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Entropy encoding, Hilbert space, and Karhunen-Loève transforms

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By introducing Hilbert space and operators, we show how probabilities, approximations, and entropy encoding from signal and image processing allow precise formulas and quantitative estimates. Our main results yield orthogonal bases which optimize distinct measures of data encoding. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2793569]

I. INTRODUCTION

Historically, the Karhunen-Loève (KL) decomposition arose as a tool from the interface of probability theory and information theory (see details with references inside the paper). It has served as a powerful tool in a variety of applications, starting with the problem of separating variables in stochastic processes, say, X_t, and processes that arise from statistical noise, for example, from fractional Brownian motion. Since the initial inception in mathematical statistics, the operator algebraic contents of the arguments have crystallized as follows: start from the process X_t, for simplicity assume zero mean, i.e., $E(X_t)=0$, and create a correlation matrix $C(s,t) = E(X_s X_t)$. (Strictly speaking, it is not a matrix, but rather an integral kernel. Nonetheless, the matrix terminology has stuck.) The next key analytic step in the Karhunen-Loève method is to then apply the spectral theorem from operator theory to a corresponding self-adjoint operator, or to some operator naturally associated with C: hence, the name the Karhunen-Loève decomposition (KLD). In favorable cases (discrete spectrum), an orthogonal family of functions $[f_n(t)]$ in the time variable arise and a corresponding family of eigenvalues. We take them to be normalized in a suitably chosen square norm. By integrating the basis functions $f_n(t)$ against X_t, we get a sequence of random variables Y_n. It was the insight of KL (Ref. 25) to give general conditions for when this sequence of random variables is independent and to show that if the initial random process X_t is Gaussian, then so are the random variables Y_n (see also Example 3.1 below.)

In the 1940s, Karhunen[21,22] pioneered the use of spectral theoretic methods in the analysis of time series and more generally in stochastic processes. It was followed up by the papers and books by Loève in the 1950s (Ref. 25) and in 1965 by Ash.[3,8] (Note that this theory precedes the surge in the interest in wavelet bases!)

As we outline below, all the settings impose rather stronger assumptions. We argue how more modern applications dictate more general theorems, which we prove in our paper. A modern tool from operator theory and signal processing which we will use is the notion of *frames* in Hilbert space. More precisely, frames are redundant "bases" in Hilbert space. They are called framed but intuitively should be thought of as generalized bases. The reason for this, as we show, is that they

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offer an explicit choice of a (nonorthogonal) expansion of vectors in the Hilbert space under consideration.[14]

In our paper, we rely on the classical literature (see, e.g., Ref. 3) and we accomplish three things: (i) we extend the original KL idea to the case of continuous spectrum; (ii) we give frame theoretic uses of the KL idea which arise in various wavelet contexts and which go beyond the initial uses of KL; and finally (iii) to give applications.[17]

These applications in our case come from image analysis, specifically from the problem of statistical recognition and detection, e.g., to nonlinear variance, for example, due to illumination effects. Then the KLD, also known as principal component analysis (PCA), applies to the intensity images. This is traditional in statistical signal detection and in estimation theory. Adaptations to compression and recognition are of a more recent vintage. In brief outline, each intensity image is converted into vector form.[26] (This is the simplest case of a purely intensity-based coding of the image, and it is not necessarily ideal for the application of KLDs.)

The ensemble of vectors used in a particular conversion of images is assumed to have a multivariate Gaussian distribution since human faces form a dense cluster in image space. The PCA method generates small set of basis vectors forming subspaces whose linear combination offer better (or perhaps ideal) approximation to the original vectors in the ensemble.[27] In facial recognition, the new bases are said to span intraface and interface variations, permitting Euclidean distance measurements to exclusively pick up changes in, for example, identity and expression.

Our presentation will start with various operator theoretic tools, including frame representations in Hilbert space. We have included more details and more explanations than is customary in more narrowly focused papers, as we wish to cover the union of four overlapping fields of specialization, operator theory, information theory, wavelets, and physics applications.

While entropy encoding is popular in engineering,[30,33,11] the choices made in signal processing are often more by trial and error than by theory. Reviewing the literature, we found that the mathematical foundation of the current use of entropy in encoding deserves closer attention.

In this paper we take advantage of the fact that Hilbert space and operator theory form the common language of both quantum mechanics and of signal/image processing. Recall first that in quantum mechanics, (pure) states as mathematical entities "are" one-dimensional subspaces in complex Hilbert space $\mathcal{H}$, so we may represent them by vectors of norm 1. Observables "are" self-adjoint operators in $\mathcal{H}$, and the measurement problem entails von Neumann's spectral theorem applied to the operators.

In signal processing, time series or matrices of pixel numbers may similarly be realized by vectors in Hilbert space $\mathcal{H}$. The probability distribution of quantum mechanical observables (state space $\mathcal{H}$) may be represented by choices of orthonormal bases (ONBs) in $\mathcal{H}$ in the usual way (see, e.g., Ref. 19). In signal/image processing, because of aliasing, it is practical to generalize the notion of ONB, and this takes the form of what is called "a system of frame vectors" (see Ref. 7).

However, even von Neumann's measurement problem, viewing experimental data as part of a bigger environment (see, e.g., Refs. 13, 36, and 15) leads to basis notions more general than ONBs. They are commonly known as positive operator valued measures (POVMs), and in the present paper we examine the common ground between the two seemingly different uses of operator theory in the separate applications. To make the paper presentable to two audiences, we have included a few more details than is customary in pure math papers.

We show that parallel problems in quantum mechanics and in signal processing entail the choice of "good" ONBs. One particular such ONB goes under the name "the KL basis." We will show that it is optimal in three ways and we will outline a number of applications.

The problem addressed in this paper is motivated by consideration of the optimal choices of bases for certain analog-to-digital problems we encountered in the use of wavelet bases in image processing (see Refs. 16, 30, 33, and 34), but certain of our considerations have an operator theoretic flavor which we wish to isolate, as it seems to be of independent interest.

There are several reasons why we take this approach. First, our Hilbert space results seem to

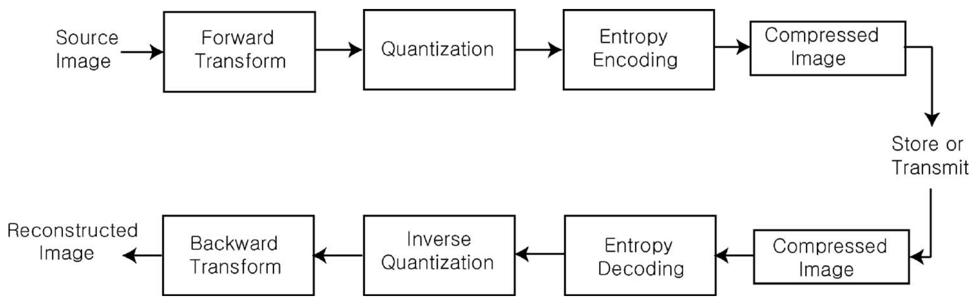

FIG. 1. Outline of the wavelet image compression process (Ref. 28).

be of general interest outside the particular applied context where we encountered them. Second, we feel that our more abstract results might inspire workers in operator theory and approximation theory.

A. Digital image compression

In digital image compression, after the quantization (see Fig. 1) entropy encoding is performed on a particular image for more efficientless memory storage. When an image is to be stored, we need either 8 bits or 16 bits to store a pixel. With efficient entropy encoding, we can use a smaller number of bits to represent a pixel in an image, resulting in less memory used to store or even transmit an image. Thus, the KL theorem enables us to pick the best basis thus to minimize the entropy and error, to better represent an image for optimal storage or transmission. Here, *optimal* means that it uses a least memory space to represent the data, i.e., instead of using 16 bits, it uses 11 bits. So, the best basis found would allow us to better represent the digital image with less storage memory.

In the next section, we give the general context and definitions from operators in Hilbert space which we shall need. We discuss the particular ONBs and frames which we use, and we recall the operator theoretic context of the KL theorem.[3] In approximation problems involving a stochastic component (for example, noise removal in time series or data resulting from image processing), one typically ends up with correlation kernels, in some cases as frame kernels (see details in Sec. IV). In some cases, they arise from systems of vectors in Hilbert space which form frames (see Definition 4.2). In some cases, parts of the frame vectors fuse (fusion frames) onto closed subspaces, and we will be working with the corresponding family of (orthogonal) projections. Either way, we arrive at a family of self-adjoint positive semidefinite operators in Hilbert space. The particular Hilbert space depends on the application at hand. While the spectral theorem does allow us to diagonalize these operators, the direct application of the spectral theorem may lead to continuous spectrum which is not directly useful in computations, or it may not be computable by recursive algorithms. As a result, we introduce in Sec. VI a weighting of the operator to be analyzed.[29]

The questions we address are the optimality of approximation in a variety of ONBs and the choice of the "best" ONB. Here, best is given two precise meanings: (1) in the computation of a sequence of approximations to the frame vectors, the error terms must be smallest possible, and similarly (2) we wish to minimize the corresponding sequence of entropy numbers (referring to von Neumann's entropy). In two theorems, we make precise an operator theoretic KL basis, which we show is optimal both in regard to criteria (1) and (2). However, before we prove our theorems, we give the two problems an operator theoretic formulation and, in fact, our theorems are stated in this operator theoretic context.

In Sec. VI, we introduce the weighting and we address a third optimality criteria, that of optimal weights. Among all the choices of weights (taking the form of certain discrete probability distributions), turning the initially given operator into trace class, the problem is then to select the particular weights which are optimal in a sense which we define precisely.

The general background references we have used for standard facts on wavelets, information theory, and Hilbert space are Refs. 8, 14, 17, 26, 27, and 29.

II. GENERAL BACKGROUND

A. From data to Hilbert space

Computing probabilities and entropy Hilbert space serves as a helpful tool. As an example, take a unit vector f in some fixed Hilbert space $\mathcal{H}$ and an orthonormal basis (ONB) ψ_i with i running over an index set I. With this, we now introduce two families of probability measures, one family $P_f(\cdot)$ indexed by $f \in \mathcal{H}$ and a second family P_T indexed by a class of operators $T:\mathcal{H} \to \mathcal{H}$.

1. The measures P_f Define

$$P_f(A) = \sum_{i\in A} |\langle \psi_i | f\rangle|^2, \tag{2.1}$$

where $A \subset I$ and where $\langle \cdot | \cdot \rangle$ denotes the inner product. Following physics convention, we make our inner product linear in the second variable. This will also let us make use of Dirac's convenient notation for rank-1 operators, see Eq. (2.5) below.

Note then that $P_f(A)$ is a probability measure on the finite subsets A of I. To begin with, we make the restriction to finite subsets. This is merely for later use in recursive systems, see, e.g., Eq. (2.2). In diverse contexts, extensions from finite to infinite is then done by means of Kolmogorov's consistency principle.[24]

By introducing a weighting, we show that this assigment also works for more general vector configurations $\mathcal{C}$ than ONBs. Vectors in $\mathcal{C}$ may represent signals or image fragments/blocks. Correlations would then be measured as inner products $\langle u | v\rangle$ with u and v representing different image pixels. In the case of signals, u and v might represent different frequency subbands.

2. The measures P_T

A second more general family of probability measures arising in the context of Hilbert space is called determinantal measures. Specifically, consider bitstreams as points in an infinite Cartesian product $\Omega = \Pi_{i\in\mathbb{N}}\{0,1\}$. Cylinder sets in Ω are indexed by finite subsets $A \subset \mathbb{N}$,

$$C_A = \{(\omega_1, \omega_2, \ldots) | \omega_i = 1 \text{ for } i \in A\}.$$

If T is an operator in $l^2(\mathbb{N})$ such that $0 \leq \langle u | Tu\rangle \leq \|u\|^2$ for all $u \in l^2$, then set

$$P_T(C_A) = \det(T(i,j))_{i,j\in A}, \tag{2.2}$$

where $T(i,j)$ is the matrix representation of T computed in some ONB in l^2. Using general principles,[24,19] it can be checked that $P_T(C_A)$ is independent of the choice of ONB.

To verify that $P_T(\cdot)$ extends to a probability measure defined on the sigma-algebra generated by $C_A s$, see, e.g., Ref. 19, Chap. 7. The argument is based on Kolmogorov's consistency principle, see Ref. 23.

Frames (Definition 4.3) are popular in analyzing signals and images. This fact raises questions of comparing two approximations: one using a frame and the other using an ONB. However, there are several possible choices of ONBs. An especially natural choice of an ONB would be that one diagonalizes the matrix $(\langle f_i | f_j\rangle)$, where (f_i) is a frame. We call such a choice of ONB Kahunen-Loève (KL) expansion. Section III deals with a continuous version of this matrix problem. The justification for why diagonalization occurs and works also when the frame (f_i) is infinite is based on the spectral theorem. For the details regarding this, see the proof of Theorem 3.3 below.

In symbols, we designate the KL ONB associated with the frame (f_i) as (ϕ_i). In computations, we must rely on finite sums, and we are interested in estimating the errors when different approximations are used and where summations are truncated. Our main results make precise how the KL

ONB yields better approximations, smaller entropy, and better synthesis. Even more, we show that infimum calculations yield minimum numbers attained at the KL ONB expansions. We emphasize that the ONB depends on the frame chosen, and this point will be discussed in detail later.

If larger systems are subdivided, the smaller parts may be represented by projections P_i and the $i-j$ correlations by the operators P_iP_j. The entire family (P_i) is to be treated as a fusion frame.[5,6] Fusion frames are defined in Definition 4.12 below. Frames themselves are generalized bases with redundancy, for example, occurring in signal processing involving multiplexing. The fusion frames allow decompositions with closed subspaces as opposed to individual vectors. They allow decompositions of signal/image processing tasks with degrees of homogeneity.

B. Definitions

Definition 2.1: Let $\mathcal{H}$ be a Hilbert space. Let (ψ_i) and (ϕ_i) be ONBs, with index set I. Usually

$$I = \mathbb{N} = \{1,2,\ldots\}. \tag{2.3}$$

If $(\psi_i)_{i\in I}$ is an ONB, we set $Q_n :=$ the orthogonal projection onto span $\{\psi_1,\ldots,\psi_n\}$.

We now introduce a few facts about operators which will be needed in the paper. In particular, we recall Dirac's terminology[10] for rank-1 operators in Hilbert space. While there are alternative notation available, Dirac's bra-ket terminology is especially efficient for our present considerations.

Definition 2.2: Let vectors $u, v \in \mathcal{H}$. Then

$$\langle u|v\rangle = \text{inner product} \in \mathbb{C}, \tag{2.4}$$

$$|u\rangle\langle v| = \text{rank-1 operator}, \quad \mathcal{H}\to\mathcal{H}, \tag{2.5}$$

where the operator $|u\rangle\langle v|$ acts as follows:

$$|u\rangle\langle v|w = |u\rangle\langle v|w\rangle = \langle v|w\rangle u \quad \text{for all } w \in \mathcal{H}. \tag{2.6}$$

Dirac's bra-ket and ket-bra notation is popular in physics, and it is especially convenient in working with rank-1 operators and inner products. For example, in the middle term in Eq. (2.6), the vector u is multiplied by a scalar (the inner product) and the inner product comes about by just merging the two vectors.

C. Facts

The following formulas reveal the simple rules for the algebra of rank-1 operators, their composition, and their adjoints:

$$|u_1\rangle\langle v_1||u_2\rangle\langle v_2| = \langle v_1|u_2\rangle|u_1\rangle\langle v_2| \tag{2.7}$$

and

$$|u\rangle\langle v|^* = |v\rangle\langle u|. \tag{2.8}$$

In particular, formula (2.7) shows that the product of two rank-1 operators is again rank-1. The inner product $\langle v_1|u_2\rangle$ is a measure of a correlation between the two operators on the left-hand side of (2.7).

If S and T are bounded operators in $\mathcal{H}$, in $B(\mathcal{H})$, then

$$S|u\rangle\langle v|T = |Su\rangle\langle T^*v|. \tag{2.9}$$

If $(\psi_i)_{i\in\mathbf{N}}$ is an ONB, then the projection

$$Q_n \coloneqq \text{proj span}\{\psi_1, \ldots, \psi_n\}$$

is given by

$$Q_n = \sum_{i=1}^{n} |\psi_i\rangle\langle\psi_i|, \tag{2.10}$$

and for each i, $|\psi_i\rangle\langle\psi_i|$ is the projection onto the one-dimensional subspace $\mathbf{C}\psi_i \subset \mathcal{H}$.

III. THE KAHUNEN-LOÈVE TRANSFORM

In general, one refers to a *KL transform* as an expansion in Hilbert space with respect to an ONB resulting from an application of the spectral theorem.

Example 3.1: Suppose that X_t is a stochastic process indexed by t in a finite interval J and takes values in $L^2(\Omega, P)$ for some probability space (Ω, P). Assume the normalization $E(X_t)=0$. Suppose that the integral kernel $E(X_t X_s)$ can be diagonalized, i.e., suppose that

$$\int_J E(X_t X_s)\phi_k(s)\mathrm{d}s = \lambda_k \phi_k(t),$$

with an ONB (ϕ_k) in $L^2(J)$. If $E(X_t)=0$, then

$$X_t(\omega) = \sum_k \sqrt{\lambda_k}\phi_k(t) Z_k(\omega), \quad \omega \in \Omega,$$

where $E(Z_j Z_k)=\delta_{j,k}$ and $E(Z_k)=0$. The ONB (ϕ_k) is called the *KL basis* with respect to the stochastic processes $\{X_t : t \in I\}$.

The KL theorem[3] states that if (X_t) is Gaussian, then so are the random variables (Z_k). Furthermore, they are $N(0,1)$, i.e., normal with mean zero and variance 1, independent and identically distributed. This last fact explains the familiar *optimality* of KL in transform coding.

Remark 3.2: Consider the case when

$$E(X_t X_s) = \tfrac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}),$$

and $H \in (0,1)$ is fixed. If $J=\mathbb{R}$ in the above application of KL to stochastic processes, then it is possible by a fractional integration to make the $L^2(\mathbb{R})$ ONB consist of wavelets, i.e.,

$$\psi_{j,k}(t) \coloneqq 2^{j/2}\psi(2^j t - k), \quad j,k \in \mathbb{Z}, \quad \text{i.e., double indexed,} \quad t \in \mathbb{R} \quad \text{for some } \psi \in L^2(\mathbb{R}),$$

see, e.g., Ref. 19. The process X_t is called H-fractional Brownian motion, as outlined in, e.g., Ref. 19, p. 57.

The following theorem makes clear the connection to the Hilbert space geometry as used in present paper:

Theorem 3.3: *Let (Ω, P) be a probability space, $J \subset \mathbb{R}$ an interval (possibly infinite), and let $(X_t)_{t\in J}$ be a stochastic process with values in $L^2(\Omega, P)$. Assume that $E(X_t)=0$ for all $t \in J$. Then $L^2(J)$ splits as an orthogonal sum*

$$L^2(J) = \mathcal{H}_d \oplus \mathcal{H}_c \tag{3.1}$$

(d is for discrete and c is for continuous) such that the following data exists.

(a) $(\phi_k)_{k\in\mathbb{N}}$ *an ONB in* $\mathcal{H}_d$.
(b) $(Z_k)_{k\in\mathbb{N}}$: *independent random variables*.
(c) $E(Z_j Z_k)=\delta_{j,k}$ *and* $E(Z_k)=0$.
(d) $(\lambda_k) \subset \mathbb{R}_{\geq 0}$.
(e) $\phi(\cdot,\cdot)$*: a Borel measure on $\mathbb{R}$ in the first variable, such that* (i) $\phi(E,\cdot) \in \mathcal{H}_c$, *where E is an open subinterval of J, and* (ii) $\langle \phi(E_1,\cdot) | \phi(E_2,\cdot)\rangle_{L^2(J)}=0$ *whenever* $E_1 \cap E_2 = \varnothing$.

(f) $Z(\cdot,\cdot)$: *a measurable family of random variables such that* $Z(E_1,\cdot)$ *and* $Z(E_2,\cdot)$ *are independent when* $E_1,E_2 \in \mathcal{B}_J$ *and* $E_1 \cap E_2 = \emptyset$,

$$E(Z(\lambda,\cdot)Z(\lambda',\cdot)) = \delta(\lambda-\lambda') \text{ and } E(Z(\lambda,\cdot)) = 0.$$

Finally, we get the following KL expansions for the $L^2(J)$ *operator with integral kernel* $E(X_tX_s)$:

$$\sum_{k\in\mathbb{N}} \lambda_k |\phi_k\rangle\langle\phi_k| + \int_J \lambda |\phi(\mathrm{d}\lambda,\cdot)\rangle\langle\phi(\mathrm{d}\lambda,\cdot)|. \tag{3.2}$$

Moreover, the process decomposes; thus,

$$X_t(\omega) = \sum_{k\in\mathbb{N}} \sqrt{\lambda_k} Z_k(\omega)\phi_k(t) + \int_J \sqrt{\lambda} Z(\lambda,\omega)\phi(\mathrm{d}\lambda,t). \tag{3.3}$$

Proof: By assumption, the integral operator in $L^2(J)$ with kernel $E(X_tX_s)$ is self-adjoint, positive semidefinite, but possibly unbounded. By the spectral theorem, this operator has the following representation:

$$\int_0^\infty \lambda Q(\mathrm{d}\lambda),$$

where $Q(\cdot)$ is a projection valued measure defined on the Borel subsets $\mathcal{B}$ of $\mathbb{R}_{\geqslant 0}$. Recall that

$$Q(S_1 \cap S_2) = Q(S_1)Q(S_2) \quad \text{for } S_1,S_2 \in \mathcal{B},$$

and $\int_0^\infty Q(\mathrm{d}\lambda)$ is the identity operator in $L^2(J)$. The two closed subspaces $\mathcal{H}_d$ and $\mathcal{H}_c$ in the decomposition (3.1) are the discrete and continuous parts of the projection valued measure Q, i.e., Q is discrete (or atomic) on $\mathcal{H}_d$, and it is continuous on $\mathcal{H}_c$.

Consider first

$$Q_d(\cdot) = Q(\cdot)|_{\mathcal{H}_d},$$

and let (λ_k) be the atoms. Then for each k, the nonzero projection $Q(\{\lambda_k\})$ is a sum of rank-1 projections $|\phi_k\rangle\langle\phi_k|$ corresponding to a choice of ONB in the λ_k subspace. [Usually, the multiplicity is 1, in which case $Q(\{\lambda_k\}) = |\phi_k\rangle\langle\phi_k|$.] This accounts for the first terms in representations (3.2) and (3.3).

We now turn to the continuous part, i.e., the subspace $\mathcal{H}_c$, and the continuous projection valued measure,

$$Q_c(\cdot) = Q(\cdot)|_{\mathcal{H}_c}.$$

The second terms in the two formulas (3.2) and (3.3) result from an application of a disintegration theorem from Ref. 12, Theorem 3.4. This theorem is applied to the measure $Q_c(\cdot)$.

We remark for clarity that the term $|\phi(\mathrm{d}\lambda,\cdot)\rangle\langle\phi(\mathrm{d}\lambda,\cdot)|$ under the integral sign in (3.2) is merely a measurable field of projections $P(\mathrm{d}\lambda)$. □

Our adaptation of the spectral theorem from books on operator theory (e.g., Ref. 19) is made with a view to the application at hand, and our version of Theorem 3.3 serves to make the adaptation to how operator theory is used for time series and for encoding. We have included it here because it is not written precisely this way elsewhere.

IV. FRAME BOUNDS AND SUBSPACES

The word "frame" in the title refers to a family of vectors in Hilbert space with basislike properties which are made precise in Definition 4.2. We will be using entropy and information as defined classically by Shannon,[28] and extended to operators by von Neumann.[18]

Reference 3 offers a good overview of the basics of both. Shannon's pioneering idea was to quantify digital "information," essentially as the negative of entropy, entropy being a measure of "disorder." This idea has found a variety of application in both signal/image processing and in quantum information theory, see, e.g., Ref. 23. A further recent use of entropy is in digital encoding of signals and images, compressing and quantizing digital information into a finite floating-point computer register. (Here, we use the word "quantizing"[30,31,28] in the sense of computer science.) To compress data for storage, an encoding is used, which takes into consideration the probability of occurrences of the components to be quantized; hence, entropy is a gauge for the encoding.

Definition 4.1: $T \in B(\mathcal{H})$ is said to be trace class if and only if $\Sigma\langle\psi_i|T\psi_i\rangle$ is absolutely convergent for some ONB (ψ_i). In this case, set

$$\mathrm{tr}(T) := \sum_{i}^{n} \langle\psi_i|T\psi_i\rangle. \tag{4.1}$$

Definition 4.2: A sequence $(h_\alpha)_{\alpha\in A}$ in $\mathcal{H}$ is called a frame if there are constants $0<c_1\leq c_2 <\infty$ such that

$$c_1\|f\|^2 \leq \sum_{\alpha\in A} |\langle h_\alpha|f\rangle|^2 \leq c_2\|f\|^2 \quad \text{for all } f\in\mathcal{H}. \tag{4.2}$$

Definition 4.3: Suppose that we are given a frame operator

$$G = \sum_{\alpha\in A} w_\alpha|f_\alpha\rangle\langle f_\alpha| \tag{4.3}$$

and an ONB (ψ_i). Then for each n, the numbers,

$$E_n^\psi = \sum_{\alpha\in A} w_\alpha\|f_\alpha - \sum_{i=1}^{n}\langle\psi_i|f_\alpha\rangle\psi_i\|^2, \tag{4.4}$$

are called the error terms.

Set $L:\mathcal{H}\to l^2$,

$$L: f \mapsto (\langle h_\alpha|f\rangle)_{\alpha\in A}. \tag{4.5}$$

Lemma 4.4: If L is as in (4.5), then $L^*:l^2\to\mathcal{H}$ is given by

$$L^*((c_\alpha)) = \sum_{\alpha\in A} c_\alpha h_\alpha, \tag{4.6}$$

where $(c_\alpha)\in l^2$, and

$$L^*L = \sum_{\alpha\in A} |h_\alpha\rangle\langle h_\alpha|. \tag{4.7}$$

Lemma 4.5: If (f_α) are the normalized vectors resulting from a frame (h_α), i.e., $h_\alpha=\|h_\alpha f_\alpha\|$, and $w_\alpha:=\|h_\alpha\|^2$, then L^*L has form (4.8).

Proof: The desired conclusion follows from the Dirac formulas (2.7) and (2.8). Indeed,

$$|h_\alpha\rangle\langle h_\alpha| = |\|h_\alpha\|f_\alpha\rangle\langle\|h_\alpha\|f_\alpha| = \|h_\alpha\|^2|f_\alpha\rangle\langle f_\alpha| = w_\alpha P_\alpha,$$

where P_α satisfies the two rules $P_\alpha=P_\alpha^*=P_\alpha^2$. □

Definition 4.6: Suppose that we are given $(f_\alpha)_{\alpha\in A}$, a frame (non-negative numbers $\{w_\alpha\}_{\alpha\in A}$), where A is an index set, with $\|f_\alpha\|=1$, for all $\alpha\in A$.

$$G := \sum_{\alpha \in A} w_\alpha |f_\alpha\rangle\langle f_\alpha| \tag{4.8}$$

is called a *frame* operator associated with (f_α).

Lemma 4.7: Note that G is trace class if and only if $\Sigma_\alpha w_\alpha < \infty$, and then

$$\mathrm{tr}\, G = \sum_{\alpha \in A} w_\alpha. \tag{4.9}$$

Proof: Identity (4.9) follows from the fact that all the rank-1 operators $|u\rangle\langle v|$ are trace class, with

$$\mathrm{tr}|u\rangle\langle v| = \sum_{i=1}^{n} \langle \psi_i|u\rangle\langle v|\psi_i\rangle = \langle u|v\rangle.$$

In particular, $\mathrm{tr}|u\rangle\langle u| = \|u\|^2$. □

We shall consider more general frame operators,

$$G = \sum_{\alpha \in A} w_\alpha P_\alpha, \tag{4.10}$$

where (P_α) is an indexed family of projections in $\mathcal{H}$, i.e., $P_\alpha = P_\alpha^* = P_\alpha^2$, for all $\alpha \in A$. Note that P_α is trace class if and only if it is finite dimensional, i.e., if and only if the subspace $P_\alpha \mathcal{H} = \{x \in \mathcal{H} | P_\alpha x = x\}$ is finite dimensional.

When (ψ_i) is given, set $Q_n := \Sigma_{i=1}^n |\psi_i\rangle\langle\psi_i|$ and $Q_n^\perp = I - Q_n$, where I is the identity operator in $\mathcal{H}$.

Lemma 4.8:

$$E_n^\psi = \mathrm{tr}(G Q_n^\perp). \tag{4.11}$$

Proof: The proof follows from the previous facts, using that

$$\|f_\alpha - Q_n f_\alpha\|^2 = \|f_\alpha\|^2 - \|Q_n f_\alpha\|^2,$$

for all $\alpha \in A$ and $n \in \mathbf{N}$. Expression (4.4) for the error term is motivated as follows. The vector components f_α in Definition 4.8 are indexed by $\alpha \in A$ and are assigned weights w_α. However, rather than computing $\Sigma_\alpha w_\alpha$ as in Lemma 4.7, we wish to replace the vectors f_α with finite approximations $Q_n f_\alpha$, then the error term (4.4) measures how well the approximation fits the data.

Lemma 4.9: $\mathrm{tr}(G Q_n) = \Sigma_{\alpha \in A} w_\alpha \|Q_n f_\alpha\|^2$.

Proof:

$$\mathrm{tr}(GQ_n) = \sum_i \langle \psi_i | G Q_n \psi_i\rangle = \sum_i \langle Q_n \psi_i | G Q_n \psi_i\rangle = \sum_{i=1}^{n} \langle \psi_i | G \psi_i\rangle = \sum_{\alpha \in A} w_\alpha \sum_{i=1}^{n} |\langle \psi_i | f_\alpha\rangle|^2 = \sum_{\alpha \in A} w_\alpha \|Q_n f_\alpha\|^2,$$

as claimed. □

Proof of Lemma 4.8 continued: The relative error is represented by the difference

$$\begin{aligned}\sum_{\alpha \in A} w_\alpha - \sum_{\alpha \in A} w_\alpha \|Q_n f_\alpha\|^2 &= \sum_{\alpha \in A} w_\alpha \|f_\alpha\|^2 - \sum_{\alpha \in A} w_\alpha \|Q_n f_\alpha\|^2 \\ &= \sum_{\alpha \in A} w_\alpha (\|f_\alpha\|^2 - \|Q_n f_\alpha\|^2) \\ &= \sum_{\alpha \in A} w_\alpha \|f_\alpha - Q_n f_\alpha\|^2 = \sum_{\alpha \in A} w_\alpha \|Q_n^\perp f_\alpha\|^2 = \mathrm{tr}(G Q_n^\perp).\end{aligned}$$

□

Definition 4.10: If G is a more general frame operator (4.10) and (ψ_i) is some ONB, we shall set $E_n^\psi := \mathrm{tr}(G Q_n^\perp)$; this is called the error sequence.

The more general case of (4.10) where

$$\operatorname{rank} P_\alpha = \operatorname{tr} P_\alpha > 1 \tag{4.12}$$

corresponds to what are called subspace frames, i.e., indexed families (P_α) of orthogonal projections such that there are $0 < c_1 \leqslant c_2 < \infty$ and weights $w_\alpha \geqslant 0$ such that

$$c_1 \|f\|^2 \leqslant \sum_{\alpha \in A} w_\alpha \|P_\alpha f\|^2 \leqslant c_2 \|f\|^2, \tag{4.13}$$

for all $f \in \mathcal{H}$.

We now make these notions precise.

Definition 4.11: A *projection* in a Hilbert space $\mathcal{H}$ is an operator P in $\mathcal{H}$ satisfying $P^* = P = P^2$. It is understood that our projections P are orthogonal, i.e., that P is a *self-adjoint* idempotent. The orthogonality is essential because, by von Neumann, we know that there is then a 1-1 correspondence between closed subspaces in $\mathcal{H}$ and (orthogonal) projections: every closed subspace in $\mathcal{H}$ is the range of a unique projection.

We shall need the following generalization of the notion, Definition 4.2, of the frame.

Definition 4.12: A *fusion frame* (or subspace frame) in a Hilbert space $\mathcal{H}$ is an indexed system (P_i, w_i) where each P_i is a projection and where (w_i) is a system of numerical weights, i.e., $w_i > 0$, such that (4.13) holds: specifically, the system (P_i, w_i) is called a fusion frame when (4.13) holds.

It is clear (see also Sec. VI) that the notion of "fusion frame" contains conventional frames, Definition 4.2, as a special case.

The property (4.13) for a given system controls the weighted overlaps of the variety of subspaces $\mathcal{H}_i (:= P_i(\mathcal{H}))$ making up the system, i.e., the intersections of subspaces corresponding to different values of the index. Typically, the pairwise intersections are nonzero. The case of zero pairwise intersections happens precisely when the projections are orthogonal, i.e., when $P_i P_j = 0$ for all pairs with i and j different. In frequency analysis, this might represent orthogonal frequency bands.

When vectors in $\mathcal{H}$ represent signals, we think of bands of signals being "fused" into the individual subspaces $\mathcal{H}_i$. Further, note that for a given system of subspaces or, equivalently, projections, there may be many choices of weights consistent with (4.13). The overlaps may be controlled or weighted in a variety of ways. The choice of weights depends on the particular application at hand.

Theorem 4.13: *The KL ONB with respect to the frame operator L^*L gives the smallest error terms in the approximation to a frame operator.*

Proof: Given the operator G which is trace class and positive semidefinite, we may apply the spectral theorem to it. What results is a discrete spectrum, with the natural order $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ and a corresponding ONB (ϕ_k) consisting of eigenvectors, i.e.,

$$G\phi_k = \lambda_k \phi_k, \quad k \in \mathbb{N}, \tag{4.14}$$

called the KL data. The spectral data may be constructed recursively starting with

$$\lambda_1 = \sup_{\phi \in \mathcal{H}, \|\phi\|=1} \langle \phi | G\phi \rangle = \langle \phi_1 | G\phi_1 \rangle \tag{4.15}$$

and

$$\lambda_{k+1} = \sup_{\substack{\phi \in \mathcal{H}, \|\phi\|=1 \\ \phi \perp \phi_1, \phi_2, \ldots, \phi_k}} \langle \phi | G\phi \rangle = \langle \phi_{k+1} | G\phi_{k+1} \rangle. \tag{4.16}$$

Now an application of Ref. 2, Theorem 4.1 yields

$$\sum_{k=1}^{n} \lambda_k \geq \operatorname{tr}(Q_n^{\psi} G) = \sum_{k=1}^{n} \langle \psi_k | G \psi_k \rangle \quad \text{for all } n, \tag{4.17}$$

where Q_n^{ψ} is the sequence of projections from (2.10), deriving from some ONB (ψ_i) and arranged such that

$$\langle \psi_1 | G \psi_1 \rangle \geq \langle \psi_2 | G \psi_2 \rangle \geq \cdots . \tag{4.18}$$

Hence, we compare ordered sequences of eigenvalues with sequences of diagonal matrix entries.

Finally, we have

$$\operatorname{tr} G = \sum_{k=1}^{\infty} \lambda_k = \sum_{k=1}^{\infty} \langle \psi_k | G \psi_k \rangle < \infty .$$

The assertion in Theorem 4.13 is the validity of

$$E_n^{\phi} \leq E_n^{\psi}, \tag{4.19}$$

for all $(\psi_i) \in \mathrm{ONB}(\mathcal{H})$ and all $n=1,2,\ldots$, and, moreover, that the infimum on the right-hand side (RHS) in (4.19) is attained for the KL-ONB (ϕ_k). However, in view of our lemma for E_n^{ψ}, Lemma 4.8, we see that (4.19) is equivalent to the system (4.17) in the Arveson-Kadison theorem. □

The Arveson-Kadison theorem is the assertion (4.17) for trace-class operators, see e.g., Refs. 1 and 2 That (4.19) is equivalent to (4.17) follows from the definitions.

Remark 4.14: Even when the operator G is not trace class, there is still a conclusion about the relative error estimates. With two (ϕ_i) and $(\psi_i) \in \mathrm{ONB}(\mathcal{H})$ and with $n < m$, m large, we may introduce the following relative error terms:

$$E_{n,m}^{\phi} = \operatorname{tr}(G(Q_m^{\phi} - Q_n^{\phi}))$$

and

$$E_{n,m}^{\psi} = \operatorname{tr}(G(Q_m^{\psi} - Q_n^{\psi})).$$

If m is fixed, we then choose a KL basis (ϕ_i) for $Q_m^{\psi} G Q_m^{\psi}$ and the following error inequality holds:

$$E_{n,m}^{\phi,\mathrm{KL}} \leq E_{n,m}^{\psi}.$$

Our next theorem gives KL optimality for sequences of entropy numbers.

Theorem 4.15: *The KL ONB gives the smallest sequence of entropy numbers in the approximation to a frame operator.*

Proof: We begin by a few facts about the entropy of trace class operators G. The entropy is defined as

$$S(G) := -\operatorname{tr}(G \log G). \tag{4.20}$$

The formula will be used on cut-down versions of an initial operator G. In some cases, only the cut-down might be trace class. Since the spectral theorem applies to G, the RHS in (4.20) is also

$$S(G) = -\sum_{k=1}^{\infty} \lambda_k \log \lambda_k . \tag{4.21}$$

For simplicity, we normalize such that $1 = \operatorname{tr} G = \Sigma_{k=1}^{\infty} \lambda_k$, and we introduce the partial sums

$$S_n^{\mathrm{KL}}(G) := -\sum_{k=1}^{n} \lambda_k \log \lambda_k \tag{4.22}$$

and

$$S_n^{\psi}(G) := -\sum_{k=1}^{n} \langle \psi_k | G\psi_k \rangle \log \langle \psi_k | G\psi_k \rangle. \tag{4.23}$$

Let $(\psi_i) \in \mathrm{ONB}(\mathcal{H})$ and set $d_k^{\psi} := \langle \psi_k | G\psi_k \rangle$; then the inequalities (4.17) take the form

$$\mathrm{tr}(Q_n^{\psi} G) = \sum_{i=1}^{n} d_i^{\psi} \leqslant \sum_{i=1}^{n} \lambda_i, \quad n = 1, 2, \ldots , \tag{4.24}$$

where, as usual, an ordering

$$d_1^{\psi} \geqslant d_2^{\psi} \geqslant \cdots \tag{4.25}$$

has been chosen.

Now the function $\beta(t) := t \log t$ is convex. Application of Remark 6.3 in Ref. 2 then yields

$$\sum_{i=1}^{n} \beta(d_i^{\psi}) \leqslant \sum_{i=1}^{n} \beta(\lambda_i), \quad n = 1, 2, \ldots . \tag{4.26}$$

Since the RHS in (4.26) is $-\mathrm{tr}(G \log G) = -S_n^{\mathrm{KL}}(G)$, the desired inequalities

$$S_n^{\mathrm{KL}}(G) \leqslant S_n^{\psi}(G), \quad n = 1, 2, \ldots , \tag{4.27}$$

follow, i.e., the KL data minimize the sequence of entropy numbers. □

A. Supplement

Let G, $\mathcal{H}$ be as before. $\mathcal{H}$ is an ∞-dimensional Hilbert space $G = \Sigma_{\alpha} P_{\alpha}$, $\omega_{\alpha} \geqslant 0$, $P_{\alpha} = P_{\alpha}^{*} = P_{\alpha}^{2}$. Suppose that $\dim \mathcal{H}_{\lambda_1}(G) > 0$, where $\dim \mathcal{H}_{\lambda_1}(G) = \{\phi \in \mathcal{H} | G\phi = \phi\}$ and $\lambda_1 := \sup\{\langle f | Gf \rangle, f \in \mathcal{H}, \|f\| = 1\}$, then define $\lambda_2, \lambda_3, \ldots$ recursively,

$$\lambda_{k+1} := \sup\{\langle f | Gf \rangle | f \perp \phi_1, \phi_2, \ldots , \phi_k\},$$

where $\dim \mathcal{H}_{\lambda_k}(G) > 0$. Set $\mathcal{K} = \vee_{k=1}^{\infty} \mathrm{span}\{\phi_1, \phi_2, \ldots , \phi_k\}$. Set $\rho := \inf\{\lambda_k | k = 1, 2, \ldots\}$, then we can apply Theorems (4.13) and (4.15) to the restriction $(G - \rho I)|_{\mathcal{K}}$, i.e., the operator $\mathcal{K} \to \mathcal{K}$ given by $\mathcal{K} \ni x \mapsto Gx - \rho x \in \mathcal{K}$.

Actually, there are two cases for G as for $G_{\mathcal{K}} := G - \rho I$: (1) compact and (2) trace class. We did (2), but we now discuss (1).

When G or $G_{\mathcal{K}}$ is given, we want to consider

$$\rho = \inf\{\langle f | Gf \rangle | \|f\| = 1\},$$

$$\lambda_1 = \sup\{\langle f | Gf \rangle | \|f\| = 1\}.$$

If $G = \Sigma_{\alpha} \omega_{\alpha} P_{\alpha}$, then $\langle f | Gf \rangle = \Sigma_{\alpha} \omega_{\alpha} \|P_{\alpha} f\|^2$. If $G = \Sigma_{\alpha} |h_{\alpha}\rangle\langle h_{\alpha}|$ where $(h_{\alpha}) \subset \mathcal{H}$ is a family of vectors, then

$$\langle f | Gf \rangle = \sum_{\alpha} |\langle h_{\alpha} | f \rangle|^2.$$

The frame bound condition takes the form

$$c_1 \|f\|^2 \leqslant \sum \omega_{\alpha} \|P_{\alpha} f\|^2 \leqslant c_2 \|f\|^2$$

or in the standard frame case $\{h_{\alpha}\}_{\alpha \in A} \in \mathrm{FRAME}(\mathcal{H})$,

$$\langle f | Gf \rangle = \sum_{\alpha} |\langle h_{\alpha} | f \rangle|^2 \leqslant c_2 \|f\|^2.$$

Lemma 4.16: If a frame system (fusion frame or standard frame) has frame bounds $0 < c_1 \leq c_2 < \infty$, *then the spectrum of the operator* G *is contained in the closed interval* $[c_1, c_2] = \{x \in \mathbb{R} \mid c_1 \leq x \leq c_2\}$.

Proof: It is clear from the formula for G that $G = G^*$. Hence, the spectrum theorem applies, and the result follows. In fact, if $z \in \mathbb{C} \setminus [c_1, c_2]$, then

$$\|zf - Gf\| \geq \operatorname{dist}(z, [c_1, c_2])\|f\|,$$

so $zI - G$ is invertible. So, z is in the resolvent set. Hence, $\operatorname{spec}(G) \subset [c_1, c_2]$. □

A frame is a system of vectors which satisfies the two *a priori* estimates (4.2), so by the Dirac notation, it may be thought of as a statement about rank-1 projections. The notion of fusion frame is the same, only with finite-rank projections, see (4.13).

V. SPLITTING OFF RANK-1 OPERATORS

The general principle in the frame analysis is to make a recursion which introduces rank-1 operators, see Definition 2.2 and Sec. II C. The theorem we will prove accomplishes that for the general class of operators in infinite dimensional Hilbert space. Our result may also be viewed as the extension of Perron-Frobenius's theorem for positive matrices. Since we do not introduce positivity in the present section, our theorem will instead include assumptions which restrict the spectrum of the operators to which our result applies.

One way rank-1 operators enter into frame analysis is through Eq. (4.7). Under the assumptions in Lemma 4.5, the operator L^*L is invertible. If we multiply Eq. (4.7) on the left and on the right with $(L^*L)^{-1}$, we arrive at the following two representations for the identity operator:

$$I = \sum_{\alpha} |\tilde{h}_\alpha\rangle\langle h_\alpha| = \sum_{\alpha} |h_\alpha\rangle\langle \tilde{h}_\alpha|, \tag{5.1}$$

where $\tilde{h}_\alpha = (L^*L)^{-1} h_\alpha$.

Truncation of the sums in (5.1) yields non-self-adjoint operators which are used in the approximation of data with frames. Starting with a general non-self-adjoint operator T, our next theorem gives a general method for splitting off a rank-1 operator from T.

Theorem 5.1: *Let* $\mathcal{H}$ *be a generally infinite dimensional Hilbert space and let* T *be a bounded operator in* $\mathcal{H}$. *Under the following three assumptions, we can split off a rank-1 operator from* T. *Specifically, assume that* $a \in \mathbb{C}$ *satisfies the following.*

(1) $0 \neq a \in \operatorname{spec}(T)$, *where* $\operatorname{spec}(T)$ *denotes the spectrum of* T.
(2) $\dim R(a-T)^{\perp} = 1$, *where* $R(a-T)$ *denotes the range of the operator* $aI - T$ *and* $\perp$ *denotes the orthogonal complement.*
(3) $\lim_{n\to\infty} a^{-n}T^n x = 0$ *for all* $x \in R(a-T)$.

Then it follows that the limit exists everywhere in $\mathcal{H}$ *in the strong topology of* $\mathcal{B}(\mathcal{H})$. *Moreover, we may pick the following representation:*

$$\lim_{n\to\infty} a^{-n}T^n = |\xi\rangle\langle w_1|, \tag{5.2}$$

for the limiting operator on $\mathcal{H}$, *where*

$$\|w_1\| = 1, \quad T^* w_1 = \bar{a} w_1, \quad \langle \xi | w_1 \rangle = 1. \tag{5.3}$$

In fact,

$$\xi - w_1 \in \overline{R(a-T)} \text{ and } T\xi = a\xi, \tag{5.4}$$

where the overbar denotes closure.

Theorem 5.1 is an analog of the Perron-Frobenius theorem (e.g., Ref. 19). Dictated by our applications, the present Theorem 5.1 is adapted to a different context where the present assumptions are different than those in the Perron-Frobenius theorem. We have not seen it stated in the literature in this version, and the proof (and conclusions) is different from that of the standard Perron-Frobenius theorem.

Remark 5.2: For the reader's benefit, we include the following statement of the Perron-Frobenius theorem in a formulation which makes it clear how Theorem 5.1 extends this theorem.

A. Perron-Frobenius

Let $d<\infty$ be given and let T be a $d\times d$ matrix with entries $T_{i,j}\geqslant 0$ and with positive spectral radius a. Then there is a column vector w with $w_i\geqslant 0$ and a row vector ξ such that the following conditions hold:

$$Tw = aw, \quad \xi T = a\xi, \quad \xi w = 1.$$

Proof of Theorem 5.1: Note that by the general operator theory, we have the following formulas:

$$R(a-T)^{\perp} = N(T^* - \bar{a}) = \{y \in \mathcal{H} | T^* y = \bar{a} y\}.$$

By assumption (2), this is a one-dimensional space, and we may pick w_1 such that $T^* w_1 = \bar{a} w_1$ and $\|w_1\| = 1$. This means that

$$\{\mathbb{C}w_1\}^{\perp} = R(a-T)^{\perp\perp} = \overline{R(a-T)}$$

is invariant under T.

As a result, there is a second bounded operator G which maps the space $\overline{R(a-T)}$ into itself and restricts T, i.e., $T|_{\overline{R(a-T)}} = G$. Further, there is a vector $\eta^{\perp} \in (\mathbb{C}w_1)^{\perp}$ such that T has the following matrix representation:

$$T = \begin{array}{cc} \begin{array}{ccc} \mathbb{C}w_1 & & (w_1)^{\perp} \end{array} & \\ \begin{pmatrix} a & | & 00\cdots \\ -- & | & --- \\ \eta^{\perp} & | & G \end{pmatrix} & \begin{array}{c} \mathbb{C}w_1 \\ \\ (w_1)^{\perp} \end{array} \end{array}.$$

The entry a in the top left matrix corner represents the following operator, $sw_1 \mapsto asw_1$. The vector $\eta_{\perp}$ is fixed and $Tw_1 = aw_1 + \eta^{\perp}$. The entry $\eta^{\perp}$ in the bottom left matrix corner represents the operator $sw_1 \mapsto s\eta^{\perp}$ or $|\eta^{\perp}\rangle\langle w_1|$.

In more detail, if Q_1 and $Q_1^{\perp} = I - Q_1$ denote the respective projections onto $\mathbb{C}w_1$ and $w_1^{\perp}$, then

$$Q_1 T Q_1 = a Q_1,$$

$$Q_1^{\perp} T Q_1 = |\eta^{\perp}\rangle\langle w_1|,$$

$$Q_1 T Q_1^{\perp} = 0,$$

$$Q_1^{\perp} T Q_1^{\perp} = G.$$

Using now assumptions in the theorem [(1) and (2)], we can conclude that the operator $a - G$ is invertible with bounded inverse

$$(a-G)^{-1} : (w_1)^{\perp} \to (w_1)^{\perp}.$$

We now turn to powers of operator T. An induction yields the following matrix representation:

$$T^n = \begin{pmatrix} a^n & \vert & 00\cdots \\ ----------- & \vert & --- \\ (a^n - G^n)(a-G)^{-1}\eta^\perp & \vert & G^n \end{pmatrix}.$$

Finally, an application of assumption (3) yields the following operator limit:

$$a^{-n}T^n \xrightarrow[n\to\infty]{} \begin{pmatrix} 1 & \vert & 00\cdots \\ ------ & \vert & --- \\ (a-G)^{-1}\eta^\perp & \vert & 00\cdots \end{pmatrix}.$$

We used that $\eta^\perp \in \overline{R(a-T)}$ and that

$$a^{-n}(a^n - G^n)(a-G)^{-1}\eta^\perp = (1 - a^{-n}G^n)(a-G)^{-1}\eta^\perp \xrightarrow[n\to\infty]{} (a-G)^{-1}\eta^\perp.$$

Further, if we set $\xi := w_1 + (a-G)^{-1}\eta^\perp$, then

$$T\xi = aw_1 + \eta^\perp + G(a-G)^{-1}\eta^\perp = aw_1 + a(a-G)^{-1}\eta^\perp = a\xi.$$

Finally, note that

$$(a-G)^{-1}\eta^\perp \in \overline{R(a-T)} = (w_1)^\perp.$$

It is now immediate from this that all of the statements in the conclusion of the theorem including (5.2)–(5.4) are satisfied for the two vectors w_1 and ξ. □

VI. WEIGHTED FRAMES AND WEIGHTED FRAME OPERATORS

In this section, we address that when frames are considered in infinite-dimensional separable Hilbert space, then the trace-class condition may not hold. There are several remedies to this, one is the introduction of a certain weighting into the analysis. Our weighting is done as follows in the simplest case: let $(h_n)_{n\in\mathbb{N}}$ be a sequence of vectors in some fixed Hilbert space and suppose that the frame condition from Definition 4.2 is satisfied for all $f \in \mathcal{H}$. We say that (h_n) is a frame. As in Sec. IV, we introduce the analysis operator L,

$$\mathcal{H} \ni f \mapsto (\langle h_n | f\rangle)_n \in l^2$$

and the two operators

$$G := L^* L : \mathcal{H} \to \mathcal{H} \tag{6.1}$$

and

$$G_R := LL^* : l^2 \to l^2 \tag{6.2}$$

(the Grammian).

As noted,

$$G = \sum_{n\in\mathbb{N}} |h_n\rangle\langle h_n|, \tag{6.3}$$

and G_R is the matrix multiplication in l^2 by the matrix $(\langle h_i | h_j\rangle)$, i.e.,

$$l^2 \ni x = (x_i) \mapsto (G_R x) = y = (y_i),$$

where

$$y_j = \sum_i \langle h_j | h_i\rangle x_i. \tag{6.4}$$

Proposition 6.1: Let $\{h_n\}$ be a set of vectors in a Hilbert space (infinite dimensional, separable) and suppose that these vectors form a frame with frame bounds c_1, c_2.

(a) *Let (v_n) be a fixed sequence of scalars in l^2. Then the frame operator $G=G_v$ formed from the weighted sequence $\{v_n h_n\}$ is trace class.*
(b) *If $\sum_{n=1}|v_n|^2=1$, then the upper frame bound for $\{v_n h_n\}$ is also c_2.*
(c) *Pick a finite subset F of the index set, typically the natural numbers $\mathbb{N}$, and then pick (v_n) in l^2 such that $v_n=1$ for all n in F. Then on this set F, the weighted frame agrees with the initial system of frame vectors $\{h_n\}$ and the weighted frame operator G_v is not changed on F.*

Proof: (a) Starting with the initial frame $\{h_n\}_{n\in\mathbb{N}}$, we form the weighted system $\{v_n h_n\}$. The weighted frame operator arises from applying (6.3) to this modified system, i.e.,

$$G_v = \sum_{n\in\mathbb{N}} |v_n h_n\rangle\langle v_n h_n| = \sum_{n\in\mathbb{N}} |v_n|^2 |h_n\rangle\langle h_n|. \tag{6.5}$$

Let (ϵ_n) be the canonical ONB in l^2, i.e., $(\epsilon_n)_k := \delta_{n,k}$. Then $h_n=L^*\epsilon_n$, so

$$\|h_n\| \leq \|L^*\|\|\epsilon_n\| = \|L^*\| = \|L\|.$$

Now apply the trace to (6.5). Suppose that $\|(v_\epsilon)\|_{l^2}=1$. Then

$$\begin{aligned}\operatorname{tr} G_v &= \sum_{n\in\mathbb{N}} |v_n|^2 \operatorname{tr}|h_n\rangle\langle h_n| \sum_{n\in\mathbb{N}} |v_n|^2\|h_n\|^2 \leq \|L\|^2\|(v_n)\|_{l^2}^2 \\ &= \|L\|^2 = \|LL^*\| = \|L^*L\| = \|G\| = \sup(\operatorname{spec}(G)).\end{aligned}$$

(Note that the estimate shows more: the sum of the eigenvalues of G_v is dominated by the top eigenvalue of G.) However, we recall (Sec. IV A) that (h_n) is a frame with frame bounds c_1, c_2. It follows (Sec. IV A) that $\operatorname{spec}(G)\subset[c_1,c_2]$. This holds also if c_1 is the largest lower bound in (4.2), and c_2 the smallest upper bound, i.e., the optimal frame bounds.

Hence, c_2 is the spectral radius of G and also $c_2=\|G\|$. The conclusion in (a) and (b) follows.

(c) The conclusion in (c) is an immediate consequence, but now

$$\operatorname{tr} G_v = \sum_{n\in\mathbb{N}} |v_n|^2\|h_n\|^2 \leq \|G\| \sum_{n\in\mathbb{N}} |v_n|^2 = c_2\left(\#F + \sum_{n\in\mathbb{N}\setminus F} |v_n|^2\right),$$

where $\#F$ is the cardinality of the set specified in (c).

Remark 6.2: Let $\{h_n\}$ and $(v_n)\in l^2$ be as in the proposition and let D_v be the diagonal operator with the sequence (v_n) down the diagonal. Then $G_v=L^*|D_v|^2L$ and $G_{R_v}=D_v^* G_R D_v$, where

$$|D_v|^2 = D_v\overline{D_v} = \begin{pmatrix} |v_1|^2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & |v_2|^2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & \dots & 0 \end{pmatrix}.$$

A. $\mathcal{B}(\mathcal{H})=\mathcal{T}(\mathcal{H})^*$

The formula $\mathcal{B}(\mathcal{H})=\mathcal{T}(\mathcal{H})^*$ summarizes the known fact[20] that $\mathcal{B}(\mathcal{H})$ is a Banach dual of the Banach space of all trace-class operators.

The conditions (4.13) and (4.2) which introduce frames (both in vector form and fusion form) may be recast with the use of this duality.

Proposition 6.3: An operator G arising from a vector system $(h_\alpha)\subset\mathcal{H}$, or from a projection system (w_α,P_α), yields a frame with frame bounds c_1 and c_2 if and only if

$$c_1 \operatorname{tr}(\rho) \leq \operatorname{tr}(\rho G) \leq c_2 \operatorname{tr}(\rho), \tag{6.6}$$

for all positive trace-class operators ρ *on* $\mathcal{H}$.

Proof: Since both (4.13) and (4.2) may be stated in the form

$$c_1\|f\|^2 \leqslant \langle f|Gf\rangle \leqslant c_2\|f\|^2$$

and

$$\mathrm{tr}(|f\rangle\langle f|) = \|f\|^2,$$

it is clear that (6.6) is sufficient.

To see that it is necessary, suppose that (4.13) holds and that ρ is a positive trace operator. By the spectral theorem, there is an ONB (f_i) and $\xi_i \geqslant 0$ such that

$$\rho = \sum_i \xi_i |f_i\rangle\langle f_i|.$$

We now use the estimates

$$c_1 \leqslant \langle f_i|Gf_i\rangle \leqslant c_2$$

in

$$\mathrm{tr}(\rho G) = \sum_i \xi_i \langle f_i|Gf_i\rangle.$$

Since $\mathrm{tr}(\rho)=\Sigma_i \xi_i$, the conclusion (6.6) follows.

Remark 6.4: Since quantum mechanical states (see Ref. 20) take the form of density matrices, the proposition makes a connection between the frame theory and quantum states. Recall that a density matrix is an operator $\rho \in \mathcal{T}(\mathcal{H})_+$ with $\mathrm{tr}(\rho)=1$.

VII. LOCALIZATION

Starting with a frame $(h_n)_{n\in\mathbb{N}}$ [nonzero vectors index set $\mathbb{N}$ for simplicity, see (4.2)], we introduce the operators

$$G := \sum_{n\in\mathbb{N}} |h_n\rangle\langle h_n|, \tag{7.1}$$

$$G_v := \sum_{n\in\mathbb{N}} |v_n|^2 |h_n\rangle\langle h_n| \quad \text{for } v \in l^2, \tag{7.2}$$

and the components

$$G_{h_n} := |h_n\rangle\langle h_n|. \tag{7.3}$$

We further note that the individual operators G_{h_n} in (7.3) are included in the l^2-index family G_v of (7.2). To see this, take

$$v = \epsilon_n = (0,0, \ldots ,0,1,0, \ldots) \quad \text{where 1 is in } n\text{th place}. \tag{7.4}$$

It is immediate that the spectrum of G_{h_n} is the singleton $\|h_n\|^2$, and we may take $\|h_n\|^{-1}h_n$ as a normalized eigenvector. Hence, for the components G_{h_n}, there are global entropy considerations. Still in applications, it is the sequence of local approximations

$$\sum_{i=1}^{m} \langle \psi_i|h_n\rangle \psi_i = Q_m^{\psi} h_n, \tag{7.5}$$

which is accessible. It is computed relative to some ONB(ψ_i). The corresponding sequence of entropy numbers is

$$S_m^{\psi}(h_n) \coloneqq -\sum_{i=1}^{m} |\langle \psi_i | h_n \rangle|^2 \log |\langle \psi_i | h_n \rangle|^2. \tag{7.6}$$

The next result shows that for every $v \in l^2$ with $\|v\|_{l^2}=1$, the combined operator G_v always is entropy improving in the following precise sense.

Proposition 7.1: Consider the operators G_v and G_{h_n} introduced in (7.2) and (7.3). *Suppose that $v \in l^2$ satisfies $\|v\|_{l^2}=1$. Then for every* ONB(ψ_i) *and for every m,*

$$S_m^{\psi}(G_v) \geq \sum_{n \in \mathbb{N}} |v_n|^2 S_m^{\psi}(G_{h_n}). \tag{7.7}$$

Proof: Let v, ψ, and m be as specified in the proposition. Introduce the convex function $\beta(t) \coloneqq t \log t$, $t \in [0,1]$ with the convention that $\beta(0)=\beta(1)=0$. Then

$$-S_m^{\psi}(G_v) = \sum_{i=1}^{m} \beta(|\langle \psi_i | G_v \psi_i \rangle|^2) \leq \sum_{i=1}^{m} \sum_{n \in \mathbb{N}} |v_n|^2 \beta(|\langle \psi_i | G_{h_n} \psi_i \rangle|^2)$$

$$= \sum_{n \in \mathbb{N}} |v_n|^2 \sum_{i=1}^{m} \beta(|\langle \psi_i | G_{h_n} \psi_i \rangle|^2) = -\sum_{n \in \mathbb{N}} |v_n|^2 S_m^{\psi}(G_{h_n}),$$

where we used that β is convex. In the last step, formula (7.6) was used. This proves (7.7) in the proposition. □

VIII. ENGINEERING APPLICATIONS

In wavelet image compression, wavelet decomposition is performed on a digital image. Here, an image is treated as a matrix of functions where the entries are pixels. The following is an example of a representation for a digitized image function:

$$\mathbf{f}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} f(0,0) & f(0,1) & \cdots & f(0,N-1) \\ f(1,0) & f(1,1) & \cdots & f(1,N-1) \\ \vdots & \vdots & \vdots & \vdots \\ f(M-1,0) & f(M-1,1) & \cdots & f(M-1,N-1) \end{pmatrix}. \tag{8.1}$$

After the decomposition quantization is performed on the image, the quantization may be lossy (meaning some information is being lost) or lossless. Then a lossless means of compression, entropy encoding, is done on the image to minimize the memory space for storage or transmission. Here, the mechanism of entropy will be discussed.

A. Entropy encoding

In most images, their neighboring pixels are correlated and thus contain redundant information. Our task is to find less correlated representation of the image, then perform redundancy reduction and irrelevancy reduction. Redundancy reduction removes duplication from the signal source (for instance, a digital image). Irrelevancy reduction omits parts of the signal that will not be noticed by the human visual system.

Entropy encoding further compresses the quantized values in a lossless manner, which gives better compression in overall. It uses a model to accurately determine the probabilities for each quantized value and produces an appropriate code based on these probabilities so that the resultant output code stream will be smaller than the input stream.

1. Some terminology

(i) Spatial redundancy refers to the correlation between neighboring pixel values.

(ii) Spectral redundancy refers to the correlation between different color planes or spectral bands.

B. The algorithm

Our aim is to reduce the number of bits needed to represent an image by removing redundancies as much as possible.

The algorithm for entropy encoding using the KL expansion can be described as follows.

(1) Perform the wavelet transform for the whole image (i.e., wavelet decomposition).
(2) Do quantization to all coefficients in the image matrix, except the average detail.
(3) Subtract the mean: subtract the mean from each of the data dimensions. This produces a data set whose mean is zero.
(4) Compute the covariance matrix,

$$\operatorname{cov}(X,Y) = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{n}.$$

(5) Compute the eigenvectors and eigenvalues of the covariance matrix.
(6) Choose components and form a feature vector (matrix of vectors),

$$(\mathrm{eig}_1, \ldots, \mathrm{eig}_n).$$

Eigenvectors are listed in decreasing order of the magnitude of their eigenvalues. Eigenvalues found in step 5 are different in values. The eigenvector with the highest eigenvalue is the principle component of the data set.
(7) Derive the new data set.

$$\text{final data} = \text{row feature matrix} \times \text{row data adjust}.$$

Row feature matrix is the matrix that has the eigenvectors in its rows with the most significant eigenvector (i.e., with the greatest eigenvalue) at the top row of the matrix. Row data adjust is the matrix with mean-adjusted data transposed. That is, the matrix contains the data items in each column with each row having a separate dimension.[31]

Inside the paper, we use (ϕ_i) and (ψ_i) to denote generic ONBs for a Hilbert space. However, in wavelet theory,[9] it is traditional to reserve ϕ for the father function and ψ for the mother function. A 1-level wavelet transform of an $N \times M$ image can be represented as

$$\mathbf{f} \mapsto \begin{pmatrix} \mathbf{a}^1 & | & \mathbf{h}^1 \\ -- & & -- \\ \mathbf{v}^1 & | & \mathbf{d}^1 \end{pmatrix}, \tag{8.2}$$

where the subimages $\mathbf{h}^1, \mathbf{d}^1, \mathbf{a}^1$, and $\mathbf{v}^1$ each have the dimension of $N/2$ by $M/2$.

$$\mathbf{a}^1 = V_m^1 \otimes V_n^1 : \phi^A(x,y) = \phi(x)\phi(y) = \sum_i \sum_j h_i h_j \phi(2x-i)\phi(2y-j),$$

$$\mathbf{h}^1 = V_m^1 \otimes W_n^1 : \psi^H(x,y) = \psi(x)\phi(y) = \sum_i \sum_j g_i h_j \phi(2x-i)\phi(2y-j),$$

$$\mathbf{v}^1 = W_m^1 \otimes V_n^1 : \psi^V(x,y) = \phi(x)\psi(y) = \sum_i \sum_j h_i g_j \phi(2x-i)\phi(2y-j),$$

$$\mathbf{d}^1 = W_m^1 \otimes W_n^1 : \psi^D(x,y) = \psi(x)\psi(y) = \sum_i \sum_j g_i g_j \phi(2x-i)\phi(2y-j), \tag{8.3}$$

where ϕ is the father function and ψ is the mother function in the sense of wavelet, V space denotes the average space, and the W spaces are the difference space from multiresolution analysis.[9] Note that, on the very right-hand side, in each of the four system of equations, we have the affinely transformed function ϕ occurring both places under each of the double summations. The reason is that each of the two functions, father function ϕ and mother function ψ, satisfies a scaling relation. So both ϕ and ψ are expressed in terms of the scaled family of functions $\phi(2\cdot -j)$ with j ranging over Z, and the numbers h_i are used in the formula for ϕ, while the numbers g_j are used for ψ. Specifically, the relations are

$$\phi(x) = \sum_i h_i \phi(2x-i) \quad \text{and} \quad \psi(x) = \sum_j g_j \phi(2x-j).$$

h_i and g_j are low-pass and high-pass filter coefficients, respectively. $\mathbf{a}^1$ denotes the first averaged image, which consists of average intensity values of the original image. Note that only ϕ function, V space, and h coefficients are used here. $\mathbf{h}^1$ denotes the first detailed image of horizontal components, which consists of intensity difference along the vertical axis of the original image. Note that ϕ function is used on y and ψ function on x, W space for x values, and V space for y values; both h and g coefficients are used accordingly. $\mathbf{v}^1$ denotes the first detailed image of vertical components, which consists of intensity difference along the horizontal axis of the original image. Note that ϕ function is used on x and ψ function on y, W space for y values, and V space for x values; both h and g coefficients are used accordingly. $\mathbf{d}^1$ denotes the first detailed image of diagonal components, which consists of intensity difference along the diagonal axis of the original image. The original image is reconstructed from the decomposed image by taking the sum of the averaged image and the detailed images and scaling by a scaling factor. It could be noted that only ψ function, W space, and g coefficients are used here (see Refs. 34 and 32).

This decomposition is not only limited to one step, but it can be done again and again on the averaged detail depending on the size of the image. Once it stops at a certain level, quantization (see Refs. 31, 30, and 33) is done on the image. This quantization step may be lossy or lossless. Then the lossless entropy encoding is done on the decomposed and quantized image.

There are various means of quantization and the commonly used one is called thresholding. Thresholding is a method of data reduction where it puts 0 for the pixel values below the thresholding value or another "appropriate" value. Soft thresholding is defined as follows:

$$T_{\text{soft}}(x) = \begin{cases} 0 & \text{if} |x| \leq \lambda \\ x-\lambda & \text{if } x > \lambda \\ x+\lambda & \text{if } x < -\lambda, \end{cases} \tag{8.4}$$

and hard thresholding as follows:

$$T_{\text{hard}}(x) = \begin{cases} 0 & \text{if} |x| \leq \lambda \\ x & \text{if} |x| > \lambda, \end{cases} \tag{8.5}$$

where $\lambda \in \mathbb{R}_+$ and x is a pixel value. It could be observed by looking at the definitions; the difference between them is related to how the coefficients larger than a threshold value λ in absolute values are handled. In hard thresholding, these coefficient values are left alone. In soft thresholding, the coefficient values area decreased by λ if positive and increased by λ if negative.[35] (also, see. Refs. 34, 16, and 32).

Starting with a matrix representation for a particular image, we then compute the covariance matrix using steps (3) and (4) in the algorithm above. Next, we compute the KL eigenvalues. As usual, we arrange the eigenvalues in decreasing order. The corresponding eigenvectors are arranged to match the eigenvalues with multiplicity. The eigenvalues mentioned here are the same eigenvalues in Theorems (4.13) and (4.15), thus yielding smallest error and smallest entropy in the computation.

The KL transform or PCA allows us to better represent each pixel on the image matrix with the smallest number of bits. It enables us to assign the smallest number of bits for the pixel that has the highest probability, then the next number to the pixel value that has the second highest probability, and so forth; thus, the pixel that has the smallest probability gets the highest value among all the other pixel values.

An example with letters in the text would better depict how the mechanism works. Suppose that we have a text with letters a, e, f, q, and r with the following probability distribution:

Letter	Probability
a	0.3
e	0.2
f	0.2
q	0.2
r	0.1

Shannon-Fano entropy encoding algorithm is outlined as follows.

- List all letters with their probabilities in decreasing order.
- Divide the list into two parts with approximately equal probability (i.e., the total of probabilities of each part sums up to approximately 0.5).
- For the letters in the first part, start the code with 0 bit and for those in the second part with 1.
- Recursively continue until each subdivision is left with just one letter.[4]

Then applying the Shannon-Fano entropy encoding scheme on the above table gives us the following assignment.

Letter	Probability	Code
a	0.3	00
e	0.2	01
f	0.2	100
q	0.2	101
r	0.1	110

Note that instead of using 8 bits to represent a letter, 2 or 3 bits are being used to represent the letters in this case.

C. Benefits of entropy encoding

One might think that the quantization step suffices for compression. It is true that the quantization does compress the data tremendously. After the quantization step, many of the pixel values are either eliminated or replaced with other suitable values. However, those pixel values are still represented with either 8 or 16 bits (see Sec. I A). So we aim to minimize the number of bits used by means of entropy encoding. KL transform or PCAs makes it possible to represent each pixel on the digital image with the least bit representation according to their probability, thus yielding the lossless optimized representation using the least amount of memory.

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