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REMARKS ON BEST APPROXIMATIONS IN GENERALIZED CONVEX SPACES

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Abstract. In this paper, we prove a best approximation theorem in generalized convex spaces. As an application, we derive a result on the existence of a maximal element and a coincidence point theorem in generalized convex spaces. The results of this paper generalize some known results in the literature.

1. Introduction and preliminaries

The notion of a generalized convex space we work with in this paper was introduced by S. Park and H. Kim in [9]. In generalized convex spaces many results on fixed points, coincidence points, equilibrium problems, variational inequalities, continuous selections, saddle points, and others, have been obtained, see for example [4, 5, 7, 9, 10, 11, 12, 13].

In this paper, we obtain a best approximation theorem for multimaps in generalized convex spaces. Some applications to the existence of a maximal elements and coincidence point theorems in generalized convex spaces are given.

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A multimap or map $F\colon X \multimap Y$ is a function from a set X into the power set of a set Y. For $A \subset X$, let $F(A) = \bigcup\{Fx\colon x \in A\}$. For any $B \subset Y$, the lower inverse and upper inverse of B under F is defined by

$$F^-(B) = \{x \in X\colon Fx \cap B \neq \emptyset\} \text{ and } F^+(B) = \{x \in X\colon Fx \subset B\},$$

respectively. The lower inverse of $F\colon X \multimap Y$ is the map $F^-\colon Y \multimap X$ defined by $x \in F^- y$ if and only if $y \in Fx$.

Let X be a metric space with metric d. For any nonnegative real number r and any subset A of X, we define the r-parallel set of A as

$$A + r = \bigcup\{B(a,r)\colon a \in A\},$$

where $B(a,r) = \{x \in X\colon d(a,x) \leq r\}$.

If A and B are nonempty subsets of X we define

$$d(A,B) = \inf\{d(a,b)\colon a \in A,\ b \in B\}.$$

For bounded and closed subsets A and B of X, the Hausdorff distance, denoted by $H(A,B)$, is defined by

$$H(A,B) = \max\{D(A,B), D(B,A)\},$$

where

$$D(A,B) = \sup_{y \in A} \inf_{x \in B} d(x,y).$$

A mapping $F\colon X \multimap Y$ is upper (lower) semicontinuous on X if and only if for every open $V \subset Y$ the set $F^+(V)$ $(F^-(V))$ is open. A mapping $F\colon X \multimap Y$ is continuous if and only if it is upper and lower semicontinuous. A mapping $F\colon X \multimap Y$ with compact values is continuous if and only if F is a continuous mapping in the Hausdorff distance.

For a nonempty subset D of X, let $\langle D \rangle$ denote the set of all nonempty finite subsets of D. Let Δ_n denote the standard n-simplex with vertices $e_1, e_2, \dots, e_{n+1}$, where e_i is the ith unit vector in $\mathbb{R}^{n+1}$.

A generalized convex space or G-convex space $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D and a function $\Gamma\colon \langle D \rangle \multimap X$ with nonempty values such that

1. for each $A, B \in \langle D \rangle$, $A \subset B$ implies $\Gamma(A) \subset \Gamma(B)$; and
2. for each $A \in \langle D \rangle$ with $|A| = n+1$, there exists a continuous function $\varphi_A\colon \Delta_n \to \Gamma(A)$, such that $\varphi_A(\Delta_J) \subset \Gamma(J)$, where Δ_J denotes the faces of Δ_n corresponding to $J \in \langle A \rangle$.

Particular forms of G-convex space are convex subset of a topological vector space, Lassonde's convex space, a metric space with Michael's convex structure, S-contractible space, H-space, Komiya's convex space, Bielawski's simplicial convexity, Joó's pseudoconvex space, see for example [4, 10, 11].

For each $A \in \langle D \rangle$, we may write $\Gamma(A) = \Gamma_A$. Note Γ_A does not need to contain A. For $(X, D; \Gamma)$, a subset C of X is said to be G-convex if for each

$A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$. If $D = X$, then $(X, D; \Gamma)$ will be denoted by (X, Γ). The G-convex hull of K, denoted by $G - \mathrm{co}\,(K)$ is the set

$$\bigcap \{B \subset X \colon B \text{ is a } G\text{-convex subset of } X \text{ containing } K\}.$$

Let C be a subset of X, a map $F\colon C \multimap X$ is called G-quasiconvex if and only if the set $F^-(S)$ is a G-convex set for each G-convex subset S of X. If X is a topological vector space and $\Gamma_A = \mathrm{co}\ A$, we obtain the class quasiconvex map, see for example [6].

Let C be a G-convex subset of X, a map $F\colon C \multimap X$ is called G-KKM map if $\Gamma_A \subset F(A)$ for each $A \in \langle C \rangle$.

The following version of G-KKM type theorem, see for example [5], will be used to prove the main result of this paper.

Theorem 1.1. *Let (X, Γ) be a G-convex space, K a nonempty subset of X and $H\colon K \multimap X$ a map with closed values and G-KKM map. If $H(x)$ is compact for at least one $x \in K$ then $\bigcap_{x \in K} H(x) \neq \emptyset$.*

2. Best approximation theorem

Theorem 2.1. *Let (X, Γ) be a metrisable G-convex space with metric d, K a nonempty G-convex compact subset of X, $F_1\colon K \multimap X$, $F_2\colon K \multimap X$ continuous maps with nonempty G-convex compact values, and let $\lambda \geq 1$ such that*

$$G - \mathrm{co}\,(F_1^-(A + r)) \subset F_1^-(A + \lambda r), \tag{2.1}$$

for all G-convex subsets A of X and nonnegative real number r. Then there exists $y_0 \in K$ such that

$$d(F_1(y_0), F_2(y_0)) \leq \lambda \inf_{x \in K} d(F_1(x), F_2(y_0)).$$

Proof. Let for every $x \in K$, $H\colon K \multimap K$ be defined by

$$H(x) = \{y \in K \colon d(F_1(y), F_2(y)) \leq \lambda d(F_1(x), F_2(y))\}.$$

The maps F_1 and F_2 are continuous, hence they are continuous in the Hausdorff distance, too. From inequality

$$|d(A, C) - d(C, B)| \leq H(A, B),$$

for each bounded and closed subsets A, B and C of X, we obtain that $H(x)$ is closed for each $x \in K$. Since K is a compact set we have that $H(x)$ is compact for each $x \in C$. We can prove that H is a G-KKM map, that is, that for every $D = \{x_1, x_2, \ldots, x_n\} \in \langle K \rangle$

$$\Gamma_D \subseteq H(D).$$

Suppose that $\Gamma_D \nsubseteq H(D)$. Then there exists $y \in \Gamma_D$ such that $y \notin H(x_i)$ for every $i \in \{1, 2, \dots, n\}$. So, we have

$$d(F_1(y), F_2(y)) > \lambda d(F_1(x_i), F_2(y)) \text{ for every } i \in \{1, 2, \dots, n\}.$$

Let $\varepsilon > 0$ be so that

$$\lambda d(F_1(x_i), F_2(y)) \leq d(F_1(y), F_2(y)) - \varepsilon \text{ for every } i \in \{1, 2, \dots, n\}.$$

Let $r = d(F_1(y), F_2(y)) - \varepsilon$. Then

$$F_1(x_i) \bigcap \left(F_2(y) + \frac{r}{\lambda}\right) \neq \emptyset \text{ for every } i \in \{1, 2, \dots, n\}.$$

So,

$$x_i \in F_1^- \left(F_2(y) + \frac{r}{\lambda}\right) \text{ for every } i \in \{1, 2, \dots, n\}.$$

This implies

$$y \in G - \text{co } F_1^- \left(F_2(y) + \frac{r}{\lambda}\right).$$

From condition (2.1) we obtain

$$y \in F_1^-(F_2(y) + r)$$

and hence

$$F_1(y) \cap (F_2(y) + r) \neq \emptyset.$$

So,

$$d(F_1(y), F_2(y)) \leq r < r + \varepsilon = d(F_1(y), F_2(y)).$$

This is a contradiction and H is G-KKM map. From Theorem 1.1 it follows that there exists $y_0 \in K$ such that

$$d(F_1(y_0), F_2(y_0)) \leq \lambda d(F_1(x), F_2(y_0)) \text{ for all } x \in K.$$

□

Example 2.2. Let X be a hyperconvex metric space, see for example [2, 3]. For a nonempty bounded subset A of X put

$$\text{co } A = \bigcap \{B \colon B \text{ is closed ball in } X \text{ containing } A\}.$$

Let $\mathcal{A}(X) = \{A \subset X \colon A = \text{co } A\}$. The elements of $\mathcal{A}(X)$ are called admissible subset of X. It is known that any hyperconvex metric space (X, d) is an G-convex space (X, Γ), with $\Gamma_A = \text{co } A$ for each $A \in \langle X \rangle$.

The r-parallel of an admissible subset of a hyperconvex metric space is also an admissible set, [2, Lemma 4.10]. Let $F_1 \colon K \multimap X$ be a G-quasi-convex map, i.e. $F_1^-(A)$ is admissible set for each admissible subset A of X. Then the map F_1 satisfies the condition (2.1) for each real number λ such that $\lambda \geq 1$.

From Theorem 2.1, we have the following best approximation theorem for hyperconvex metric space.

Theorem 2.3. *Let X be a hyperconvex metric space and K a nonempty admissible compact subset of X, $F_1\colon K \multimap X$, $F_2\colon K \multimap X$ continuous maps with nonempty admissible compact values, and F_1 be a G-quasiconvex map. Then there exists $y_0 \in K$ such that*

$$d(F_1(y_0), F_2(y_0)) = \inf_{x\in K} d(F_1(x), F_2(y_0)).$$

Corollary 2.4 ([3])**.** *Let X be a hyperconvex metric space and K nonempty admissible compact. Let $f\colon K \to X$ be continuous. Then there exists $y_0 \in K$ such that*

$$d(y_0, f(y_0)) = \inf_{x\in K} d(x, f(y_0)).$$

3. Applications

From Theorem 2.1, we have the following coincidence point theorem.

Theorem 3.1. *Let (X, Γ) be a metrisable G-convex space, K a nonempty G-convex compact subset of X, $F_1\colon K \multimap X$, $F_2\colon K \multimap X$ continuous maps with nonempty G-convex compact values, and let $\lambda \geq 1$ such that*

$$G - \mathrm{co}\,(F_1^-(A + r)) \subset F_1^-(A + \lambda r),$$

for all G-convex subsets A of X and nonnegative real number r. If for every $x \in K$, with $F_1(x) \cap F_2(x) = \emptyset$ there exists $\alpha \in (0, 1/\lambda)$ such that $F_1(K) \cap (F_2(x) + \alpha d(F_1(x), F_2(x))) \neq \emptyset$ then, there exists $y_0 \in K$ such that $F_1(y_0) \cap F_2(y_0) \neq \emptyset$.

Proof. By the Theorem 2.1, there exists $y_0 \in K$ such that

$$d(F_1(y_0), F_2(y_0)) \leq \lambda \inf_{x\in K} d(F_1(x), F_2(y_0)).$$

We claim that such y_0 is a coincidence point, i.e. $F_1(y_0) \cap F_2(y_0) \neq \emptyset$. Suppose not, i.e. $F_1(y_0) \cap F_2(y_0) = \emptyset$. Then we have the existence of $\alpha \in (0, 1/\lambda)$ such that

$$F_1(K) \cap (F_2(y_0) + \alpha d(F_1(y_0), F_2(y_0))) \neq \emptyset.$$

Let $u_1 \in F_1(K) \cap (F_2(y_0) + \alpha d(F_1(y_0), F_2(y_0)))$. Then we obtain that there exists $u_2 \in F_2(y_0)$ such that $u_1 \in B(u_2, \alpha d(F_1(y_0), F_2(y_0)))$, and

$$d(u_1, u_2) \leq \alpha d(F_1(y_0), F_2(y_0)) < \frac{d(F_1(y_0), F_2(y_0))}{\lambda}.$$

Hence,

$$d(F_1(y_0), F_2(y_0)) \le \lambda \inf_{x \in K} d(F_1(x), F_2(y_0)) \le \lambda d(u_1, u_2) < d(F_1(y_0), F_2(y_0)),$$

which is a contradiction. Therefore, $F_1(y_0) \cap F_2(y_0) \neq \emptyset$. □

Corollary 3.2. *Let X be a hyperconvex metric space and K a nonempty admissible compact subset of X, $F\colon K \multimap X$ a continuous map with nonempty admissible compact values. If for every $x \in K$, with $x \notin F(x)$ there exists $\alpha \in (0,1)$ such that $K \cap B[x, \alpha d(F(x), x)] \neq \emptyset$, then there exists $x_0 \in K$ such that $x_0 \in F(x_0)$.*

Remark 3.3. Note that, if $x_0 \notin F(x_0)$ then $x_0 \in \operatorname{Bd} K$. Namely, if $x_0 \in \operatorname{Int} K$, then there exists $r > 0$ such that

$$B(x_0, r) \subset K \text{ and } r < d(F(x_0), x_0) \le d(F(x_0), x) \text{ for all } x \in B(x_0, r).$$

We show that

$$B(x_0, r) \cap (F(x_0) + d(F(x_0), x_0) - r) \neq \emptyset.$$

Let $F(x_0) = \bigcap_{\alpha \in \Lambda} B(x_\alpha, r_\alpha)$, $x_\alpha \in F(x_0)$. Then by [2, Lemma 4.10], we have

$$F(x_0) + d(F(x_0), x_0) - r = \bigcap_{\alpha \in \Lambda} B(x_\alpha, r_\alpha + d(F(x_0), x_0) - r).$$

We show that

$$d(x_0, x_\alpha) \le r_\alpha + d(F(x_0), x_0).$$

Namely,

$$\inf_{x \in K} d(F(x_0), x) \ge d(F(x_0), x_0) = d(u, x_0), \text{ for any } u \in F(x_0),$$

and

$$d(u, x_0) \ge d(x_0, x_\alpha) - d(x_\alpha, u) \ge d(x_0, x_\alpha) - r_\alpha.$$

So,

$$d(x_0, x_\alpha) \le r_\alpha + d(x_0, u) \le r_\alpha + d(F(x_0), x_0)$$

and

$$B(x_0, r) \cap (F(x_0) + d(F(x_0), x_0) - r) \neq \emptyset.$$

Let $z \in K$ be such that

$$z \in B(x_0, r) \cap (F(x_0) + d(F(x_0), x_0) - r),$$

we obtain

$$d(F(x_0), x_0) \le d(F(x_0), z) \le d(F(x_0), x_0) - r < d(F(x_0), x_0),$$

which is a contradiction. Therefore, $x_0 \in \operatorname{Bd} K$.

Corollary 3.4 ([8]). *Let X be a hyperconvex metric space, K a nonempty admissible compact subset of X, $f\colon K \to X$ continuous function and for every $x \in \operatorname{Bd} K$, with $x \neq f(x)$ there exists $\alpha \in (0,1)$ such that $K \cap B(f(x), \alpha d(x, f(x))) \neq \emptyset$. Then f has a fixed point.*

As an application of Theorem 2.1, we obtain the result of existence of maximal elements for G-convex space. Let $F\colon K \multimap X$. An element $x \in K$ is a maximal element of K if $F(x) = \emptyset$, see for example [1]. The F-maximal set of F is defined as $M_F = \{x \in K\colon F(x) = \emptyset\}$.

Theorem 3.5. *Let (X, Γ) be a metrisable G-convex space, K a nonempty G-convex compact subset of X, $F_1\colon K \multimap X$, $F_2\colon K \multimap X$ continuous maps with G-convex compact values, and let $\lambda \geq 1$ such that $G-\operatorname{co}(F_1^-(A+r)) \subset F_1^-(A+\lambda r)$, for all G-convex subsets A of X and nonnegative real number r. If $F_2(x) \subset F_1(K \backslash \{x\})$ for each $x \in K$, then $M_{F_1} \cup M_{F_2}$ is a nonempty set.*

Proof. Suppose that $M_{F_1} \cup M_{F_2} = \emptyset$. Then by Theorem 2.1, there exists an $y_0 \in K$ such that

$$d(F_1(y_0), F_2(y_0)) \leq \lambda \inf_{x \in K} d(F_1(x), F_2(y_0)).$$

Since $F_2(x) \subset F_1(K \backslash \{x\})$ for each $x \in K$, we obtain

$$\inf_{x \in K} d(F_1(x), F_2(y_0)) = 0,$$

and $d(F_1(y_0), F_2(y_0)) = 0$. This implies that $F_1(y_0) \cap F_2(y_0) \neq \emptyset$. This contradicts to the assumption $F_2(x) \subset F_1(K \backslash \{x\})$ for each $x \in K$. Hence, $M_{F_1} \cup M_{F_2}$ is nonempty. □

Corollary 3.6. *Let X be a hyperconvex metric space, K a nonempty admissible compact subset of X, $F\colon K \multimap K$ a continuous map with admissible compact values and $x \notin F(x)$ for each $x \in K$. Then F has a maximal element.*

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