sin\gamma}, \qquad f_2(\alpha,\beta,\gamma) = \frac{\cos\alpha\cos\beta + \cos\gamma}{\cos\gamma + 1 - \sin\gamma} - 1.$$

Then

$$f(\alpha,\beta,\gamma) = \cos^2\beta + [f_1(\alpha,\beta,\gamma)]^2 - [f_2(\alpha,\beta,\gamma)]^2.$$

Simple computations show

$$\frac{\partial f_1}{\partial\gamma} = \frac{-\cos\beta\sin^2\beta - (\cos\beta\cos\gamma + \cos\alpha)\cos\gamma}{\sin^2\gamma} < 0,$$

$$\frac{\partial f_2}{\partial\gamma} = \frac{1 - \sin\gamma + \cos\alpha\cos\beta(\sin\gamma + \cos\gamma)}{(\cos\gamma + 1 - \sin\gamma)^2} > 0.$$

Since for all $(\alpha,\beta,\gamma) \in F$, clearly $f_1 > 0$, and by Lemma 5, $f_2 > 0$, we get that

$$\frac{\partial f}{\partial\gamma} = 2f_1\frac{\partial f_1}{\partial\gamma} - 2f_2\frac{\partial f_2}{\partial\gamma} < 0.$$

□

**Lemma 14.** *Let $\gamma \in [B, \pi/2)$. Then all isosceles triangles with angles $\alpha$, $\alpha$, and $\gamma$ fail the strong triangle inequality. Furthermore, these triangles fail with inequality, that is, $(\alpha, \alpha, \gamma) \in N_\gamma$.*

*Proof.* We will use Lemma 4 with $\alpha = \beta$ and $\gamma = B$. In that case, $\cos\gamma = 7/25$ and $\sin\gamma = 24/25$; also $\cosh h = \cos\alpha/\sin(B/2) = \frac{5}{3}\cos\alpha$. It is elementary to see that to satisfy the inequality of the lemma, even with equality, $\cos\alpha \le 3/5$ is necessary, so $\alpha, \beta \ge \cos^{-1}(3/5)$, and then $\alpha + \beta + \gamma \ge 2\cos^{-1}(3/5) + \tan^{-1}(24/7) = \pi$.

So if $\gamma = B$, then all $(\alpha, \alpha) \in N_\gamma$, and by Lemma 13, this remains true for $\gamma > B$. □

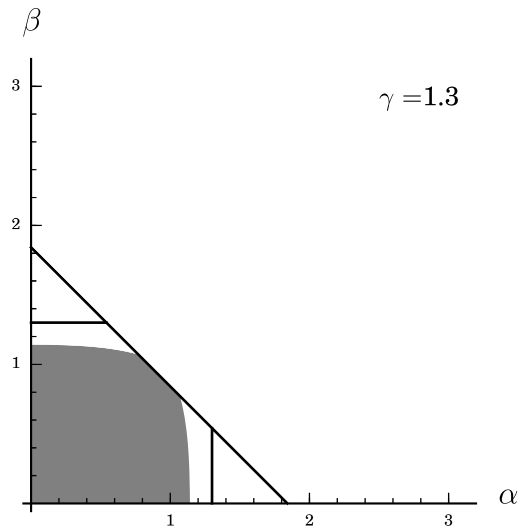


Figure 3. Similar to Figure 2, the shaded region represents $N_\gamma$, this time for $\gamma = 1.3$. See caption of Figure 2 for additional explanation of features.

**Lemma 15.** *Let $\gamma \in [B, \pi/2)$. Then the set of values of $\alpha$ for which $\alpha$ is negative-positive is the union of two open intervals $(0, e_\gamma)$ and $(\pi - \gamma - e_\gamma, i_\gamma)$, and $z$ is continuous and decreasing on these intervals. No value $\alpha$ is positive-negative. Furthermore, the region $N_\gamma$ is the region under $z$ and under the line $\alpha+\beta+\gamma = \pi$.*

See Figure 3 for illustration.

*Proof.* Let $\gamma \in [B, \pi/2)$. By [1], there exists $(\alpha, \beta) \notin \overline{P}_\gamma$ on the diagonal $\alpha+\beta+\gamma = \pi$. It is implicit in [3] that the set $\{(\alpha, \beta) \notin \overline{P}_\gamma : \alpha + \beta + \gamma = \pi\}$ is a closed line segment of the line $\alpha + \beta + \gamma = \pi$, and the endpoints of this line segment are the only points of $\overline{Z}_\gamma$ of the line. Also, this line segment is symmetric in $\alpha$ and $\beta$. Let the two endpoints of the line segment have coordinates $(e_\gamma, \pi - \gamma - e_\gamma)$, and $(\pi - \gamma - e_\gamma, e_\gamma)$.

By Lemma 14, the open line segment from $(0, 0)$ to $(\frac{\pi-\gamma}{2}, \frac{\pi-\gamma}{2})$ is entirely in $N_\gamma$. Let

$$T = \{(\alpha, \beta) : e_\gamma \leq \alpha, \beta \leq \pi - \gamma - e_\gamma \text{ and } \alpha + \beta + \gamma < \pi\}.$$

We will show that $T \subseteq N_\gamma \cup Z_\gamma$. Indeed, suppose a point $(\alpha, \beta)$ in the interior of $T$ belongs to $P_\gamma$. Without loss of generality $\alpha > \beta$. Then there are $\alpha_0 < \alpha < \alpha_1$ with $(\alpha_0, \beta), (\alpha_1, \beta) \in T \cap N_\gamma$, and so by continuity, there are $\alpha_0'$ and $\alpha_1'$ with $(\alpha_0', \beta), (\alpha_1', \beta) \in T \cap Z_\gamma$, contradicting the fact that $Z_\gamma$ is a function. The statement for the boundary of $T$ follows from Corollary 10.

If $\alpha < e_\gamma$, then $\alpha$ is negative-positive. This is because $(\alpha, \alpha) \in N_\gamma$ and $(\alpha, \min\{\pi - \gamma - \alpha, \gamma\}) \in \overline{P}_\gamma$. So $z_\gamma(\alpha)$ is defined on $(0, e_\gamma)$, and therefore it is continuous on this interval.

Now we will show that $\lim_{\alpha\to e_\gamma^-} z(\alpha) = \pi - \gamma - e_\gamma$. If this is not true, there is $\epsilon > 0$ and a sequence $\alpha_1, \alpha_2, \ldots$ with $\alpha_n \to e_\gamma$ such that $z(\alpha_n) < \pi - \gamma - e_\gamma - \epsilon$. Let $\beta_n = \pi - \gamma - e_\gamma - \epsilon/2$ (a constant sequence). Now the sequence $(\alpha_n, \beta_n)$ converges to the point $(e_\gamma, \pi - \gamma - e_\gamma - \epsilon/2)$, so a sequence of points in $P_\gamma$, converges to a point in $N_\gamma \cup Z_\gamma$. The only way this can happen if $(e_\gamma, \pi - \gamma - e_\gamma - \epsilon/2) \in Z_\gamma$. But the argument can be repeated with $\epsilon/3$ instead of $\epsilon/2$, so $(e_\gamma, \pi - \gamma - e_\gamma - \epsilon/3) \in Z_\gamma$, and this contradicts the fact that $Z_\gamma$ is a function.

Since $z(\alpha)$ is continuous and bijective on $(0, e_\gamma)$, it is monotone. We will show it must be decreasing. First we note that for $\gamma = B$, $z$ is clearly decreasing, because in that case $e_\gamma = \frac{\pi-\gamma}{2}$, and by symmetry, the function is "copied over" to the interval $(e_\gamma, i_\gamma)$, so it can not be increasing and bijective. Then, since $f$ is continuous, $z_\gamma(\alpha)$ is continuous in $\gamma$, so if $z_\gamma(\alpha_1) > z_\gamma(\alpha_2)$ for some $\alpha_1 < \alpha_2$ and $z_{\gamma'}(\alpha_1) < z_{\gamma'}(\alpha_2)$ for some $\gamma' > \gamma$, then by the Intermediate Value Theorem, there is a $\gamma < \gamma_0 < \gamma'$ for which $z_{\gamma_0}(\alpha_1) = z_{\gamma_0}(\alpha_2)$, a contradiction. Informally speaking, the function $z$ can not flip its monotonicity without failing injectivity at some point.

We have already seen that $\alpha$ is negative-positive on $(0, e_\gamma)$. By the fact that $z$ is decreasing on this interval, it is implied that $\alpha$ is all-negative on $[e_\gamma, \pi - \gamma - e_\gamma]$, and $\alpha$ is again negative-positive on $(\pi - \gamma - e_\gamma, i_\gamma)$. Finally, $\alpha$ is all-positive on $[i_\gamma, \gamma)$.

The last statement of the lemma is now clear. □

For the actual computations, we will need the value of $e_\gamma$. From [3], which describes the equality case for Euclidean geometry, we know that $e_\gamma$ is the value of $\alpha$ for which

$$\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = 1,$$

and since the triangle is Euclidean, we have $\alpha/2 + \beta/2 = \pi/2 - \gamma/2$. These equations yield two symmetric solutions for $\alpha$ and $\beta$; by our choice in the lemma, we need the smaller of these. We conclude

$$e_\gamma = 2\tan^{-1}\left(\frac{1}{2} - \sqrt{\tan\left(\frac{\gamma}{2}\right) - \frac{3}{4}}\right).$$

For the proof of the next result we let

$$\overline{S} = \{(\alpha, \beta, \gamma) \in F : f(\alpha, \beta, \gamma) \le 0\}$$

and note that $\overline{S}$ is the set of points in $F$ where the strong triangle inequality fails.

**Theorem 16.** *The probability that the strong triangle inequality holds is*

$$\frac{7}{8} - \frac{6}{\pi^3}\Bigg(\int_\Gamma^B \int_0^{i_\gamma} z_\gamma(\alpha)\,d\alpha d\gamma + \int_B^{\pi/2}\left(\frac{(\pi - \gamma - 2e_\gamma)^2}{2} - e_\gamma^2 + 2\int_0^{e_\gamma} z_\gamma(\alpha)\,d\alpha\right)d\gamma\Bigg).$$

*Proof.* We break up the integral

$$\int_{\Gamma}^{\pi/2} \mu(N_\gamma)\, d\gamma \tag{4}$$

over two intervals: $(\Gamma, B)$ and $(B, \pi/2)$. By Lemma 12, in the former interval, $N_\gamma$ is the region under the function $z_\gamma$. So if $\gamma \in (\Gamma, B)$, then $\int_\Gamma^B \mu(N_\gamma)\, d\gamma = \int_\Gamma^B \int_0^{i_\gamma} z_\gamma(\alpha)\, d\alpha\, d\gamma$. If $\gamma \in (B, \pi/2)$, then, by Lemma 15 and symmetry,

$$\int_B^{\pi/2} \mu(N_\gamma)\, d\gamma = \int_B^{\pi/2} \left( 2\int_0^{e_\gamma} z_\gamma(\alpha)\, d\alpha - e_\gamma^2 + (\pi - \gamma - 2e_\gamma)^2/2 \right) d\gamma.$$

Thus,

$$\mathrm{Vol}(\overline{S}) = \int_\Gamma^B \int_0^{i_\gamma} z_\gamma(\alpha)\, d\alpha d\gamma + \int_B^{\pi/2} \left( \frac{(\pi - \gamma - 2e_\gamma)^2}{2} - e_\gamma^2 + 2\int_0^{e_\gamma} z_\gamma(\alpha)\, d\alpha \right) d\gamma.$$

By Proposition 2, the strong triangle inequality does not hold if $\gamma \geq \frac{\pi}{2}$. The volume of the tetrahedron for $\gamma \geq \pi/2$ is $\pi^3/48$. Since the volume of the tetrahedron with $\gamma \geq 0$ is $\pi^3/6$ it follows that the required probability is

$$1 - \left( \frac{\mathrm{Vol}(\overline{S})}{\pi^3/6} + \frac{1}{8} \right),$$

and the formula follows. □

## 5. Theoretical error estimates

We are almost ready to use our favorite computer algebra system to compute the actual number. However, numerical integration will not guarantee accurate results in general. To make sure that we can (theoretically) control the error of computation, we need one more theorem.

**Theorem 17.** *The volume of $\overline{S}$ may be approximated by arbitrary precision. More precisely, for all $\epsilon > 0$ there is an algorithm to compute a numerical upper bound $M$ and a lower bound $m$ such that $m < \mathrm{Vol}(\overline{S}) < M$ and $M - m < \epsilon$.*

*Proof.* Lemma 13 implies that in (4) we integrate a monotone increasing function, because $\mu(N_\gamma)$ is the measure of the level set of $f$ at $\gamma$. Recall that for a monotone decreasing (respectively, increasing) function, the left Riemann sum overestimates (underestimates) the integral, and the right Riemann sum underestimates (overestimates) it. That is, it is possible to know how precise the the numerical estimate is, and if necessary, it is possible to repeat the computation with higher resolution.

In the actual computation given by Theorem 16, both terms in the parenthesis involve computations of integrals of monotone functions, and the inner integrals in those terms are also computing integrals of monotone functions. So, in essence, the numerical computation involves the integration of a monotone increasing function, whose values may be approximated at arbitrary precision. □

## 6. Conclusion

We can now use Theorem 16 and the computer algebra system Sage to get the following result.

**Corollary 18.** *Under the assumption that $\alpha, \beta, \gamma$ can be chosen uniformly in the interval $(0, \pi)$ and $\alpha + \beta + \gamma < \pi$, the strong triangle inequality $a + b > c + h$ holds approximately 78.67% of the time.*

Since we know that the strong triangle inequality fails when $\gamma \geq \pi/2$, we could restrict our attention to triangles where $\gamma < \pi/2$. In this case, the inequality $a + b > c + h$ holds approximately 90% of the time. For the Euclidean case, where $\alpha + \beta + \gamma = \pi$ and $\gamma < \pi/2$, it was shown in [3] that the strong triangle inequality holds approximately 92% of the time. Since the calculations in this paper involved volumes and the calculations in [3] involved areas, it is hard to directly compare the hyperbolic and Euclidean probabilities of the strong triangle inequality. We can say, however, that in both planes the strong triangle inequality is likely to hold.

## Appendix A. Sage code

The following code will visualize the value $a + b - c - h$ (referred as "strength") of a labelled triangle depending on the angles. It generates $2000$ pictures (or "frames"), and each frame will correspond to a fixed value of the angle $\gamma$, which grows throughout the frames from $0$ to $\pi/2$. The number of frames is defined with the variable `number`. For each frame, the strength is indicated for the angles $\alpha$, $\beta$, as the color of a point in the $(\alpha, \beta)$ coordinate system. Small positive strength is indicated by blue colors, high positive strength is indicated by red colors. The contours are changing from $0$ to $1$. Negative strength will be simply the darkest blue. To make the frames more informative, this darkest blue color may be replaced by a distinctive color outside of Sage (e.g. using Imagemagick). A black square on the bottom left corner indicates the points for which $\gamma$ is the greatest angle. Outside of this square, the strength is proven to be positive. The pictures are saved as numbered `png` files.

```
sage: def strength(al,be,ga): #this is a+b-c-h
...       cha=(cos(be)*cos(ga)+cos(al))/(sin(be)*sin(ga))
...       chb=(cos(al)*cos(ga)+cos(be))/(sin(al)*sin(ga))
...       chc=(cos(al)*cos(be)+cos(ga))/(sin(al)*sin(be))
...       a=arccosh(cha)
...       b=arccosh(chb)
...       c=arccosh(chc)
...       shb=sqrt(chb^2-1)
...       shh=shb*sin(al)
...       h=arcsinh(shh)
...       expression=a+b-c-h
...       return expression
...
sage: def defect(al,be,ga): return pi-al-be-ga
...
sage: var("al be ga")
sage: con=[]
sage: for i in xrange(50): con.append(i/50)
```

```
sage: map=sage.plot.colors.get_cmap('coolwarm')
sage: number=2000
sage: for i in xrange(number):
...       gamma=(i+1)*(pi/2)/(number)
...       p=contour_plot(strength(al,be,ga=gamma),(al,0,pi),(be,0,pi),
...           contours=con,cmap=map,plot_points=1000,
...           figsize=[10,10],region=defect(al,be,ga=gamma))
...       p+=line([(0,pi-gamma),(pi-gamma,0)],color='black')
...       p+=line([(0,gamma),(min(pi-2*gamma,gamma),gamma)],color='black')
...       p+=line([(gamma,0),(gamma,min(pi-2*gamma,gamma))],color='black')
...       p+=text("$\\gamma=$"+str(float(gamma)),(2.5,3),
...           vertical_alignment='top',horizontal_alignment='left')
...       p.save('hyper'+str(i).zfill(4)+'.png')
```

A video generated by this code can be found at

http://www.math.louisville.edu/~biro/movies/sti.mp4.

In this video, negative strength is represented by the color green. To generate the video, the following commands were executed in Bash (Linux Mint 17.1, ImageMagick and libav-tools installed). The reason of cropping in the second line is that the default mp4 encoder for avconv (libx264) requires even height and width.

```
for i in hyper*.png; do convert $i -fill green -opaque "#3b4cc0" x$i; done
avconv -i xhyper%04d.png -r 25 -vf "crop=2*trunc(iw/2):2*trunc(ih/2):0:0"
-b:v 500k sti.mp4
```

The following code performs the numerical computation of the integral. We are trying to follow the paper as close as possible, including notations. Note that the numerical integration is performed by Gaussian quadrature, so error bounds are not guaranteed in this code. We use the `mpmath` package and we store 100 decimal digits.

```
sage: from mpmath import *
sage: mp.dps=100
sage: Gamma=findroot(lambda x: -1-cos(x)+sin(x)+sin(x/2)*sin(x),1.15)
sage: Beta=atan(24/7)
...
sage: def i(gamma):
...       return acos(((sin(gamma)-1)^2+cos(gamma))/(2*sin(gamma)-cos(gamma)-1))
...       #return z(gamma,0) #This should give the same result
...
sage: def e(gamma):
...       D=tan(gamma/2)-3/4
...       if D<0:
...           sol=1/2
...       else:
...           sol=1/2-sqrt(D)
...       return 2*atan(sol)
...
sage: def z(gamma,alpha):
...       denominator=cos(gamma)+1-sin(gamma)
...       a=csc(gamma)^2-(cos(alpha)/denominator)^2
...       b=cos(alpha)*(cos(gamma)+1)/sin(gamma)^2
...       c=(cos(alpha)/sin(gamma))^2-((sin(gamma)-1)/denominator)^2
...       d=b^2-4*a*c
...       if d>=0:
...           sol=(-b-sqrt(d))/(2*a)
```

```
...        else:
...            sol=-b/(2*a)
...        if sol>1 or sol<-1:
...            result=0
...        else:
...            result=min(acos(sol),pi-alpha-gamma)
...        return result
...
sage: f = lambda gamma: quad(lambda alpha: z(gamma,alpha),[0,i(gamma)])
sage: g = lambda gamma: (pi-gamma-2*e(gamma))^2/2-e(gamma)^2+
....       2*quad(lambda alpha: z(gamma,alpha),[0,e(gamma)])
sage: int1=quad(f,[Gamma,Beta])
sage: int2=quad(g,[Beta,pi/2])
sage: print "Probability:", 7/8-(6/pi^3)*(int1+int2)
```

**Computer code**

A version of this paper extended with computer code is available on arXiv.

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