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Structural Optimization by Density Distribution for Maximization of Natural Frequency

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This paper proposes a basic method for designing light and rigid structures that have a maximum natural frequency for a designated mode. A design variable "density," related to the material properties of a three-dimensional solid element, is introduced into the finite element method (FEM). Thus, a structure is expressed as a density distribution inside its design domain, and the optimal structure is obtained by searching for the most suitable such distribution.

Introduction

Various kinds of mobile mechanisms need to be designed lightly and rigidly. For example, accurate hard-disk head positioners must be designed to have higher natural frequencies to avoid interference with their control systems. In such cases, the natural frequency of a certain vibration mode needs to be increased in order to reduce the maximum displacement of a certain point on the structure. The purpose of the proposed method is to increase the natural frequency of the most critical mode for a given application by optimizing the shape of a structure.

Precise dimensions of a structure can be optimized by computation. However, topology, such as choices between tubes and solid cylinders or the best number of holes to be made in an object, is designed on an empirical basis.

To determine optimal topology, Bendsøe and Kikuchi proposed a material property-driven topology-optimization method using homogenization (1988). Homogenization is a method of estimating the effective properties of materials having microstructures.

A design variable, "density" is introduced in this paper. This value is related to the material properties of a three-dimensional solid element. The relation between the density and the Young's modulus is expressed by an estimate based on the Voigt and the Reuss models. The estimation assumes that the microstructures are cubes. Only the one variable, density, serves as an indicator of material properties of an element, while the homogenization method requires at least 6 variables to determine material properties. We show here that the natural frequency of any designated mode can be increased by repeatedly modifying the density distribution according to the sensitivities of each individual element's densities to that natural frequency.

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Structural Optimization

Optimum Criteria and Voluminal Sensitivity. We have formulated an optimization problem as follows, and will introduce a new concept, voluminal sensitivity, to solve it:

$$\begin{aligned}
&\mathbf{K}(\mathbf{v}) - \mu\mathbf{M}(\mathbf{v}) = 0,\\
&(2\pi f)^2 = \mu,\\
&\text{maximize} \quad f_j(\mathbf{v}),\\
&\text{with respect to} \quad \mathbf{v} = (v_1, v_2, \ldots, v_n) \quad \text{for} \quad (0 < v < 1),\\
&\text{subject to} \quad m_{\text{total}}(\mathbf{v}) = m_0,
\end{aligned} \tag{1}$$

where f_j denotes the natural frequency of the designated mode, m_{total} denotes the total mass of the structure, $\mathbf{v}$ denotes a vector whose components are design variables, $\mathbf{K}$ and $\mathbf{M}$ denotes the stiffness and mass matrices, respectively, for the FEM model. The first equation represents the generalized eigenvalue problem of three-dimensional elasticity. The design variable v is the effective density of a porous material defined in the next section. The energy function is

$$\mathcal{L}' = \tfrac{1}{2}\mathbf{y}^T\mathbf{K}\mathbf{y} - \mu(\tfrac{1}{2}\mathbf{y}^T\mathbf{M}\mathbf{y} - \tfrac{1}{2}). \tag{2}$$

When μ is the eigenvalue of eigenvector $\mathbf{y}$, this functional is stationary, and the natural frequency f_j of designated mode j is derived from eigenvalue μ_j, via the second equation in (1).

When the set of variables $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ maximizes the natural frequency $f_j(\mathbf{v})$ with mass held constant, it will make the following functional stationary:

$$\mathcal{L} = \tfrac{1}{2}\mathbf{y}^T\mathbf{K}\mathbf{y} - \mu(\tfrac{1}{2}\mathbf{y}^T\mathbf{M}\mathbf{y} - \tfrac{1}{2}) - \lambda(m_{\text{total}} - m_0). \tag{3}$$

Therefore, the optimum criteria (Gallagher, 1973) will satisfy the following equations:

$$\begin{aligned}
&\mathbf{K}(\mathbf{v})\mathbf{y} = \mu\mathbf{M}(\mathbf{v})\mathbf{y},\\
&m_{\text{total}}(\mathbf{v}) = m_0,\\
&\tfrac{1}{2}\mathbf{y}^T\mathbf{M}(\mathbf{v})\mathbf{y} = \tfrac{1}{2},\\
&\frac{1}{2}\mathbf{y}^T\left(\frac{\partial\mathbf{K}(\mathbf{v})}{\partial v_i}\right)\mathbf{y} - \frac{1}{2}\mu\mathbf{y}^T\left(\frac{\partial\mathbf{M}(\mathbf{v})}{\partial v_i}\right)\mathbf{y}\\
&\quad - \lambda\left(\frac{\partial m_{\text{total}}(\mathbf{v})}{\partial v_i}\right) = 0, \quad \text{for} \quad (i = 1, 2, \ldots, n).
\end{aligned} \tag{4}$$

To find the numerical solution $\mathbf{v}_{opt}$ that satisfies Eqs. (4) simultaneously, the voluminal sensitivity s_{ij} is defined as the following:

$$s_{ij} = \frac{(\partial f_j/\partial v_i)}{(\partial m_{\text{total}}/\partial v_i)} = \frac{1}{4\pi\sqrt{\mu_j}}\frac{(\partial\mu_j/\partial v_i)}{(\partial m_{\text{total}}/\partial v_i)},$$

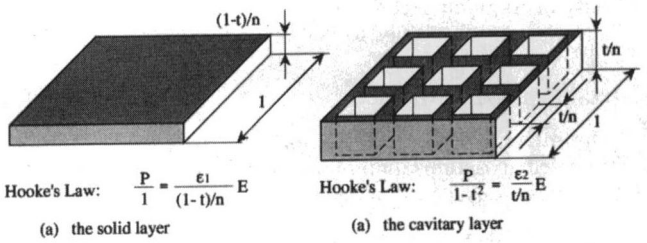
where

$$\frac{\partial\mu_j}{\partial v_i} = \mathbf{y}_j^T\left(\frac{\partial\mathbf{K}(\mathbf{v})}{\partial v_i}\right)\mathbf{y}_j - \mu_j\mathbf{y}_j^T\left(\frac{\partial\mathbf{M}(\mathbf{v})}{\partial v_i}\right)\mathbf{y}_j. \tag{5}$$

Voluminal sensitivity shows the total increase in the jth natural frequency if the density value of the corresponding element were to increase from 0 to 1. All of the elements' voluminal sensitivities become uniform value for all of the variables $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ when the optimum criteria are satisfied. We define the convergence value ϵ of the voluminal sensitivities as

$$\epsilon = \frac{1}{n}\sum(s_i - s_{\text{ave}})^2, \tag{6}$$

where n is the number of design variables, and s_{ave} is the average



where P: the force load on the cross section
ε: strain
n: number of holes per unit length
t: the edge length of the cubic cavities of the unit cell

Fig. 1 Young's modulus of cavitary layer (estimated by the Voigt model)

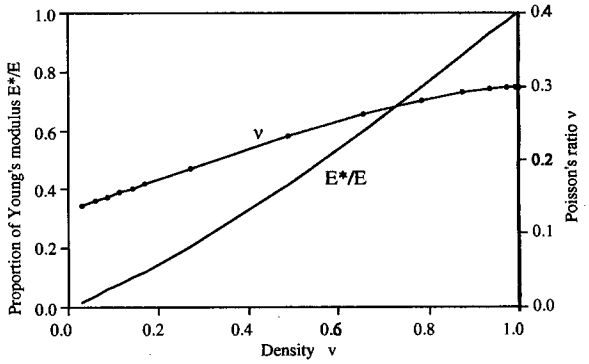


Fig. 3 Material properties vs density

voluminal sensitivity. The value ϵ will approach 0 as the objective function $f_j(\mathbf{v})$ increases to its maximum value.

Density. We defined a variable "density" for a hypothetical material that has porous homogeneous microstructure. Density is defined to mean the proportion of the solid part within a unit of substructure. Density will take a value in the range between 0 and 1. If the design domain of a structure is divided as in an FEM model, that structure is expressed as elements, the densities of which are equal to 1. And "vacancies" can be expressed as a group of elements the densities of which are 0. Our objective, is to find within a limited weight, the optimum density distribution giving the highest natural frequency for the designated vibration mode.

We have estimated the relation between the density and the equivalent Young's modulus by using a combination of the Voigt and the Reuss models (Kingery, 1976). The Voigt model is introduced to evaluate the equivalent Young's modulus of a single cavitary layer, Fig. 1(b), and the Reuss model is introduced to evaluate the equivalent Young's modulus of the multiply-layered substructure, Fig. 2. The strain on each layer is assumed to be independent of neighboring layers. Therefore, the total of the longitudinal strains of the component layers makes up the longitudinal strain of a unit of substructure. The Young's modulus of the unit substructure is expressed as

$$E^* = \frac{1 - (1 - v)^{2/3}}{2 - v - (1 - v)^{2/3}} E. \tag{7}$$

The curves in Fig. 3 are the graphs of density vs. Young's modulus and Poisson's ratio. The latter is calculated by interpolating from the results of FEM simulations

$$\nu^* = -0.082721v^3 + 0.054451v^2 + 0.19598v + 0.13288. \tag{8}$$

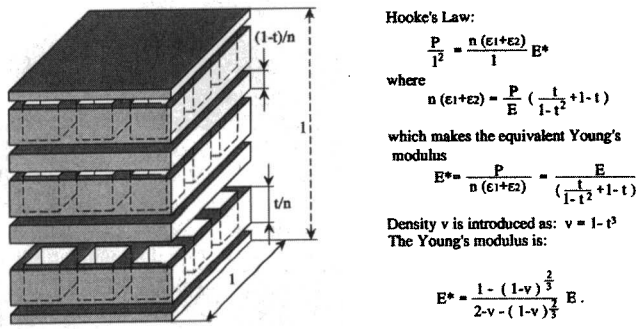


Fig. 2 Young's modulus of a multi-layer unit cell (estimated by the Reuss model)

Voluminal Sensitivity of an Element. The stiffness matrix $\mathbf{K}$ and the mass matrix $\mathbf{M}$ of an FEM model are composed of submatrices $\mathbf{k}_i$ and $\mathbf{m}_i$

$$\mathbf{K} = \sum \mathbf{k}_i$$

$$\mathbf{M} = \sum \mathbf{m}_i \quad (i = 1, 2, \ldots, n). \tag{9}$$

The submatrices $\mathbf{k}_i$ and $\mathbf{m}_i$ for the ith element are defined by the following:

$$k_i = \int B_i^T D_i B_i \, dV,$$

$$m_i = \rho_i \int B_i^T B_i \, dV, \tag{10}$$

where

$$\mathbf{D}_i = \mathbf{D}_i \, (E^*(v_i), \nu^*(v_i)), \tag{11}$$

and ρ_i denotes the weight of one unit of substructure for element i. The strain-displacement matrix $\mathbf{B}_i$ is defined by the constituent nodes of the ith element and is independent of its material properties. The stress-strain matrix $\mathbf{D}_i$, on the other hand, is independent of the nodes of the element. To prevent the complication of calculation that would be caused if we allowed for the movement of the nodes because they are shared with neighboring elements, the differentials of the matrices $\mathbf{k}_i$ and $\mathbf{m}_i$ are obtained simply by differentiating the material properties according to the density of the element, so they do not involve any shape-defining variables.

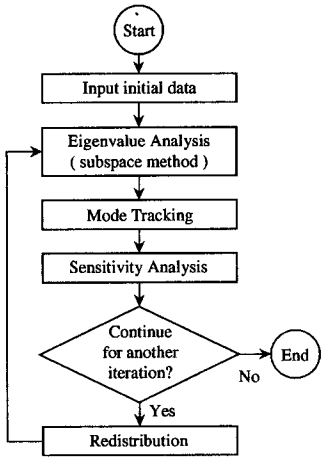


Fig. 4 Flowchart for calculating density distribution according to voluminal sensitivities

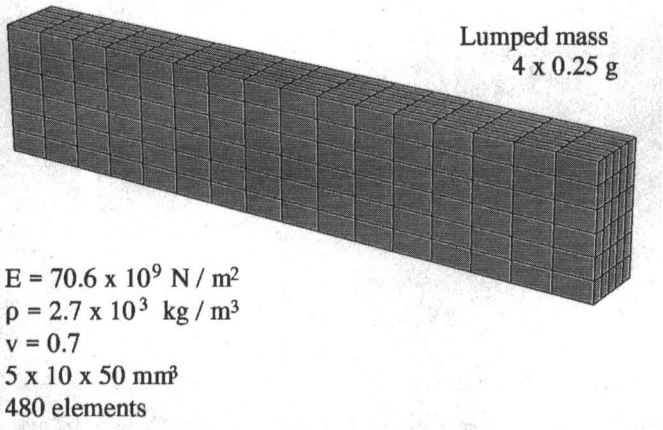


Fig. 5 Initial density distribution

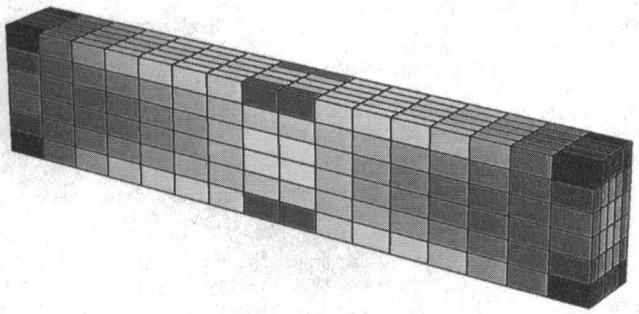


Fig. 6 Initial sensitivity distribution

Optimization Program

The flowchart of the optimization program is shown in Fig. 4. Initial data includes node data, element data defining the design domain, material properties, constraint conditions, lumped masses, and the initial, uniform density distribution. As the order of the designated mode may shift during the iterative calculations, it is necessary to identify which of the eigenvectors is that of the designated mode. Whether to continue on to another iteration is determined by the convergence value ϵ. If this value becomes less than a certain threshold value, the calculation is discontinued, otherwise the density distribution is modified and another eigenvalue analysis is performed. The density values are revised by the vector space method according to the values of corresponding voluminal sensitivities. Both minimum value v_{min}, and maximum value v_{max}, are imposed on all of the values v_i so that they will take values between 0 and 1.

Example

We applied this method to a rectangular beam structure (Fig. 5) made of aluminum with both ends fixed. The initial model had a uniform density of $v = 0.7$, and the four nodes at the center of the beam's edges had extra lumped mass. The objective function was the natural frequency of the torsional vibration mode, which was initially at the third mode, 15.10 kHz. The voluminal sensitivity distribution of the initial model (Fig. 6) shows the areas where densities should be increased or decreased. The convergence value ϵ was 0.1159.

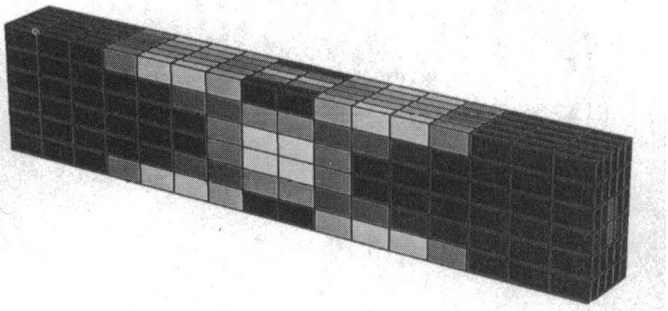


Fig. 7 Final density distribution

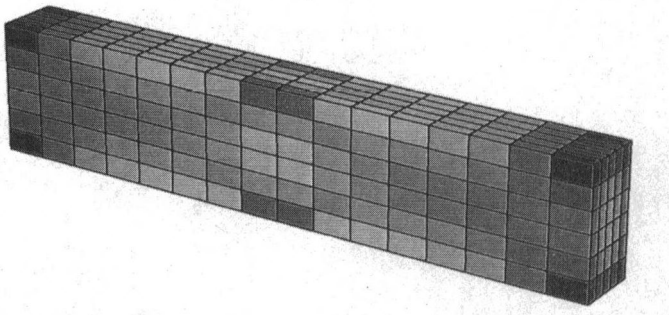


Fig. 8 Final sensitivity distribution

The final density distribution after 20 iterative calculations is shown in Fig. 7. With this density distribution, the natural frequency was increased to 16.66 kHz. The densities were limited to be between 0 and 1. The black parts of Fig. 7 represent areas where the densities have become 1, and the white parts represent areas where the densities have become 0. The final density distribution suggests a possible structure with a topology that differs from that of the initial model, i.e., a bar with the lighter areas carved away. Figure 8 shows the voluminal sensitivity distribution of the final model, with its convergence value $\epsilon = 0.0016$. The voluminal sensitivities of the final model have become less disparate than those of the initial model.

Summary

We have proposed a new method for structural optimization. A design variable "density" was defined and related to the elastic properties. By differentiating the eigenvalue according to the density, the "voluminal sensitivity" for the natural frequency of a designated mode was calculated for each element. This voluminal sensitivity of an element showed whether its density should be increased or reduced to maximize the natural frequency. After iterated calculations to homogenize the voluminal sensitivities of the elements, the optimal density distribution for a designated vibration mode was obtained. The final density distribution suggested an optimum structure with a topology differing from that of the initial model.

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