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GABOR TRANSFORM IN QUANTUM CALCULUS AND APPLICATIONS

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Abstract

In this work, using the q-Jackson integral and some elements of the q-harmonic analysis associated with zero order q-Bessel operator, for a fixed $q \in]0,1[$, we study the q analogue of the continuous Gabor transform associated with the q-Bessel operator of order zero. We give some q-harmonic analysis properties (a Plancherel formula, an $L^2_q(\mathbb{R}_{q,+}, xd_qx)$ inversion formula, etc), and a weak uncertainty principle for it. Then, we show that the portion of the q-Bessel Gabor transform lying outside some set of finite measure cannot be arbitrarily too small. Finally, using the kernel reproducing theory, given by Saitoh [13], we give the q analogue of the practical real inversion formula for q-Bessel Gabor transform.

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Key Words and Phrases: q-Gabor transform, practical real inversion formula, kernel reproducing, uncertainty principles

1. Introduction

Time-Frequency analysis plays a central role in signal analysis. Since years ago, it has been recognized that the global Fourier transform of a long time signal has a little practical value to analyze the frequency spectrum of a signal. That is why, the Gabor method is preferred to the classical

Fourier method, whenever the time dependence of the analyzed signal is of the same importance as its frequency dependence.

However, there exist strict limits to the maximal Time-Frequency resolution of this transform, similar to Heisenberg's uncertainty principles in the Fourier analysis.

In fact, Gabor [4] was the first to introduce the Gabor transform, in which he uses translations and modulations of a single Gaussian to represent one dimensional signal. Other names for this transform used in literature, are: short time Fourier transform, Weyl-Heisenberg transform, windowed Fourier transform (cf. Grochening [6] for more details).

In the present paper we show the q-analogue of the continuous Gabor transform associated with the q-Bessel operator of order zero, giving a definition and some q-analysis properties for it. We discuss some uncertainty principles, which basically claim that the support of this transform of a function cannot be too small, and we conclude by some applications.

The paper is organized as follows. In §2, we recall the main results about the harmonic analysis related to the third basic zero order Bessel function. In §3, we introduce the q-analogue of the continuous Gabor transform associated with the q-Bessel operator and give some q-harmonic properties for it (Plancherel formula, L^2_q inverse formula, weak uncertainty for it). §4 is devoted to some applications. More precisely, using the kernel reproducing theory given by Saitoh [13] we study the problems of approximative concentration, practical real inversion formulas and extremal function for the q-Bessel Gabor transform.

2. Preliminaries

Throughout this paper, we fix $q \in]0,1[$. In this section we provide some notations and results used in the q-theory. We refer to the book by Gasper and Rahman [5], for the definitions, notations and properties of the q-shifted factorials and the q-hypergeometric functions.

2.1. Notations.

For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1;\ (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),\ n = 1,\ 2, ...;\ (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{2.1}$$

We also denote

$$(a_1, a_2, .., a_p : q)_n = (a_1;q)_n(a_2;q)_n...(a_p;q)_n,\ \ n = 0, 1, .., \infty, \tag{2.2}$$

$$[x]_q = \frac{1-q^x}{1-q},\ x \in \mathbb{C} \tag{2.3}$$

and

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n},\ n \in \mathbb{N}. \tag{2.4}$$

$$\mathbb{R}_{q,+} = \Big\{ q^n \ :\ n \in \mathbb{Z} \Big\},\ \ \widetilde{\mathbb{R}}_{q,+} = \mathbb{R}_{q,+} \cup \{0\} \text{ and } \mathbb{R}_q = \Big\{ \pm q^n \ :\ n \in \mathbb{Z} \Big\}. \tag{2.5}$$

The q-Jakson integrals from 0 to a and from 0 to ∞ are defined by

$$\int_0^a f(x) d_q x = (1-q) a \sum_{n=0}^{\infty} f(aq^n) q^n, \tag{2.6}$$

$$\int_0^\infty f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n, \tag{2.7}$$

provided the sums converge absolutely.

The q-Jakson integral in a generic interval $[a,b]$ is given by (cf. [7])

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \tag{2.8}$$

The q-derivatives $D_q f$ and $D_q^+ f$ of a function f are given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x},\ \ (D_q^+ f)(x) = \frac{f(q^{-1}x) - f(x)}{(1-q)x} \text{ if } x \neq 0, \tag{2.9}$$

$(D_q f)(0) = f'(0)$ and $(D_q^+ f)(0) = q^{-1} f'(0)$ provided $f'(0)$ exists.

Using these two q-derivatives, we put

$$\Delta_q = \frac{(1-q)^2}{x} D_q [x D_q^+]. \tag{2.10}$$

2.2. The q-Bessel function

In [10], Koornwinder and Swarttouw, in order to study a q-analogue of the Hankel transform and to give its inversion formula and a Plancherel formula, defined the third Jackson's q-Bessel function using the q-hypergeometric function ${}_1\varphi_1$, as follows

$$J_\alpha(z;q^2) = \frac{z^\alpha (q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty}\ {}_1\varphi_1(0;q^{2\alpha+2};q^2,q^2z^2). \tag{2.11}$$

It follows that for all $\lambda \in \mathbb{C}$, the function $x \mapsto J_0(\lambda x; q^2)$ is the solution of the q-problem

$$\begin{cases} \Delta_q u(x) = -\lambda^2 u(x), \\ u(0) = 1, \quad u'(0) = 0. \end{cases} \tag{2.12}$$

We denote by

• $S_{*q}(\mathbb{R}_q)$ the space of all functions f on $\mathbb{R}_{q,+}$ such that for all $m,\ n \in \mathbb{N}$:

$$\sup_{x\in\mathbb{R}_{q,+}} |x^{2m}\Delta_q^n f(x)| < \infty, \quad \text{and} \quad (D_q^+(\Delta_q^n f)(x)) \to 0 \text{ as } x \downarrow 0 \text{ in } \mathbb{R}_{q,+}.$$

• $\mathcal{D}_{*q}(\mathbb{R}_q)$ the space of all functions f on $\mathbb{R}_{q,+}$ with bounded support such that for all $n \in \mathbb{N}$, we have $(D_q^+(\Delta_q^n(f))(x)) \to 0$ as $x \downarrow 0$ in $\mathbb{R}_{q,+}$.

• $C_{*q,0}(\mathbb{R}_q)$ the space of all functions f on $\widetilde{\mathbb{R}}_{q,+}$ for which $f(x) \to 0$ as $x \to \infty$ in $\mathbb{R}_{q,+}$ and $f(x) \to f(0)$ as $x \downarrow 0$ in $\mathbb{R}_{q,+}$.

Using the q-Jackson integrals, we note that for $p > 0$:

• $L_q^p(\mathbb{R}_{q,+}, xd_qx) = \Big\{f : \|f\|_{p,q} = \Big(\int_0^\infty |f(x)|^p x d_q x\Big)^{\frac{1}{p}} < \infty.\Big\}$

• $L_q^\infty(\mathbb{R}_{q,+}, xd_qx) = \Big\{f : \|f\|_{\infty,q} = \sup_{x\in\mathbb{R}_{q,+}} |f(x)| < \infty.\Big\}$.

The following lemma shows some properties for the third Jackson's $q-$Bessel function of order zero.

Lemma 2.1. *For $x \in \mathbb{R}_{q,+}$, we have*

i) $|J_0(x; q^2)| \leq 1$

ii) $x \mapsto J_0(\lambda x; q^2) \in S_{*q}(\mathbb{R}_q)$, *for all* $\lambda \in \mathbb{R}_{q,+}$.

In the following subsections we collect some notations and results on the q-generalized translation, q-Bessel Fourier transform and q-generalized convolution product (cf. [2,3]).

2.3. The q-generalized translation

Let f be a function defined on $\mathbb{R}_{q,+}$, the $q-$generalized translation of f is given by

$$\tau_{q,x}(f)(y) = \sum_{k=-\infty}^{\infty} K(x, y, q^k) f(q^k), \quad x,\ y \in \mathbb{R}_{q,+}, \tag{2.13}$$

provided the sum absolutely converges and

$$\tau_{q,0}(f) = f, \tag{2.14}$$

where

$$K(q^m, q^n, q^k) = [J_{m-k}(q^{n-k}; q^2)]^2, \quad \text{for } m, n, k \in \mathbb{Z}. \tag{2.15}$$

The q-generalized translation is positive and verifies the following properties:

$$\tau_{q,x}(f)(y) = \tau_{q,y}(f)(x), \ \ x, \ y \in \mathbb{R}_{q,+}. \tag{2.16}$$

For $f \in L^1_q(\mathbb{R}_{q,+}, xd_qx)$, we have

$$\int_0^\infty \tau_{q,x}(f)(y)yd_qy = \int_0^\infty f(y)yd_qy, \ \ x \in \mathbb{R}_{q,+}. \tag{2.17}$$

For $f, g \in L^1_q(\mathbb{R}_{q,+}, xd_qx)$, we have

$$\int_0^\infty \tau_{q,x}(f)(y)g(y)yd_qy = \int_0^\infty f(y)\tau_{q,x}(g)(y)yd_qy, \ \ x \in \mathbb{R}_{q,+}. \tag{2.18}$$

$$\tau_{q,x}J_0(.; q^2)(y) = J_0(x; q^2)J_0(y; q^2), \ \ x, \ y, \in \mathbb{R}_{q,+}. \tag{2.19}$$

2.4. The q-Bessel Fourier transform

For $f \in L^1_q(\mathbb{R}_{q,+}, xd_qx)$, we define the q-Bessel Fourier transform by:

$$\mathcal{F}_q(f)(\lambda) = \frac{1}{1-q}\int_0^\infty f(x)J_0(\lambda x; q^2)xd_qx, \quad \lambda \in \widetilde{\mathbb{R}}_{q,+}. \tag{2.20}$$

This transform satisfies the following properties:

i) For $f \in L^1_q(\mathbb{R}_{q,+}, xd_qx)$,

$$\|\mathcal{F}_q(f)\|_{\infty,q} \le \frac{1}{1-q}\|f\|_{1,q}. \tag{2.21}$$

ii) For $f \in L^1_q(\mathbb{R}_{q,+}, xd_qx)$ we have

$$\mathcal{F}_q(\tau_{q,x}f)(\lambda) = J_0(\lambda x; q^2)\mathcal{F}_q(f)(\lambda), \ x, \ \lambda \in \widetilde{\mathbb{R}}_{q,+}. \tag{2.22}$$

iii) If f, D_q^+f, $\Delta_q f \in L^1_q(\mathbb{R}_{q,+}, xd_qx)$ and $xD_q^+f(x) \to 0$ as $x \downarrow 0$ in $\mathbb{R}_{q,+}$, then

$$\mathcal{F}_q(\Delta_q f)(\lambda) = -\lambda^2\mathcal{F}_q(f)(\lambda), \ \ \lambda \in \mathbb{R}_{q,+}. \tag{2.23}$$

THEOREM 2.1. *For all f in $L^1_q(\mathbb{R}_{q,+}, xd_qx)$, we have*

$$f(x) = \frac{1}{1-q}\int_0^\infty \mathcal{F}_q(f)(\lambda)J_0(\lambda x; q^2)\lambda d_q\lambda, \quad x \in \mathbb{R}_{q,+}. \tag{2.24}$$

THEOREM 2.2. i) *Plancherel formula for $\mathcal{F}_q$:*
a) *$\mathcal{F}_q$ is an isomorphism from $S_{*,q}(\mathbb{R}_q)$ onto itself and $\mathcal{F}_q^{-1} = \mathcal{F}_q$.*
b) *For all f in $S_{*q}(\mathbb{R}_q)$, we have*

$$\|\mathcal{F}_q(f)\|_{2,q} = \|f\|_{2,q}. \tag{2.25}$$

ii) *Plancherel theorem for $\mathcal{F}_q$:*
The q-Bessel Fourier transform $f \to \mathcal{F}_q$ can be uniquely extended to an isometric isomorphism on $L^2_q(\mathbb{R}_{q,+}, xd_qx)$.

2.5. The q-convolution product

We define the q-convolution product for two suitable functions f and g by

$$f *_B g(x) = \frac{1}{1-q}\int_0^\infty \tau_{q,x} f(y)g(y)yd_qy, \quad x \in \mathbb{R}_{q,+}. \tag{2.26}$$

This q-convolution product is commutative, associative and satisfies the following properties:

i) Let $1 \le p, k, r \le +\infty$, such that $\frac{1}{p}+\frac{1}{k}-\frac{1}{r} = 1$. If f is in $L^p_q(\mathbb{R}_{q,+}, xd_qx)$ and g an element of $L^k_q(\mathbb{R}_{q,+}, xd_qx)$, then $f *_B g$ belongs to $L^r_q(\mathbb{R}_{q,+}, xd_qx)$ and we have

$$\|f *_B g\|_{r,q} \le \frac{1}{1-q}\|f\|_{p,q}\|g\|_{k,q}. \tag{2.27}$$

ii) Let f be in $L^1_q(\mathbb{R}_{q,+}, xd_qx)$ and g in $L^2_q(\mathbb{R}_{q,+}, xd_qx)$.
The q-convolution product of f and g is the function $f *_B g$ of $L^2_q(\mathbb{R}_{q,+}, xd_qx)$ satisfying

$$\mathcal{F}_q(f *_B g) = \mathcal{F}_q(f)\mathcal{F}_q(g). \tag{2.28}$$

Moreover, for f, g in $L^2_q(\mathbb{R}_{q,+}, xd_qx)$, the function $f *_B g$ belongs to $L^2_q(\mathbb{R}_{q,+}, xd_qx)$ if and only if the function $\mathcal{F}_q(f)\mathcal{F}_q(g)$ belongs to $L^2_q(\mathbb{R}_{q,+}, xd_qx)$ and (2.28) holds.

iii) Let f and g be in $L^2_q(\mathbb{R}_{q,+}, xd_qx)$. Then, we have

$$||f *_B g||^2_{2,q} = \int_0^\infty |\mathcal{F}_q(f)(\xi)|^2|\mathcal{F}_q(g)(\xi)|^2\xi d_q\xi, \tag{2.29}$$

both members being finite or infinite.

3. The q-Bessel Gabor transform

In this section we show the q-analogue of the continuous Gabor transform associated with the zero order q-Bessel operator, and discuss their properties.

Notation. We denote by $X_q^p,\ p \in [1,\infty]$ the space of all functions f defined on $\mathbb{R}_{q,+} \times \mathbb{R}_{q,+}$ with respect to the measure $d\mu_q(x,y) = xyd_qxd_qy$ such that

$$\|f\|_{p,\mu_q} = \left(\int_0^\infty \int_0^\infty |f(x,y)|^p d\mu_q(x,y) \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty$$

and

$$\|f\|_{\infty,\mu_q} \;=\; ess \sup_{x,y \in \mathbb{R}_{q,+}} |f(x,y)|.$$

Definition 3.1. For any function g in $L_q^2(\mathbb{R}_{q,+}, xd_qx)$ and any ν in $\mathbb{R}_{q,+}$, we define the modulation of g by ν as:

$$\mathcal{M}_{q,\nu} g := g_{q,\nu} := \mathcal{F}_q(\sqrt{\tau_{q,\nu}(g^2)}). \tag{3.1}$$

Remark 3.1. For a function g in $L_q^2(\mathbb{R}_{q,+}, xd_qx)$ we have

$$\|g_{q,\nu}\|_{2,q} = \|g\|_{2,q}. \tag{3.2}$$

Let us define the family $g_{q,y}^{\nu}(x) = \tau_{q,y} g_{q,\nu}(x)$, for $x \in \mathbb{R}_{q,+}$.

Definition 3.2. Let g be a function in $L_q^2(\mathbb{R}_{q,+}, xd_qx)$. We define the q-Bessel Gabor transform $\mathcal{G}_g^q$ for a function f in $L_q^2(\mathbb{R}_{q,+}, xd_qx)$ by

$$\mathcal{G}_g^q f(y,\nu) := \frac{1}{1-q} \int_0^\infty f(x) g_{q,y}^{\nu}(x) x d_q x, \quad y,\nu \in \mathbb{R}_{q,+} \tag{3.3}$$

which can also be written in the form

$$\mathcal{G}_g^q f(y,\nu) := f *_B g_{q,\nu}(y). \tag{3.4}$$

From the relation (2.27) we have the following proposition:

Proposition 3.1. *For functions f, g in $L_q^2(\mathbb{R}_{q,+}, xd_qx)$, we have*

$$\|\mathcal{G}_g^q f\|_{\infty,\mu_q} \leq \frac{1}{1-q} \|f\|_{2,q} \|g\|_{2,q}.$$

Proposition 3.2. *(Plancherel formula)*
Let g be in $L^2_q(\mathbb{R}_{q,+}, xd_qx)$. Then, for all f in $L^2_q(\mathbb{R}_{q,+}, xd_qx)$, we have

$$||\mathcal{G}^q_g f||_{2,\mu_q} = \|g\|_{2,q}\|f\|_{2,q}. \tag{3.5}$$

P r o o f. The relation (2.29), Fubini's theorem, Theorem 2.2 and the relation (3.1) give the result. ∎

Remark 3.2. Let $g \in L^2_q(\mathbb{R}_{q,+}, xd_qx)\backslash\{0\}$. From Proposition 3.2 we can see that the normalized q-Bessel Gabor transform $\frac{1}{\|g\|_{2,q}}\mathcal{G}^q_g$ is an isometry from the Hilbert space $L^2_q(\mathbb{R}_{q,+}, xd_qx)$ into the Hilbert space X^2_q.

As in the classical case, the q-Bessel Gabor transform preserves the orthogonality relation, which is shown below.

Corollary 3.1. *Let g be a function in $L^2_q(\mathbb{R}_{q,+}, xd_qx)$. Then, for all f, h in $L^2_q(\mathbb{R}_{q,+}, xd_qx)$, we have*

$$\int_0^\infty\int_0^\infty \mathcal{G}^q_g f(y,\nu)\overline{\mathcal{G}^q_g h(y,\nu)}d\mu_q(y,\nu) = \|g\|^2_{2,q}\int_0^\infty f(x)\overline{h(x)}xd_qx. \tag{3.6}$$

Theorem 3.1. *(L^2_q inversion formula)*
Let g be a function in $\Big(L^2_q(\mathbb{R}_{q,+}, xd_qx) \cap L^\infty_q(\mathbb{R}_{q,+}, xd_qx)\Big) \setminus \{0\}$. Then, for any function f in $L^2_q(\mathbb{R}_{q,+}, xd_qx)$, we have

$$f = \lim_{N\to+\infty}\int_0^N\int_0^\infty \mathcal{G}^q_g(\mathcal{F}_q f)(y,\nu)\mathcal{F}_q(\tau_{q,y}g_{q,\nu})(.)\frac{d\mu_q(\nu,y)}{\|g\|^2_{2,q}} \tag{3.7}$$

in $L^2_q(\mathbb{R}_{q,+}, xd_qx)$.

P r o o f. Using the relations (3.4), (3.1), Theorem 2.2 i) and the fact that

$$\int_0^\infty \tau_{q,\nu}(|g|^2)(x)xd_qx = \|g\|^2_{2,q},$$

we get

$$f = \frac{1}{\|g\|^2_{2,q}}\int_0^\infty \mathcal{F}_q\Big(\mathcal{G}^q_g(\mathcal{F}_q f)(.,\nu)\Big)(.)\mathcal{F}_q(g_{q,\nu})(.)\nu d_q\nu\ ,\ a.e. \tag{3.8}$$

From the Cauchy-Schwarz inequality and Fubini's theorem, we obtain

$$\int_0^N\int_0^\infty |\mathcal{G}_g^q(\mathcal{F}_q f)(y,\nu)|\sqrt{\tau_{q,\nu}|g|^2(y)}d\mu_q(\nu,y)$$

$$\leq \|\mathcal{G}_g^q(\mathcal{F}_q f)\|_{2,\mu_q}\|g\|_{2,q}\|\chi_{[0,N]}\|_{1,q}<\infty.$$

Thus, from Fubini's theorem, the relation (2.22) and Theorem 2.2 i), we deduce that

$$f_N(x)=\frac{1}{\|g\|_{2,q}^2}\int_0^N\int_0^\infty \mathcal{G}_g^q(\mathcal{F}_q f)(y,\nu)\mathcal{F}_q(\tau_{q,y}g_{q,\nu})(x)\nu y d_q\nu\, d_q y$$

$$=\frac{1}{\|g\|_{2,q}^2}\int_0^\infty \mathcal{F}_q\Big(\chi_{[0,N]}\mathcal{G}_g^q(\mathcal{F}_q f)(.,\nu)\Big)(x)\mathcal{F}_q(g_{q,\nu})(x)\nu d_q\nu.$$

On the other hand, using the relation (3.8), we get

$$||f-f_N||_{2,q}^2=\frac{1}{\|g\|_{2,q}^4}\int_0^\infty\Big|\int_0^\infty \mathcal{F}_q\Big((1-\chi_{[0,N]})\mathcal{G}_g^q(\mathcal{F}_q(f)(.,\nu)\Big)(x)$$

$$\mathcal{F}_q(g_{q,\nu})(x)\nu d\nu\Big|^2 x d_q x.$$

Applying the Cauchy-Schwarz inequality, Fubini's theorem and Plancherel formula for the q-Bessel transform we obtain

$$||f-f_N||_{2,q}^2\leq\frac{1}{\|g\|_{2,q}^2}\|(1-\chi_{[0,N]})\mathcal{G}_g^q(\mathcal{F}_q f)\|_{2,\mu_q}^2.$$

Taking this result into consideration and by applying the dominated convergence theorem to it, we find that

$$||f-f_N||_{2,q}\to 0 \text{ as } N\to\infty.$$

This end the proof. ∎

COROLLARY 3.2. *(Coherent states).*
Let g be in $L_q^2(\mathbb{R}_{q,+},xd_qx)\backslash\{0\}$. Then, $\mathcal{G}_g^q\Big(L_q^2(\mathbb{R}_{q,+},xd_qx)\Big)$ is a reproducing kernel Hilbert space in X_q^2 with kernel function $\mathcal{K}_g(y,\nu;y',\nu')$ defined by

$$\mathcal{K}_g(y,\nu;y',\nu')=\frac{1}{\|g\|_{2,q}^2}\int_0^\infty \tau_{q,y}g_{q,\nu}(x)\tau_{q,y'}g_{q,\nu'}(x)xd_qx$$

$$= \frac{1-q}{\|g\|_{2,q}^2} \tau_{q,y} g_{q,\nu} *_B g_{q,\nu'}(x) \tag{3.9}$$

is pointwise bounded such that:

$$|\mathcal{K}_g(y',\nu';y,\nu)| \leq 1; \quad (y',\nu'),\ (y,\nu) \in \mathbb{R}_{q,+} \times \mathbb{R}_{q,+}. \tag{3.10}$$

P r o o f. From the representation of the q-Bessel Gabor transform given by the relations (3.4), (3.1) and the inversion formula (3.8), we get

$$\mathcal{G}_g^q f(y,\nu) = \frac{1}{\|g\|_{2,q}^2} \int_0^\infty \Big(\int_0^\infty \mathcal{F}_q(\mathcal{G}_g^q(f(.,\nu'))(x) \mathcal{F}_q(g_{q,\nu'})(x)\nu' d_q\nu' \Big)$$
$$\times\ \mathcal{F}_q(g_{q,\nu})(x) J_0(xy;q^2) x d_q x.$$

Thus, Fubini's theorem and the relation (2.22), Theorem 2.2 i) and the relation (2.28) give

$$\mathcal{G}_g^q f(y,\nu) = \frac{1}{\|g\|_{2,q}^2} \int_0^\infty \int_0^\infty \mathcal{F}_q(\mathcal{G}_g^q(f(.,\nu'))(x) \mathcal{F}_q(\tau_{q,y} g_{q,\nu} *_B g_{q,\nu'})(x) d\mu_q(\nu',x).$$

On the other hand, one can easily see that for every $y,\nu,\nu' \in \mathbb{R}_{q,+}$, the function

$$y' \mapsto \tau_{q,y} g_{q,\nu} *_B g_{q,\nu'}(y')$$

belongs to $L_q^2(\mathbb{R}_{q,+}, x d_q x)$. Therefore, the result follows by applying the Parseval formula for the q-Bessel Fourier transform. ■

To simplify the notation, we shall indicate $|\,.\,|_q$ the product measure $d\mu_q(x,y)$ in $\mathbb{R}_{q,+} \times \mathbb{R}_{q,+}$

Proposition 3.3. *Let f and g be two functions in $L_q^2(\mathbb{R}_{q,+}, x d_q x)$ such that $\|g\|_{2,q} = 1$. Suppose that $\|f\|_{2,q} = 1$. Then, for $U \subset \mathbb{R}_{q,+} \times \mathbb{R}_{q,+}$ and $\varepsilon > 0$ satisfying*

$$\int\int_U |\mathcal{G}_g^q f(y,\nu)|^2 d\mu_q(y,\nu) \geq 1 - \varepsilon,$$

we have

$$|U|_q \geq (1-\varepsilon)(1-q)^2.$$

P r o o f. Using Proposition 3.1 we obtain

$$\|\mathcal{G}_g^q f\|_{\infty,q} \leq \frac{1}{1-q}.$$

Hence,

$$1-\varepsilon \leq \int\int_{U} |\mathcal{G}_g^q f(y,\nu)|^2 d\mu_q(y,\nu) \leq \|\mathcal{G}_g^q f\|_{\infty,q}^2 |U|_q \leq (\frac{1}{1-q})^2 |U|_q.$$

Therefore,

$$(1-\varepsilon)(1-q)^2 \leq |U|_q.$$

■

Proposition 3.4. *Let f be in $L_q^2(\mathbb{R}_{q,+}, xd_qx)$ and g be a function in $L_q^2(\mathbb{R}_{q,+}, xd_qx)$ such that $\|g\|_{q,2} = 1$ and $p \in [2,\infty[$. Then,*

$$\int_0^\infty\int_0^\infty |\mathcal{G}_g^q f(y,\nu)|^p d_q(y,\nu) \leq \frac{1}{(1-q)^{p-2}} \|f\|_{2,q}^p. \tag{3.11}$$

P r o o f. Using Propositions 3.1 and 3.2, the result follows by applying the Riesz-Thorin interpolation theorem. ■

As a consequence of the inequality (3.11), we deduce that if the q-Bessel Gabor transform is essentially supported on a set $U \subset \mathbb{R}_{q,+} \times \mathbb{R}_{q,+}$ (example, when $\mathcal{G}_g^q f = |U|_q^{-\frac{1}{2}} \chi_U$), then $|U|_q \geq (1-q)^2$.

4. Applications

4.1. Approximative concentration of Gabor transform in quantum calculus

In order to prove a concentration result for the q-Bessel Gabor transform, we need the following notations:

$P_R : X_q^2 \longrightarrow X_q^2$ the orthogonal projection from X_q^2 onto $\mathcal{G}_g^q\Big(L_q^2(\mathbb{R}_{q,+}, xd_qx)\Big)$.

$P_M : X_q^2 \longrightarrow X_q^2$ the orthogonal projection from X_q^2 onto the subspace of function supported in M, where $M \subset \mathbb{R}_{q,+} \times \mathbb{R}_{q,+}$ with $|M|_q < \infty$.

We put

$$\|P_M P_R\|_q = \sup\Big\{\|P_M P_R v\|_{2,\mu_q},\ v \in X_q^2;\ \|v\|_{2,\mu_q} = 1\Big\}. \tag{4.1}$$

The aim result of this subsection is the following.

Theorem 4.1. *(Concentration of $\mathcal{G}_g^q f$ in small sets.)*
Let g be a function in $L^2(\mathbb{R}_{q,+}, xd_qx)$ and $M \subset \mathbb{R}_{q,+} \times \mathbb{R}_{q,+}$ with $(1-q)\sqrt{|M|_q} < 1$. Then, for all f in $L_q^2(\mathbb{R}_{q,+}, xd_qx)$ we have

$$\|\mathcal{G}_g^q f - \chi_M \mathcal{G}_g^q f\|_{2,\mu_q} \geq \left(1 - (1-q)\sqrt{|M|_q}\right) \|g\|_{2,q} \|f\|_{2,q}. \tag{4.2}$$

P r o o f. From the definition of P_M and P_R we have

$$\|\mathcal{G}_g^q f - \chi_M \mathcal{G}_g^q f\|_{2,\mu_q} = ||(I - P_M P_R)\mathcal{G}_g^q f||_{2,\mu_q}.$$

Thus, using Proposition 3.2 we obtain

$$\begin{aligned} \|\mathcal{G}_g^q f - \chi_M \mathcal{G}_g^q f\|_{2,\mu_q} &\geq \|\mathcal{G}_g^q f\|_{2,\mu_q}(1 - \|P_M P_R\|) \\ &\geq \|g\|_{2,q}\|f\|_{2,q}\|(1 - \|P_M P_R\|). \end{aligned} \tag{4.3}$$

As P_R is a projection onto a reproducing kernel Hilbert space, then, from Saitoh [13], P_R can be represented by

$$P_R F(y,\nu) = \int_0^\infty \int_0^\infty F(y',\nu')\mathcal{K}_g(y',\nu';y,\nu)d\mu_q(y',\nu')$$

with $\mathcal{K}_g$ defined by (3.9). Hence, for $F \in X_q^2$ arbitrary, we have

$$P_M P_R F(y,\nu) = \int_0^\infty \int_0^\infty \chi_M(y,\nu) F(y',\nu')\mathcal{K}_g(y',\nu';y,\nu)d\mu_q(y',\nu')$$

and its Hilbert-Schmidt norm

$$\|P_M P_R\|_{HS} = \Big(\int_0^\infty \int_0^\infty |\chi_M(y,\nu)|^2 |\mathcal{K}_g(y',\nu';y,\nu)|^2 d\mu_q(y',\nu')d\mu_q(y,\nu)\Big)^{\frac{1}{2}}.$$

By the Cauchy-Schwarz inequality we see that

$$\|P_M P_R\|_{HS} \geq \|P_M P_R\|_q. \tag{4.4}$$

On the other hand, from (3.9), Plancherel's formula for q-Bessel Fourier transform and Fubini's theorem, it is easy to see that

$$\|P_M P_R\|_{HS} \leq (1-q)\sqrt{|M|_q}. \tag{4.5}$$

Thus, from the relations (4.3), (4.4) and (4.5) we obtain the result.

■

4.2. Practical real inversion formulas for $\mathcal{G}_g^q$

In this paragraph we give practical real inversion formulas.

Let $s \in \mathbb{R}$. We define the space $H_q^s(\mathbb{R}_{q,+})$ by

$$H_q^s(\mathbb{R}_{q,+}) := \Big\{ f \in L_q^2(\mathbb{R}_{q,+}, xd_qx) \,:\, (1+\xi^2)^{s/2}\mathcal{F}_q(f) \in L_q^2(\mathbb{R}_{q,+}, xd_qx) \Big\}.$$

The space $H_q^s(\mathbb{R}_{q,+})$ provided with the inner product

$$\langle f, g\rangle_{H_q^s} = \int_0^{+\infty} (1+\xi^2)^s \mathcal{F}_q(f)(\xi)\overline{\mathcal{F}_q(g)}(\xi)\xi d_q\xi, \tag{4.6}$$

and the norm $\|f\|^2_{H_q^s} = \langle f, f\rangle_{H_q^s}$, is a Hilbert space.

Proposition 4.1. *Let g be a function in $L^2(\mathbb{R}_{q,+}, xd_qx) \bigcap L^\infty(\mathbb{R}_{q,+}, xd_qx)$ and $\nu \in \mathbb{R}_{q,+}$. The integral transform $\mathcal{G}_g^q(., \nu)$, is a bounded linear operator from $H_q^s(\mathbb{R}_{q,+})$, s in $\mathbb{R}_+$, into $L^2(\mathbb{R}_{q,+}, xd_qx)$, and we have*

$$\|\mathcal{G}_g^q f(., \nu)\|_{2,q} \leq ||g||_{\infty,q}\|f\|_{H_q^s}.$$

P r o o f. Let f be in $H_q^s(\mathbb{R}_{q,+})$. Using Theorem 2.2 we have

$$\|\mathcal{G}_g^q f(., \nu)\|^2_{2,q} = \|\mathcal{F}_q(\mathcal{G}_g^q f(., \nu))\|^2_{2,q}.$$

Involving the relationships (3.4),(3.1) and (2.25), we can write

$$\|\mathcal{G}_g^q f(., \nu)\|^2_{2,q} = \int_0^{+\infty} |\mathcal{F}_q(f)(\xi)|^2 \tau_{q,\nu}(g^2)(\xi)\xi d_q\xi.$$

Therefore

$$\|\mathcal{G}_g^q f(., \nu)\|_{2,q} \leq ||g||_{\infty,q}\|f\|_{H_q^s}.$$

∎

Definition 4.1. Let g be a function in $L^2(\mathbb{R}_{q,+}, xd_qx) \bigcap L^\infty(\mathbb{R}_{q,+}, xd_qx)$. Let $r > 0$, $\nu \in \mathbb{R}_{q,+}$ and $s \in \mathbb{R}_+$. We define the Hilbert space $H_q^{r,s}(\mathbb{R}_{q,+})$ as the subspace of $H_q^s(\mathbb{R}_{q,+})$ with the inner product:

$$\langle f, h\rangle_{H_q^{r,s}} = r\langle f, h\rangle_{H_q^s} + \langle \mathcal{G}_g^q f(., \nu), \mathcal{G}_g^q h(., \nu)\rangle_{2,q}, \quad f, h \in H_q^s(\mathbb{R}_{q,+}).$$

The norm associated to the inner product is defined by:

$$\|f\|^2_{H_q^{r,s}} := r\|f\|^2_{H_q^s} + \|\mathcal{G}_g^q f(., \nu)\|^2_{2,q}.$$

Proposition 4.2. *Let g be a function in $L^2(\mathbb{R}_{q,+}, xd_qx) \bigcap L^\infty(\mathbb{R}_{q,+}, xd_qx)$. For $s \geq 0$, the Hilbert space $H_q^{r,s}(\mathbb{R}_{q,+})$ admits the following reproducing kernel:*

$$P_r(x,y) = \frac{1}{(1-q)^2} \int_0^{+\infty} \frac{J_0(x\xi;q^2)J_0(y\xi;q^2)\xi d_q\xi}{r(1+\xi^2)^s + \tau_{q,\nu}(g^2)(\xi)}.$$

P r o o f. i) Let y be in $\mathbb{R}_{q,+}$, from Theorem 2.1 we can prove that there exists a function $x \mapsto P_r(x,y)$ belongs to $L^2(\mathbb{R}_{q,+}, xd_qx)$ such that we have

$$\mathcal{F}_q\Big(P_r(.,y)\Big)(\xi) = \frac{1}{1-q}\frac{J_0(y\xi;q^2)}{r(1+\xi^2)^s + \tau_{q,\nu}(g^2)(\xi)}. \tag{4.7}$$

On the other hand we have

$$\forall\, \xi \in \mathbb{R}_{q,+},\ \mathcal{F}_q\Big(\mathcal{G}_g^q(P_r(.,y))(.,\nu)\Big)(\xi) = \sqrt{\tau_{q,\nu}(g^2)(\xi)}\mathcal{F}_q\Big(P_r(.,y)\Big)(\xi). \tag{4.8}$$

Hence from Theorem 2.2 ii), we obtain

$$\begin{aligned}\|\mathcal{G}_g^q(P_r(.,y))(.,\nu)\|_{2,q}^2 &= \int_0^{+\infty} \tau_{q,\nu}(g^2)(\xi)|\mathcal{F}_q\Big(P_r(.,y)\Big)(\xi)|^2\xi d_q\xi \\ &\le \frac{C}{r^2}\int_0^{\infty} \frac{\tau_{q,\nu}(g^2)(\xi)|J_0(y\xi;q^2)|^2}{(1+\xi^2)^{2s}}\xi d_q\xi < \infty.\end{aligned}$$

Therefore we conclude that $\|P_r(.,y)\|^2_{H_q^{r,s}} < \infty$.

ii) Let f be in $H_q^{r,s}(\mathbb{R}_{q,+})$ and y in $\mathbb{R}_{q,+}$. Then

$$\langle f, P_r(.,y)\rangle_{H_q^{r,s}} = rI_1 + I_2, \tag{4.9}$$

where

$$I_1 = \langle f, P_r(.,y)\rangle_{H_q^s} \quad \text{and} \quad I_2 = \langle \mathcal{G}_g^q f(.,\nu), \mathcal{G}_g^q(P_r(.,y))(.,\nu)\rangle_{2,q}.$$

From (4.6) and (4.7), we have

$$I_1 = \frac{1}{1-q}\int_0^{+\infty} \frac{(1+\xi^2)^s\mathcal{F}_q(f)(\xi)J_0(y\xi;q^2)\xi d_q\xi}{r(1+\xi^2)^s + \tau_{q,\nu}(g^2)(\xi)}.$$

From (4.8),(4.7) and Theorem 2.2 ii) we have

$$I_2 = \frac{1}{1-q}\int_0^{+\infty} \frac{\tau_{q,\nu}(g^2)(\xi)\mathcal{F}_q(f)(\xi)J_0(y\xi;q^2)\xi d_q\xi}{r(1+\xi^2)^s + \tau_{q,\nu}(g^2)(\xi)}.$$

The relations (4.9) and (2.24) imply that

$$\langle f, P_r(.,y)\rangle_{H_q^{r,s}} = f(y).$$

■

4.3. Extremal function for q-Gabor transform

In this subsection, we prove for a given function g in $L^2(\mathbb{R}_{q,+}, xd_qx) \bigcap L^\infty(\mathbb{R}_{q,+}, xd_qx)$ that the infinitum of

$$\left\{ r\|f\|^2_{H^s_q} + \|h - \mathcal{G}^q_g f(.,\nu)\|^2_{2,q},\ f \in H^s_q(\mathbb{R}_{q,+}) \right\}$$

is attained at some function denoted by $f^*_{r,h}$, which is unique, called the extremal function. We start it with the following fundamental theorem (cf. [13]).

THEOREM 4.2. *Let H^r_K be a Hilbert space admitting the reproducing kernel $K_r(p,q)$ on a set E and H a Hilbert space. Let $L : H^r_K \to H$ be a bounded linear operator on H_K into H. For $r > 0$, we introduce the inner product in H^r_K and we call it H_{K_r} as*

$$\langle f_1, f_2 \rangle_{H_{K_r}} = r\langle f_1, f_2 \rangle_{H^r_K} + \langle Lf_1, Lf_2 \rangle_H.$$

Then:

i) *H_{K_r} is a Hilbert space with the reproducing kernel $K_r(p,q)$ on E and satisfying the equation*

$$K_r(.,q) = (rI + L^*L)K_r(.,q),$$

where L^* is the adjoint operator of $L : H_K \to H$.

ii) *For any $r > 0$ and for any h in H, the infinitum*

$$\inf_{f \in H_K} \left\{ r\|f\|^2_{H^r_K} + \|Lf - h\|^2_H \right\}$$

is attained by a unique function $f^*_{r,h}$ in H_K and this extremal function is given by

$$f^*_{r,h}(p) = \langle h, LK_r(.,p) \rangle_H. \tag{4.10}$$

We can now state the main result of this paragraph.

THEOREM 4.3. *Let g be a function in $L^2(\mathbb{R}_{q,+}, xd_qx) \bigcap L^\infty(\mathbb{R}_{q,+}, xd_qx)$. Let $s \geq 0$. For any h in $L^2(\mathbb{R}_{q,+}, xd_qx)$ and for any $r > 0$, the infinitum*

$$\inf_{f \in H^s_q} \left\{ r\|f\|^2_{H^s_q} + \|h - \mathcal{G}^q_g f(.,\nu)\|^2_{2,q} \right\} \tag{4.11}$$

is attained by a unique function $f^*_{r,h}$ given by

$$f^*_{r,h}(x) = \int_0^{+\infty} h(y) Q_r(x,y) y d_q y, \tag{4.12}$$

where

$$Q_r(x,y) = Q_{r,s}(x,y) = \frac{1}{(1-q)^2}\int_0^{+\infty} \frac{\sqrt{\tau_{q,\nu}(g^2)(\xi)}J_0(x\xi;q^2)J_0(y\xi;q^2)}{r(1+\xi^2)^s + \tau_{q,\nu}(g^2)(\xi)}\xi d_q\xi. \tag{4.13}$$

P r o o f. By Proposition 4.2 and Theorem 4.2 ii), the infinitum given by (4.11) is attained by a unique function $f^*_{r,h}$, and from (4.10) the extremal function $f^*_{r,h}$ is represented by

$$f^*_{r,h}(y) = \langle h, \mathcal{G}^q_g(P_r(.,y))(.,\nu)\rangle_{2,q}, \quad y \in \mathbb{R}_{q,+},$$

where P_r is the kernel given by Proposition 4.2. On the other hand we have

$$\mathcal{G}^q_g f(x,\nu) = \frac{1}{1-q}\int_0^{+\infty} \sqrt{\tau_{q,\nu}(g^2)(\xi)}\mathcal{F}_q(f)(\xi)J_0(x\xi;q^2)\xi d_q\xi, \quad \text{for all } x \in \mathbb{R}_{q,+}.$$

Hence by (4.8), we obtain

$$\mathcal{G}^q_g\Big(P_r(.,y)\Big)(.,\nu)(x) = \frac{1}{(1-q)^2}\int_0^{+\infty} \frac{\sqrt{\tau_{q,\nu}(g^2)(\xi)}J_0(x\xi;q^2)J_0(y\xi;q^2)}{r(1+\xi^2)^s + \tau_{q,\nu}(g^2)(\xi)}\xi d_q\xi$$
$$= Q_r(x,y).$$

This gives (4.13). ∎

Corollary 4.1. *Let g be a function in $L^2(\mathbb{R}_{q,+}, xd_qx)\bigcap L^\infty(\mathbb{R}_{q,+}, xd_qx)$, $s \geq 0$, $r,\delta > 0$ and h, h_δ in $L^2(\mathbb{R}_{q,+}, xd_qx)$ such that*

$$\|h - h_\delta\|_{2,q} \leq \delta.$$

Then

$$\|f^*_{r,h} - f^*_{r,h_\delta}\|_{H^s_q} \leq \frac{\delta}{2\sqrt{r}}.$$

P r o o f. From (4.13) and Fubini's theorem we have

$$\mathcal{F}_q(f^*_{r,h})(\xi) = \frac{\sqrt{\tau_{q,\nu}(g^2)(\xi)}\mathcal{F}_q(h)(\xi)}{r(1+\xi^2)^s + \tau_{q,\nu}(g^2)(\xi)}. \tag{4.14}$$

Hence

$$\mathcal{F}_q(f^*_{r,h} - f^*_{r,h_\delta})(\xi) = \frac{\sqrt{\tau_{q,\nu}(g^2)(\xi)}\mathcal{F}_q(h-h_\delta)(\xi)}{r(1+\xi^2)^s + \tau_{q,\nu}(g^2)(\xi)}.$$

Using the inequality $(x+y)^2 \geq 4xy$, we obtain

$$(1+\xi^2)^s\Big|\mathcal{F}_q(f^*_{r,h}-f^*_{r,h_\delta})(\xi)\Big|^2 \leq \frac{1}{4r}|\mathcal{F}_q(h-h_\delta)(\xi)|^2.$$

Thus and from Theorem 2.2 ii) we obtain

$$\|f^*_{r,h}-f^*_{r,h_\delta}\|^2_{H^s_q} \leq \frac{1}{4r}\|\mathcal{F}_q(h-h_\delta)\|^2_{2,q} = \frac{1}{4r}\|h-h_\delta\|^2_{2,q},$$

which gives the desired result. ■

COROLLARY 4.2. *Let g be a function in $L^2(\mathbb{R}_{q,+},xd_qx)\bigcap L^\infty(\mathbb{R}_{q,+},xd_qx)$. Let $s \geq 0$ and $r > 0$. If f is in $H^s_q(\mathbb{R}_{q,+})$ and $h = \mathcal{G}^q_g f(.,\nu)$. Then*

$$\|f^*_{r,h}-f\|^2_{H^s_q} \to 0 \quad \text{as } r \to 0.$$

P r o o f. From (4.4), we have

$$\mathcal{F}_q(f^*_{r,h})(\xi) = \frac{\sqrt{\tau_{q,\nu}(g^2)(\xi)}\mathcal{F}_q(h)(\xi)}{r(1+\xi^2)^s+\tau_{q,\nu}(g^2)(\xi)}.$$

Hence

$$\mathcal{F}_q(f^*_{r,h}-f)(\xi) = \frac{-r(1+\xi^2)^s\mathcal{F}_q(f)(\xi)}{r(1+\xi^2)^s+\tau_{q,\nu}(g^2)(\xi)}.$$

Then we obtain

$$\|f^*_{r,h}-f\|^2_{H^s_q} = \int_0^\infty h_{r,t,s}(\xi)|\mathcal{F}_q(f)(\xi)|^2\xi d_q\xi,$$

with

$$h_{r,t,s}(\xi) = \frac{r^2(1+\xi^2)^{3s}}{\Big(r(1+\xi^2)^s+\tau_{q,\nu}(g^2)(\xi)\Big)^2}.$$

Since

$$\lim_{r\to 0} h_{r,t,s}(\xi) = 0, \quad \text{and} \quad |h_{r,t,s}(\xi)| \leq (1+\xi^2)^s,$$

we obtain the result from the dominated convergence theorem. ■

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