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A new robust optimization approach for scheduling under uncertainty: I. Bounded uncertainty

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Abstract

The problem of scheduling under bounded uncertainty is addressed. We propose a novel robust optimization methodology, which when applied to mixed-integer linear programming (MILP) problems produces "robust" solutions which are in a sense immune against bounded uncertainty. Both the coefficients in the objective function, the left-hand-side parameters and the right-hand-side parameters of the inequalities are considered. Robust optimization techniques are developed for two types of uncertain data: bounded uncertainty and bounded and symmetric uncertainty. By introducing a small number of auxiliary variables and constraints, a deterministic robust counterpart problem is formulated to determine the optimal solution given the (relative) magnitude of uncertain data, feasibility tolerance, and "reliability level" when a probabilistic measurement is applied. The robust optimization approach is then applied to the scheduling under uncertainty problem. Based on a novel and effective continuous-time short-term scheduling model proposed by Floudas and coworkers [Ind. Eng. Chem. Res. 37 (1998a) 4341; Ind. Eng. Chem. Res. 37 (1998b) 4360; Ind. Eng. Chem. Res. 38 (1999) 3446; Comp. Chem. Engng. 25 (2001) 665; Ind. Eng. Chem. Res. 41 (2002) 3884; Ind. Eng. Chem. Res. (2003)], three of the most common sources of bounded uncertainty in scheduling problems are addressed, namely processing times of tasks, market demands for products, and prices of products and raw materials. Computational results on several small examples and an industrial case study are presented to demonstrate the effectiveness of the proposed approach.

Keywords: Process scheduling; Uncertainty; Robust optimization; MILP; Bounded uncertainty

1. Introduction

The research area of production scheduling has received considerable attention from both the academia and the chemical processing industries over the past decade. Most of the work in the literature assumes that all data are deterministic, that is, they are of constant known values. However, in reality, uncertainty is prevalent in the context of scheduling due to lack of accurate process models and variability of process and environmental data. Therefore, it is of crucial importance to develop systematic methods to address the problem of scheduling under uncertainty, in order to create efficient and reliable schedules (Floudas & Lin, 2003).

The issue of robustness in scheduling under uncertainty has received relatively little attention, in spite of its importance and the fact that there has been a substantial amount of work to address the problem of design and operation of batch plants under uncertainty. Most of the existing work has followed the scenario-based framework, in which the uncertainty is modeled through the use of a number of scenarios, using either discrete probability distributions or the discretization of continuous probability distribution functions, and the expectation of a certain performance criterion, such as the expected profit which is optimized with respect to the scheduling decision variables. Bassett, Pekny, and Reklaitis (1997) considered process uncertainties in processing time fluctuations, equipment reliability/availability, process yields, demands, and manpower changes. They used Monte Carlo sampling to generate random instances, determined a schedule for each instance, and generated distribution of aggregated properties to infer operating policies. Ierapetritou and Pistikopoulos (1996) addressed the scheduling of single-stage and multistage multiproduct continuous plants with a single production line at each stage when uncertainty in product demands is involved. They used Gaussian quadrature integration to evaluate the expected profit and formulated MILP models. Vin and Ierapetritou (2001) considered demand uncertainty for the short-term scheduling of multiproduct and multipurpose batch plants. They introduced several metrics to evaluate the robustness of a schedule and

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proposed a multiperiod programming model using extreme points of the demand range as scenarios to generate a single sequence of tasks with the minimal average makespan over all scenarios. Balasubramanian and Grossmann (2002) proposed a multiperiod MILP model for scheduling multistage flowshop plants with uncertain processing times. They minimized expected makespan and developed a special branch and bound algorithm with an aggregated probability model. The scenario-based approaches provide a straightforward way to implicitly incorporate uncertainty. However, they inevitably enlarge the size of the problem significantly as the number of scenarios increases exponentially with the number of uncertain parameters. This main drawback limits the application of these approaches to solve practical problems with a large number of uncertain parameters.

Sanmarti, Espuña, and Puigjaner (1997) presented a different approach for the scheduling of production and maintenance tasks in multipurpose batch plants in the face of equipment failure uncertainty. They computed a reliability index for each unit and for each scheduled task and formulated a nonconvex MINLP model to maximize the overall schedule reliability. Because of the significant difficulty in the rigorous solution of the resulting problem, a heuristic method was developed to find solutions that improve the robustness of an existing schedule. There have also been attempts to transform a stochastic model to direct deterministic equivalent representation. Orçun, Altinel, and Hortaçsu (1996) considered uncertain processing times in batch processes and employed chance constraints to account for the risk of violation of timing constraints under certain conditions such as uniform distribution functions.

An alternative approach for scheduling under uncertainty is reactive scheduling. It is carried out to adjust a schedule, which is usually obtained a priori in a deterministic manner, upon realization of the uncertain parameters or occurrence of unexpected events. Due to the "on-line" nature of reactive scheduling, it is required to generate updated schedules in a timely manner and often, heuristic approaches are developed for schedule modifications (e.g., Cott & Macchietto, 1989; Kanakamedala et al., 1994; Sanmarti et al., 1996; Rodrigues et al., 1996; Honkomp et al., 1999; Vin & Ierapetritou, 2000). A recent review on scheduling approaches that includes uncertainty issues can be found in Floudas & Lin (2003).

In this work, we propose a novel robust optimization approach to address the problem of scheduling under bounded uncertainty. The underlying framework is based on a robust optimization methodology first introduced for linear programming (LP) problems by Ben-Tal and Nemirovski (2000) and extended in this work for mixed-integer linear programming (MILP) problems. The approach produces "robust" solutions which are in a sense immune against uncertainties in both the coefficients and right-hand-side parameters of the inequality constraints. The approach can be applied to address the problem of production scheduling with uncertain processing times, market demands, and/or prices of products and raw materials. The rest of this paper is organized as follows. We will first define the problem under investigation and present a motivating example. Then a robust optimization approach is proposed for general MILP problems with uncertain parameters in the inequality constraints. Subsequently, the robust optimization approach is further applied to three classes of scheduling problems with uncertainty in processing times, product demands, and market prices, respectively. Finally, computational results are presented, followed by concluding remarks.

2. Problem statement

The scheduling problem of chemical processes is defined as follows. Given

(i) production recipes (i.e. the processing times for each task at the suitable units, and the amount of the materials required for the production of each product),
(ii) available equipment and the ranges of their capacities,
(iii) material storage policy,
(iv) production requirement, and
(v) time horizon under consideration,

determine

(i) the optimal sequence of tasks taking place in each unit,
(ii) the amount of material being processed at each time in each unit,
(iii) the processing time of each task in each unit,

so as to optimize a performance criterion, for example, to minimize the makespan or to maximize the overall profit.

The most common sources of uncertainty in the aforementioned scheduling problem are:

(i) the processing times of tasks,
(ii) the market demands for products, and
(iii) the prices of products and/or raw materials.

An uncertain parameter can be described using discrete or continuous distributions. In some cases, only limited knowledge about the distribution is available, for example, the uncertainty is bounded, or the uncertainty is symmetrically distributed in a certain range. In the best situation, the distribution function for the uncertain parameter is given, for instance, as a normal distribution with known mean and standard deviation. In this paper, we will focus on bounded uncertainty.

3. A motivating example

Consider the following example process that was first presented by Kondili, Pantelides, and Sargent (1993a) and has been widely studied in the literature. Two products can be produced from three feeds according to the State-Task Network as shown in Fig. 1 through one heating, three reaction and one separation tasks, and four intermediate materials.

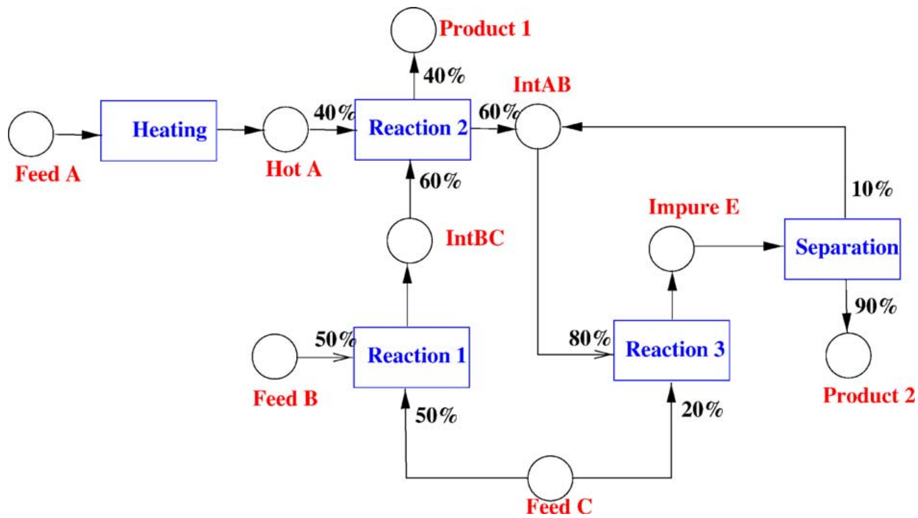

Fig. 1. State-Task Network for the motivating example.

Table 1
Data for the motivating example

Units	Capacity	Suitability	Processing time
Heater	100	Heating	1.0
Reactor 1	50	Reactions 1–3	2.0, 2.0, 1.0
Reactor 2	80	Reactions 1–3	2.0, 2.0, 1.0
Separator	200	Separation	2.0
States	Storage capacity	Initial amount	Price
Feed A	Unlimited	Unlimited	0.0
Feed B	Unlimited	Unlimited	0.0
Feed C	Unlimited	Unlimited	0.0
Hot A	100	0.0	0.0
IntAB	200	0.0	0.0
IntBC	150	0.0	0.0
ImpureE	200	0.0	0.0
Product 1	Unlimited	0.0	10.0
Product 2	Unlimited	0.0	10.0

One heater, two reactors and one separator are available, of which the data is given in Table 1. The data of the states involved is also provided in Table 1. The objective is to maximize the profit from sales of products manufactured in a time horizon of 12 h.

The continuous-time formulation proposed by Floudas and coworkers (Ierapetritou & Floudas, 1998a,b; Ierapetritou et al., 1999; Lin & Floudas, 2001; Lin et al., 2002; Lin, Chajakis, & Floudas, 2003) is used to solve this simple scheduling problem. The "nominal" solution is shown in Fig. 2, which features intensive utilization of the two reactors and an objective value (profit) of 3639. However, this solution can become completely infeasible when there is uncertainty in the processing times of the tasks. That is, when a task requires longer processing time than its nominal value, it will not be able to finish processing within the time interval assigned in the nominal schedule. In this example, even a very small perturbation may make the schedule infeasible and have a substantial effect on the scheduling decisions. For instance, if the processing time of each task is increased by simply 0.1% of its nominal value, then the nominal schedule will become infeasible and the optimal schedule with the slightly increased processing times will be significantly different from the nominal schedule, as shown in Fig. 3. In the heater and the separator, the number of tasks as well as processing amounts changes, while in the two reactors, even the task sequences are different. Furthermore, the profit is reduced considerably to 3265.

It is clear that solving a scheduling problem at the nominal values of the uncertain data is not enough. To obtain reliable and efficient schedules, systematic and effective approaches to take into account uncertainty are required. In this work, we propose a new robust optimization framework to generate schedules that are reliable in the presence of uncertainty arising from various sources. The framework utilizes a continuous-time MILP formulation for the

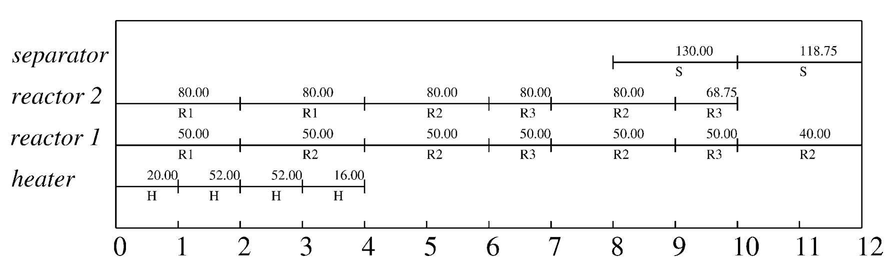

Fig. 2. Optimal solution with nominal processing times (profit = 3639).

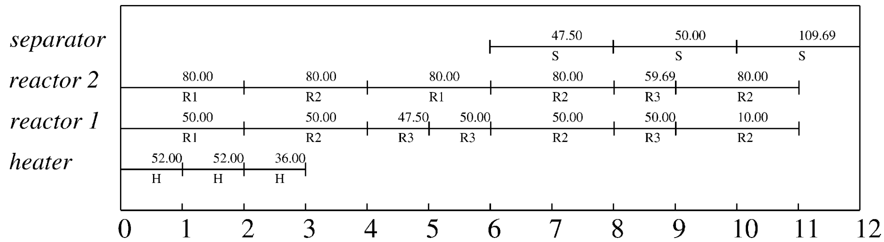

Fig. 3. Optimal solution with processing times increased by 0.1% (profit = 3265).

scheduling of general chemical processes proposed by Floudas and coworkers (Ierapetritou & Floudas, 1998a,b; Ierapetritou et al., 1999; Lin & Floudas, 2001; Lin et al., 2002; Lin, Chajakis, & Floudas, 2003).

4. Robust optimization for MILP problems

Consider the following generic mixed-integer linear programming (MILP) problem:

$$\begin{aligned} &\operatorname*{Min/Max}_{x,y}\; c^{\mathrm{T}}x + d^{\mathrm{T}}y \\ &\text{s.t.}\quad Ex + Fy = e \\ &\qquad Ax + By \le p \\ &\qquad \underline{x} \le x \le \bar{x} \\ &\qquad y = 0, 1 \end{aligned} \tag{1}$$

Assume that the uncertainty arises from both the coefficients and the right-hand-side parameters of the inequality constraints, namely, a_{lm}, b_{lk} and p_l. We are concerned about the feasibility of the following inequality.

$$\sum_m a_{lm}x_m + \sum_k b_{lk}y_k \le p_l \tag{2}$$

As shown in the previous section with the motivating example on scheduling, the optimal solution of an MILP program may become infeasible, that is, one or more constraints are violated substantially, if the nominal data is slightly perturbed. Our objective here is to develop a robust optimization methodology to generate "reliable" solutions to the MILP program, which are immuned against uncertainty. This robust optimization methodology was first introduced for Linear Programming (LP) problems with uncertain linear coefficients by Ben-Tal and Nemirovski (2000) and is extended in this work to MILP problems under uncertainty. Two types of uncertainty are addressed: (i) bounded uncertainty and (ii) bounded and symmetric uncertainty.

4.1. Bounded uncertainty

Suppose that the uncertain data range in the following intervals:

$$|\tilde{a}_{lm} - a_{lm}| \le \epsilon|a_{lm}|, \quad |\tilde{b}_{lk} - b_{lk}| \le \epsilon|b_{lk}|, \\ |\tilde{p}_l - p_l| \le \epsilon|p_l| \tag{3}$$

where $\tilde{a}_{lm}$, $\tilde{b}_{lk}$ and $\tilde{p}_l$ are the "true" values, a_{lm}, b_{lk} and p_l are the nominal values, and $\epsilon > 0$ is a given (relative) uncertainty level.

We call a solution (x, y) robust if it satisfies the following conditions:

(i) (x, y) is feasible for the nominal problem;
(ii) whatever are the true values of the coefficients and right-hand-side parameters within the corresponding intervals, (x, y) must satisfy the l-th inequality constraint with an error of at most $\delta\max[1, |p_l|]$, where δ is a given infeasibility tolerance.

Theorem 1. *Given an infeasibility tolerance* (δ), *to generate robust solutions, the following so-called* (ϵ, δ)*-Interval Robust Counterpart* ($IRC[\epsilon, \delta]$) *of the original uncertain MILP problem can be derived.*

$$\begin{aligned} &\operatorname*{Min/Max}_{x,y,u}\; c^{\mathrm{T}}x + d^{\mathrm{T}}y \\ &\text{s.t.}\quad Ex + Fy = e \\ &\qquad Ax + By \le p \\ &\qquad \sum_m a_{lm}x_m + \epsilon\sum_{m\in M_l}|a_{lm}|u_m + \sum_{k\notin K_l} b_{lk}y_k \\ &\qquad\quad + \sum_{k\in K_l}(b_{lk} + \epsilon|b_{lk}|)y_k \\ &\qquad \le p_l - \epsilon|p_l| + \delta\max[1, |p_l|], \quad \forall l \\ &\qquad -u_m \le x_m \le u_m, \quad \forall m \\ &\qquad \underline{x} \le x \le \bar{x} \\ &\qquad y_k = 0, 1, \quad \forall k \end{aligned} \tag{4}$$

where M_l and K_l are the set of indices of the x and y variables, respectively, with uncertain coefficients in the l-th inequality constraint.

Proof. We want to find a robust solution (x, y) which satisfies condition (i) and condition (ii), that is:

$$\begin{aligned} &\forall l \quad \forall(\tilde{a}_{lm} : |\tilde{a}_{lm} - a_{lm}| \le \epsilon|a_{lm}|,\ \tilde{b}_{lk} : |\tilde{b}_{lk} - b_{lk}| \le \epsilon|b_{lk}|, \\ &\quad \text{and } \tilde{p}_l : |\tilde{p}_l - p_l| \le \epsilon|p_l|) \\ &\quad : \sum_{m\notin M_l} a_{lm}x_m + \sum_{m\in M_l}\tilde{a}_{lm}x_m + \sum_{k\notin K_l} b_{lk}y_k + \sum_{k\in K_l}\tilde{b}_{lk}y_k \\ &\quad \le \tilde{p}_l + \delta\max[1, |p_l|] \end{aligned} \tag{5}$$

where M_l and K_l are the set of indices of the x and y variables, respectively, with uncertain coefficients in the l-th inequality constraint.

Using the worst-case values of the uncertain parameters:

$$\tilde{a}_{lm}x_m \le a_{lm}x_m + \epsilon|a_{lm}||x_m|, \; \tilde{b}_{lk}y_k \le b_{lk}y_k + \epsilon|b_{lk}|y_k, \quad \text{and} \quad \tilde{p}_l \ge p_l - \epsilon|p_l| \tag{6}$$

and substituting into Eq. (5) and rearranging terms, it is clear that a solution (x, y) is robust if and only if it is a feasible solution of the following optimization problem:

$$\begin{aligned} &\underset{x,y}{\text{Min/Max}}\; c^{\mathrm{T}}x + d^{\mathrm{T}}y \\ &\text{s.t.} \quad Ex + Fy = e \\ &\qquad Ax + By \le p \\ &\qquad \sum_m a_{lm}x_m + \epsilon \sum_{m\in M_l} |a_{lm}||x_m| + \sum_{k\notin K_l} b_{lk}y_k \\ &\qquad\quad + \sum_{k\in K_l} (b_{lk} + \epsilon|b_{lk}|)y_k \\ &\qquad \le p_l - \epsilon|p_l| + \delta\max[1, |p_l|], \quad \forall l \\ &\qquad \underline{x} \le x \le \bar{x} \\ &\qquad y_k = 0, 1, \quad \forall k. \end{aligned} \tag{7}$$

The above problem is equivalent to problem (4) which represents the absolute value operator with a set of auxiliary variables (u_m) and a set of additional constraints. □

For each inequality constraint that involves uncertain coefficients and/or right-hand-side parameters, an additional constraint is introduced to incorporate the uncertainty and maintain the relationships among the relevant binary and continuous variables under the uncertainty level and the given infeasibility tolerance. Essentially, this constraint considers the worst case values of the uncertain parameters which make the inequality the most difficult to maintain; at the same time, a certain degree of relaxation is introduced to allow tolerable violations of the constraint.

Note that mathematical model (4) remains an MILP model. Compared to the original deterministic MILP problem, the robust counterpart has a set of auxiliary variables (u_m) and a set of additional constraints relating the variables x_m and u_m.

4.2. Bounded and symmetric uncertainty

Now assume that the uncertain data are distributed around the nominal values randomly and symmetrically as follows.

$$\tilde{a}_{lm} = (1 + \epsilon\xi_{lm})a_{lm}, \quad \tilde{b}_{lk} = (1 + \epsilon\xi_{lk})b_{lk}, \quad \tilde{p}_l = (1 + \epsilon\xi_l)p_l \tag{8}$$

where ξ_{lm}, ξ_{lk} and ξ_l are random variables distributed symmetrically in the interval $[-1, 1]$.

In this situation, it makes sense to define a probabilistic version of the condition (ii) in the previous section for a robust solution as follows: (ii′) For the l-th inequality, the probability of the event of constraint violation, i.e.

$$\sum_m \tilde{a}_{lm}x_m + \sum_k \tilde{b}_{lk}y_k > p_l + \delta\max[1, |p_l|],$$

is at most κ, where $\delta > 0$ is a given feasibility tolerance and $\kappa > 0$ is a given "reliability level".

The following lemma is used in the proof of next theorem, Theorem 2, which gives the $(\epsilon, \delta, \kappa)$-Robust Counterpart (RC$[\epsilon, \delta, \kappa]$) of the original uncertain MILP problem for the case of bounded and symmetric uncertainty.

Lemma 1. *Let q_h be given reals and η_h be independent random variables symmetrically distributed in $[-1, 1]$. Then for every $\Omega > 0$ the following inequality holds*:

$$\Pr\left\{\sum_h \eta_h q_h > \Omega\sqrt{\sum_h q_h^2}\right\} \le \exp\{-\Omega^2/2\}. \tag{9}$$

For the sake of completeness, we provide here the proof of Lemma 1 presented by Ben-Tal and Nemirovski (2000). By homogeneity arguments, it suffices to consider the case of $\sum_h q_h^2 = 1$. In this case

$$\begin{aligned} \Pr\left\{\sum_h \eta_h q_h > \Omega\right\} &\underbrace{\le}_{(a)} \exp\{-\Omega^2\}\mathbb{E}\left\{\exp\left\{\Omega\sum_h \eta_h q_h\right\}\right\} \\ &\underbrace{=}_{(b)} \exp\{-\Omega^2\}\prod_h \mathbb{E}\{\exp\{\Omega\eta_h q_h\}\} \\ &\underbrace{\le}_{(c)} \exp\{-\Omega^2\}\prod_h \left[\sum_{r=0}^{\infty}\frac{(\Omega q_h)^{2r}}{(2r)!}\right] \\ &\le \exp\{-\Omega^2\}\prod_h \left[\sum_{r=0}^{\infty}\frac{(\Omega^2 q_h^2/2)^r}{r!}\right] \\ &\underbrace{=}_{(d)} \exp\{-\Omega^2\}\prod_h \exp\{\Omega^2 q_h^2/2\} \\ &\underbrace{=}_{(e)} \exp\{-\Omega^2/2\} \end{aligned}$$

with

(a) Tschebyshev inequality;
(b) independence of η_h;
(c) Taylor's expansion and symmetric distribution of $\eta_h \in [-1, 1]$;
(d) Taylor's expansion;
(e) $\sum_h q_h^2 = 1$.

Theorem 2. *Given an infeasibility tolerance (δ) and a reliability level (κ), to generate robust solutions, the following $(\epsilon, \delta, \kappa)$-Robust Counterpart (RC$[\epsilon, \delta, \kappa]$) of the original uncertain MILP problem can be derived.*

$$\begin{aligned} &\underset{x,y,u,z}{\text{Min/Max}}\; c^{\mathrm{T}}x + d^{\mathrm{T}}y \\ &\text{s.t.} \quad Ex + Fy = e \\ &\qquad Ax + By \le p \end{aligned}$$

$$\sum_m a_{lm}x_m + \sum_k b_{lk}y_k + \epsilon \left[\sum_{m\in M_l} |a_{lm}|u_{lm} + \Omega \sqrt{\sum_{m\in M_l} a_{lm}^2 z_{lm}^2 + \sum_{k\in K_l} b_{lk}^2 y_k + p_l^2} \right] \le p_l + \delta \max[1, |p_l|], \quad \forall l$$
$$-u_{lm} \le x_m - z_{lm} \le u_{lm}, \quad \forall l, m$$
$$\underline{x} \le x \le \bar{x}$$
$$y_k = 0, 1, \quad \forall k \tag{10}$$

where Ω is a positive parameter with $\kappa = \exp\{-\Omega^2/2\}$.

Proof. We introduce two sets of variables, u_{lm} and z_{lm}, and let (x, y, u, z) satisfy:

$$\sum_m a_{lm}x_m + \sum_k b_{lk}y_k + \epsilon \left[\sum_{m\in M_l} |a_{lm}|u_{lm} + \Omega \sqrt{\sum_{m\in M_l} a_{lm}^2 z_{lm}^2 + \sum_{k\in K_l} b_{lk}^2 y_k + p_l^2} \right] \le p_l + \delta \max[1, |p_l|], \quad \forall l \tag{11}$$

$$-u_{lm} \le x_m - z_{lm} \le u_{lm}, \quad \forall l, m \tag{12}$$

where Ω is a positive parameter with $\kappa = \exp\{-\Omega^2/2\}$.

Note that (11) is equivalent to:

$$\epsilon\Omega \sqrt{\sum_{m\in M_l} a_{lm}^2 z_{lm}^2 + \sum_{k\in K_l} b_{lk}^2 y_k + p_l^2} \le p_l + \delta \max[1, |p_l|] - \sum_m a_{lm}x_m - \epsilon \sum_{m\in M_l} |a_{lm}|u_{lm} - \sum_k b_{lk}y_k, \quad \forall l \tag{13}$$

and (12) leads to:

$$\xi_{lm}(x_m - z_{lm}) \le u_{lm} \Rightarrow \xi_{lm}x_m \le \xi_{lm}z_{lm} + u_{lm} \tag{14}$$

Then

$$\Pr\left\{ \sum_m \tilde{a}_{lm}x_m + \sum_k \tilde{b}_{lk}y_k > \tilde{p}_l + \delta\max[1, |p_l|] \right\}$$
$$= \Pr\left\{ \sum_m a_{lm}x_m + \epsilon \sum_{m\in M_l} \xi_{lm}|a_{lm}|x_m + \sum_k b_{lk}y_k + \epsilon \sum_{k\in K_l} \xi_{lk}|b_{lk}|y_k > p_l + \epsilon\xi_l|p_l| + \delta\max[1, |p_l|] \right\}$$
$$\underbrace{\le}_{(14)} \Pr\left\{ \sum_m a_{lm}x_m + \epsilon \sum_{m\in M_l} \xi_{lm}|a_{lm}|z_{lm} + \epsilon \sum_{m\in M_l} |a_{lm}|u_{lm} + \sum_k b_{lk}y_k + \epsilon \sum_{k\in K_l} \xi_{lk}|b_{lk}|y_k - \epsilon\xi_l|p_l| > p_l + \delta\max[1, |p_l|] \right\}$$
$$= \Pr\left\{ \epsilon \left[\sum_{m\in M_l} \xi_{lm}|a_{lm}|z_{lm} + \sum_{k\in K_l} \xi_{lk}|b_{lk}|y_k - \xi_l|p_l| \right] > p_l + \delta\max[1, |p_l|] - \sum_m a_{lm}x_m - \epsilon \sum_{m\in M_l} |a_{lm}|u_{lm} - \sum_k b_{lk}y_k \right\}$$
$$\underbrace{\le}_{(13)} \Pr\left\{ \sum_{m\in M_l} \xi_{lm}|a_{lm}|z_{lm} + \sum_{k\in K_l} \xi_{lk}|b_{lk}|y_k - \xi_l|p_l| > \Omega \sqrt{\sum_{m\in M_l} a_{lm}^2 z_{lm}^2 + \sum_{k\in K_l} b_{lk}^2 y_k + p_l^2} \right\}$$
$$\underbrace{\le}_{(9)} \exp\{-\Omega^2/2\}. \quad \square$$

Note the last step is based on the fact that $y_k^2 = y_k$, as $y_k = 0, 1\, \forall k$.

Therefore, assume that (x, y) can be extended to a feasible solution (x, y, u, z) of problem (10), then (x, y) is a robust solution that satisfies (i) and (ii′) with $\kappa = \exp\{-\Omega^2/2\}$.

Note that (10) is a convex MINLP problem with additional auxiliary variables (u_{lm} and z_{lm}), which can still be solved efficiently using MINOPT (Schweiger & Floudas, 1998) or GAMS/DICOPT (Viswanathan & Grossmann, 1990).

It should be pointed out that in Lemma 1 concerning the probability estimation, the inequality is desirably tight only when a large number of random variables are involved and hence, the above robust counterpart problem is effective, that is, it generates solutions that are not too conservative only when a large number of uncertain parameters, including both coefficients and right-hand-sides, appear in each inequality constraint under uncertainty. Under this circumstance, RC[ϵ, δ, κ] is "less conservative" than IRC[ϵ, δ].

In the discussion above, for simplicity, we have assumed that there is a single common uncertainty level (ϵ), infeasibility tolerance (δ), and reliability level (κ) in each MILP problem with uncertain parameters. The proposed robust optimization techniques can be easily extended to account for the more general case in which the uncertainty level varies from one parameter to another and the infeasibility tolerance and reliability level are dependent on the constraint of interest. Furthermore, note that for each type of uncertainty addressed above, one additional constraint is introduced for each inequality constraint that contains uncertain parameter(s) and auxiliary variables are added if needed. Because the transformation is carried out at the level of constraints, in principle, the two robust optimization techniques presented can be applied to a single MILP problem involving different types of uncertainties. More specifically, for each inequality constraint, as long as all of its uncertain parameters are of the same type, an additional constraint that corresponds to the uncertainty type can be introduced to obtain the deterministic counterpart problem.

Note that the aforementioned robust optimization methodology circumvents any need for explicit or implicit discretization or sampling of the uncertain data, avoiding undesirable increase of the problem size. This renders the methodology capable of handling problems with a large number of uncertain parameters.

5. Robust optimization for scheduling under uncertainty

The robust optimization methodology proposed in the previous section can be applied to address the problem of scheduling under uncertainty. In this work, we employ the continuous-time formulation presented by Floudas and coworkers (Ierapetritou & Floudas, 1998a,b; Ierapetritou et al., 1999; Lin & Floudas, 2001), which leads to MILP models (see Appendix A for the complete scheduling formulation), to develop new robust scheduling approaches for the following three classes of uncertainties:

(i) uncertainty in processing times/rates of tasks,
(ii) uncertainty in market demands for products,
(iii) uncertainty in market prices of products and raw materials.

5.1. Uncertainty in processing times

The parameters of processing times/rates of tasks participate in the duration constraint as linear coefficients of the binary variable (i.e., α_{ij}) and the continuous variable (i.e., β_{ij}):

$$T^{\mathrm{f}}(i,j,n) - T^{\mathrm{s}}(i,j,n) = \alpha_{ij}\cdot wv(i,n) + \beta_{ij}\cdot b(i,j,n) \tag{15}$$

where $wv(i,n)$ is a binary variable indicating whether or not task (i) starts at event point (n); $b(i,j,n)$ is a continuous variable determining the batch-size of the task; $T^{\mathrm{s}}(i,j,n)$ and $T^{\mathrm{f}}(i,j,n)$ are continuous variables representing the starting and finishing time of the task, respectively. Note that this is an equality constraint. To apply the robust optimization techniques proposed in the previous section for inequality constraints with uncertain parameters, two approaches are developed.

5.1.1. Approach 1

In the first approach, the duration constraint is relaxed to an inequality constraint as follows:

$$T^{\mathrm{f}}(i,j,n) - T^{\mathrm{s}}(i,j,n) \geq \alpha_{ij}\cdot wv(i,n) + \beta_{ij}\cdot b_{ij}. \tag{16}$$

Consequently, the variable $T^{\mathrm{f}}(i,j,n)$ represents the lower bound on the finishing time of the task, instead of the exact finishing time as determined by the original duration constraint. Now the various robust optimization techniques can be applied to this inequality constraint with uncertain parameters α_{ij} and β_{ij}.

For example, consider a task with parameters α_{ij} and β_{ij} exhibiting bounded uncertainty in the following ranges:

$$\alpha_{ij}^{\mathrm{L}} \leq \tilde{\alpha}_{ij} \leq \alpha_{ij}^{\mathrm{U}}, \quad \beta_{ij}^{\mathrm{L}} \leq \tilde{\beta}_{ij} \leq \beta_{ij}^{\mathrm{U}}. \tag{17}$$

According to Theorem 1, to obtain the deterministic robust counterpart problem, the following constraint is added to the original scheduling model:

$$T^{\mathrm{f}}(i,j,n) - T^{\mathrm{s}}(i,j,n) \geq \alpha_{ij}^{\mathrm{U}}\cdot wv(i,n) + \beta_{ij}^{\mathrm{U}}\cdot b(i,j,n) - \delta. \tag{18}$$

Note that no auxiliary variables need to be introduced because the variable $b(i,j,n)$ (batch-size of the task) is non-negative by definition.

5.1.2. Approach 2

In the second approach, the original duration constraint (15) is eliminated from the scheduling model and the variable $T^{\mathrm{f}}(i,j,n)$ is substituted as follows:

$$T^{\mathrm{f}}(i,j,n) = T^{\mathrm{s}}(i,j,n) + \alpha_{ij}\cdot wv(i,n) + \beta_{ij}\cdot b(i,j,n). \tag{19}$$

The constraints related to the variable $T^{\mathrm{f}}(i,j,n)$ include the sequencing constraints and the time horizon constraints:

$$T^{\mathrm{s}}(i,j,n+1) \geq T^{\mathrm{f}}(i,j,n) \quad \forall i\in I,\ j\in J_i,\ n\in N,\ n\not\equiv N. \tag{20}$$

$$T^{\mathrm{s}}(i,j,n+1) \geq T^{\mathrm{f}}(i',j,n) - H[1-wv(i',n)] \quad \forall j\in J,\ i\in I_j,\ i'\in I_j,\ i\not\equiv i',\ n\in N,\ n\not\equiv N. \tag{21}$$

$$T^{\mathrm{s}}(i,j,n+1) \geq T^{\mathrm{f}}(i',j',n) - H[1-wv(i',n)], \quad \forall j,j'\in J,\ i\in I_j,\ i'\in I_{j'},\ i\not\equiv i',\ n\in N,\ n\not\equiv N. \tag{22}$$

$$T^{\mathrm{f}}(i,j,n) \leq H, \quad \forall i\in I,\ j\in J_i,\ n\in N. \tag{23}$$

After the substitution, they become:

$$T^{\mathrm{s}}(i,j,n+1) \geq T^{\mathrm{s}}(i,j,n) + \alpha_{ij}\cdot wv(i,n) + \beta_{ij}\cdot b(i,j,n), \quad \forall i\in I,\ j\in J_i,\ n\in N,\ n\not\equiv N. \tag{24}$$

$$T^{\mathrm{s}}(i,j,n+1) \geq T^{\mathrm{s}}(i',j,n) + \alpha_{i'j}\cdot wv(i',n) + \beta_{i'j}\cdot b(i',j,n) - H[1-wv(i',n)], \quad \forall j\in J,\ i\in I_j,\ i'\in I_j,\ i\not\equiv i',\ n\in N,\ n\not\equiv N. \tag{25}$$

$$T^{s}(i,j,n+1) \geq T^{s}(i',j',n) + \alpha_{i'j'} \cdot wv(i',n) + \beta_{i'j'} \cdot b(i',j',n) - H[1-wv(i',n)],$$
$$\forall j,\ j' \in J,\ i \in I_j,\ i' \in I_{j'},\ i \not\equiv i',\ n \in N,\ n \not\equiv N. \quad (26)$$

$$T^{s}(i,j,n) + \alpha_{ij} \cdot wv(i,n) + \beta_{ij} \cdot b(i,j,n) \leq H, \quad \forall i \in I,\ j \in J_i,\ n \in N. \quad (27)$$

Now the uncertain parameters α_{ij} and β_{ij} participate in these inequality constraints and the robust optimization techniques can be readily applied.

5.2. Uncertainty in product demands

The parameters of product demands (i.e., dem_s) appear as the right-hand-side parameters in the demand constraints:

$$\mathrm{STF}(s) \geq \mathrm{dem}_s, \quad \forall s \in S_\mathrm{p} \quad (28)$$

where $\mathrm{STF}(s)$ is a continuous variable representing the amount of state (s) accumulated at the end of the time horizon and S_p is the set of final products.

The robust optimization techniques can be directly applied to these inequality constraints with uncertain parameters. For example, in the case of bounded uncertainty:

$$\mathrm{dem}_s^\mathrm{L} \leq \widetilde{\mathrm{dem}}_s \leq \mathrm{dem}_s^\mathrm{U}, \quad (29)$$

according to Theorem 1, the constraint to be added to the original scheduling model to derive the deterministic robust counterpart problem is as follows:

$$\mathrm{STF}(s) \geq \mathrm{dem}_s^\mathrm{U} - \delta. \quad (30)$$

5.3. Uncertainty in market prices

The parameters of market prices (i.e., p_s) participate in the objective function for the calculation of the overall profit:

$$\text{Maximize} \quad \text{Profit} = \sum_{s \in S_\mathrm{p}} p_s \cdot \mathrm{STF}(s) - \sum_{s \in S_\mathrm{r}} p_s \cdot \mathrm{STI}(s) \quad (31)$$

where S_p and S_r are the sets of final products and raw materials, respectively; $\mathrm{STI}(s)$ and $\mathrm{STF}(s)$ are continuous variables representing the initial amount of state (s) at the beginning and the final amount of state (s) at the end, respectively. This objective function can be expressed in an equivalent way as follows:

$$\begin{aligned} &\text{Maximize} \quad \text{Profit} \\ &\text{s.t.} \quad \text{Profit} \leq \sum_{s \in S_\mathrm{p}} p_s \cdot \mathrm{STF}(s) - \sum_{s \in S_\mathrm{r}} p_s \cdot \mathrm{STI}(s). \end{aligned} \quad (32)$$

Now the uncertain parameters p_s appear as linear coefficients in the above inequality constraint and the robust optimization techniques can be easily applied. For example, if the uncertainty is bounded and symmetric:

$$\tilde{p}_s = (1 + \epsilon\xi_s) p_s \quad (33)$$

where ξ_s is distributed symmetrically in $[-1, 1]$, then according to Theorem 2, the deterministic robust counterpart problem can be obtained by introducing the following constraint to the original scheduling model:

$$-\sum_{s \in S_\mathrm{p}} p_s \cdot \mathrm{STF}(s) + \sum_{s \in S_\mathrm{r}} p_s \cdot \mathrm{STI}(s) + \text{Profit}(1-\delta) + \epsilon \left[\sum_{s \in S_\mathrm{r}} p_s y(s) + \sum_{s \in S_\mathrm{p}} p_s y(s) + \Omega \sqrt{\sum_{s \in S_\mathrm{r}} p_s^2 z(s)^2 + \sum_{s \in S_\mathrm{p}} p_s^2 z(s)^2} \right] \leq 0 \quad (34)$$

$$-y(s) \leq \mathrm{STI}(s) - z(s) \leq y(s), \quad \forall s \in S_\mathrm{r}$$
$$-y(s) \leq \mathrm{STF}(s) - z(s) \leq y(s), \quad \forall s \in S_\mathrm{p}$$

where $\kappa = \exp\{-\Omega^2/2\}$. Note that the additional constraint (34) is convex and hence the resulting problem is a convex MINLP problem.

6. Computational studies

The above robust optimization formulation is applied to three examples. The first two examples are implemented with GAMS (Brooke, Kendrick, & Meeraus, 1988) and the third is implemented with MINOPT (Schweiger & Floudas, 1998). The MILP problems are solved using CPLEX 7.0 while the MINLP problems are solved using DICOPT (Viswanathan & Grossmann, 1990). All computations are done on a HP-J2240 workstation.

6.1. Motivating example: bounded uncertainty in processing times

Let us revisit the motivating example in Section 3. Assume that the uncertainty of the processing times is bounded and the (relative) uncertainty level (ϵ) is 15%, that is,

$$0.85\alpha \leq \tilde{\alpha} \leq 1.15\alpha \quad (35)$$

and the infeasibility tolerance level (δ) is 10%.

By solving the IRC[ϵ, δ] problem, a "robust" schedule is obtained, as shown in Fig. 4, which takes into account uncertainty in the processing times. The nominal schedule can be seen in Fig. 2 in Section 3.

Compared to the nominal solution which is obtained at the nominal values of the processing times, the robust solution exhibits very different scheduling strategies, such as in task-unit assignments and task timings. For example, even the sequences of tasks in the two reactors in Fig. 4 deviates

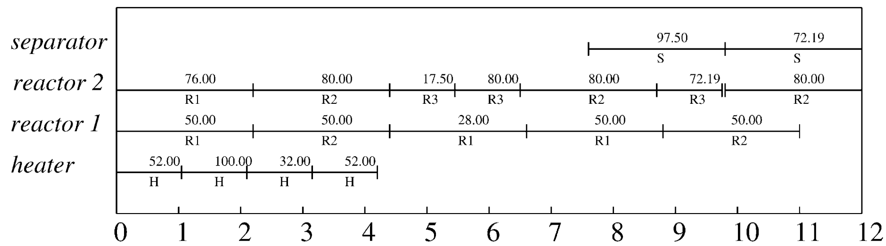

Fig. 4. Robust solution of the motivating example ($\epsilon = 15\%$, $\delta = 10\%$, profit = 2887.19).

Table 2
Model and solution statistics of the motivating example

	Nominal solution	Robust solution
Profit	3638.75	2887.19
CPU time (s)	2.68	114.47
Binary variables	96	96
Continuous variables	378	378
Constraints	553	713

significantly from those in the nominal solution in Fig. 2. The robust solution ensures that the robust schedule obtained is feasible with the specified uncertainty level and infeasibility tolerance. However, the resulting profit is reduced, from 3638.75 to 2887.19, which reflects the effect of uncertainty on overall production. A comparison of the model and solution statistics for the nominal and robust solutions can be found in Table 2.

Fig. 5 summarizes the results of the IRC problem with three different levels of uncertainty. It is shown that with a given infeasibility tolerance, the maximal profit that can be achieved decreases as the uncertainty level increases, which indicates more "conservative" scheduling decisions because of the existence of uncertainty. On the other hand, at a given uncertainty level, the profit increases as the infeasibility tolerance is increased, which means more "aggressive" scheduling arrangements can be incorporated if violations of related timing constraints can be tolerated to a larger extent. These results are consistent with intuition and other approaches, however, with the robust optimization approach, the effects of uncertainty and the trade-offs between conflicting objectives are quantified rigorously and efficiently. It should be noted that at a given uncertainty level, the objective value of profit as well as the corresponding schedule changes dramatically at discrete points as the infeasibility tolerance increases. This results from the following characteristics of the example problem: the time horizon and the processing times of tasks are both fixed.

6.2. Example 2: bounded and symmetric uncertainty in market prices

In this example, we consider bounded and symmetric uncertainty in market prices for the same process in the motivating example. However, the processing time parameters are taken from Example 2 in Ierapetritou and Floudas

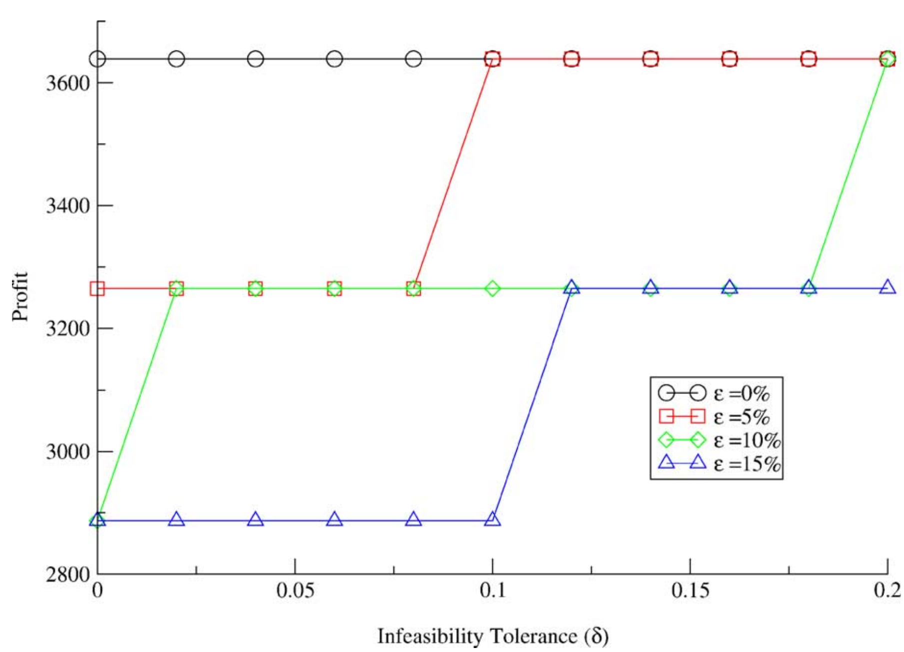

Fig. 5. Profit vs. infeasibility tolerance at different uncertainty levels.

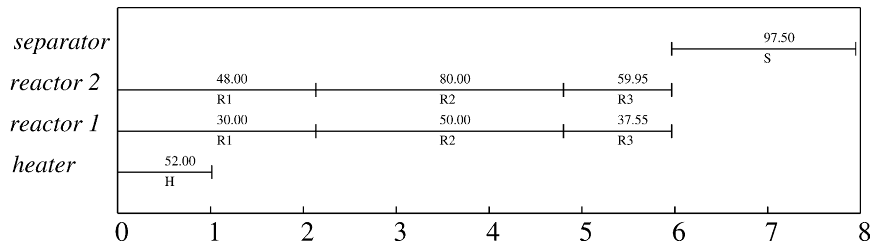

Fig. 6. Nominal solution of example 2 (profit: 1088.75).

(1998a) (i.e., the mean processing times are the same as those in Table 1 and they vary between 2/3 and 4/3 of their mean values) and the time horizon is 8 h. Assume that the nominal prices of the three raw materials and the two products are 5.0, 5.0, 5.0, 10.0, and 15.0, respectively. The uncertainty level is $\epsilon = 5\%$; the infeasibility tolerance is $\delta = 0\%$; and the reliability level $\kappa = 10\%$. The nominal schedule is shown in Fig. 6 with a profit of 1088.75. The robust schedule is obtained by solving the robust counterpart problem, as shown in Fig. 7, and the corresponding profit is 961.73. By executing this schedule, the profit is guaranteed to be above 961.73 with a probability of 90% in the presence of the 5% uncertainty in the prices of the raw materials and products. A comparison of the model and solution statistics for the nominal and robust solutions can be found in Table 3.

The results for different uncertainty levels at a given infeasibility tolerance and reliability level are shown in Fig. 8. The higher the uncertainty level is, the lower the achievable profit becomes. By applying the robust optimization methodology, the effect of the uncertainty on the schedule is clearly quantified.

6.3. *An industrial case study*

6.3.1. *Problem description*

This example is based on an industrial case study presented by Lin, Floudas, Modi, and Juhasz (2002) for a multiproduct chemical plant that manufactures ten different products according to a basic three-stage recipe and its variations by employing ten pieces of equipment. We consider the first sub-horizon in the original case study that consists of five days and involves eight different products. The objective function is the maximization of overall production defined by the weighted sum of materials accumulated at the end of the sub-horizon minus a penalty term for not meeting demands at the intermediate due dates. For each of the eight products, one of the processing recipes shown in Fig. 9 is applied.

Table 3
Model and solution statistics for example 2

	Nominal solution	Robust solution
Profit	1088.75	961.73
CPU time (s)	0.11	1.15
Binary variables	60	60
Continuous variables	280	290
Constraints	334	377

Corresponding to the three basic operations, the plant has three types of units: four type 1 units (units 1–4) for operation 1, three type 2 units (units 5–7) for operation 2, and three type 3 units (units 8–10) for operation 3. Type 1 units and type 3 units are utilized in batch mode, while type 2 units are operated in a continuous mode. The nominal processing time or processing rate of each task in its suitable units is shown in Table 4.

To determine the form of the uncertainties in processing times/rates, actual plant data was analyzed. Due to the wide variability between parameters, bounded uncertainty was chosen and the range of each uncertain parameter determined. There are a total of twenty-seven uncertain parameters identified. Eight in units 1–4, five in units 5–7, and fourteen in units 8–10. The summary of the nominal values and ranges for each uncertain parameter are presented in Table 5.

Approach 2 for uncertainty in processing times/rates in Section 5.1 is applied for this case study. In addition to the basic sequencing constraints, the processing time also appears in two additional constraints related to the timing

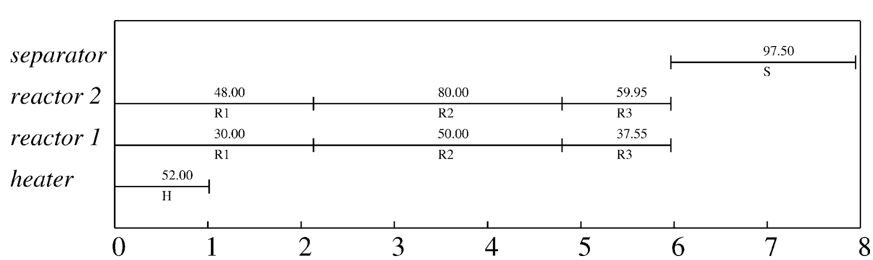

Fig. 7. Robust solution of example 2 (profit: 961.73).

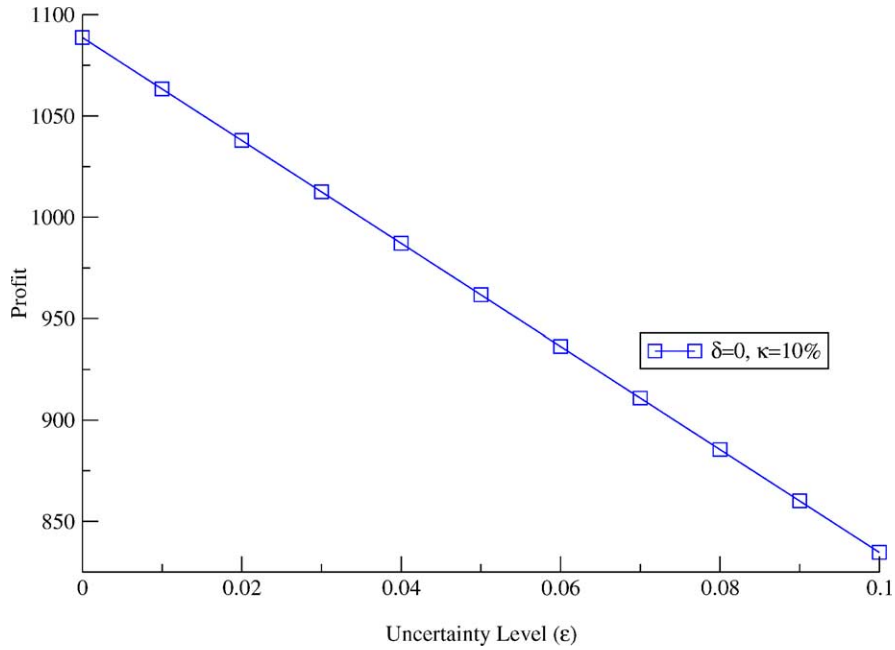

Fig. 8. Profit vs. uncertainty level ($\delta = 0\%$, $\kappa = 10\%$).

of operation 1 tasks.

$$T^{s}(i, j, n+1) \le T^{f}(i, j, n) + H(2 - wv(i, j, n) - wv(i, j, n+1)), \quad \forall i \in I_r,\ j \in J_i,\ n \in N,\ n \not\equiv N \tag{36}$$

$$T^{s}(i, j, n+1) \le T^{f}(i', j, n) + tcl_{ii'} + H(2 - wv(i, j, n) - wv(i', j, n+1)), \quad \forall j \in J_r,\ i \in I_j,\ i' \in I_j,\ i \not\equiv i',\ n \in N,\ n \not\equiv N \tag{37}$$

where I_r is the set of operation 1 tasks and J_r is the set of type 1 units suitable for operation 1 tasks. Upon substitution of the $T^{f}(i, j, n)$ variables, the following additional constraints are introduced to obtain the robust counterpart problem.

$$T^{s}(i, j, n+1) - T^{s}(i, j, n) \le \alpha^{L}_{ij} \cdot wv(i, j, n) + \beta^{L}_{ij} \cdot B(i, j, n) + H(2 - wv(i, j, n) - wv(i, j, n+1)) + \delta 2 \tag{38}$$

$$T^{s}(i, j, n+1) - T^{s}(i', j, n) \le \alpha^{L}_{i'j} \cdot wv(i', j, n) + \beta^{L}_{i'j} \cdot B(i', j, n) + tcl_{ii'} + H(2 - wv(i, j, n) - wv(i', j, n+1)) + \delta 2 \tag{39}$$

Table 4
Nominal processing times and rates in the case study

Task	Unit 1	Unit 2	Unit 3	Unit 4	Unit 5	Unit 6	Unit 7	Unit 8	Unit 9	Unit 10
1	0	0	0	9.5	–	–	–	–	–	–
2	–	–	–	–	0	0	0.95	–	–	–
3	–	–	–	–	–	–	–	12	12.8	12.5
4	0	10	10	10	–	–	–	–	–	–
5	–	–	–	–	0.575	0.575	0.725	–	–	–
6	–	–	–	–	–	–	–	12	12.8	12.5
7	6.09	6.09	6.09	11.1	–	–	–	–	–	–
8	–	–	–	–	0.6	0.6	0.8	–	–	–
9	–	–	–	–	–	–	–	12.5	13.8	12.9
10	6.09	6.09	6.09	11.1	–	–	–	–	–	–
11	–	–	–	–	0.6	0.6	0.8	–	–	–
12	–	–	–	–	–	–	–	12.5	13.8	12.9
13	6.09	6.09	6.09	11.1	–	–	–	–	–	–
14	–	–	–	–	0.6	0.6	0.8	–	–	–
15	–	–	–	–	–	–	–	12.5	13.8	12.9
16	–	–	–	–	0.6	0.6	0.8	–	–	–
17	–	–	–	–	–	–	–	12.5	13.8	12.9
18	0	8.5	8.5	0	–	–	–	–	–	–
19	–	–	–	–	–	–	–	0	15	16
20	0	0	8.38	9.5	–	–	–	–	–	–

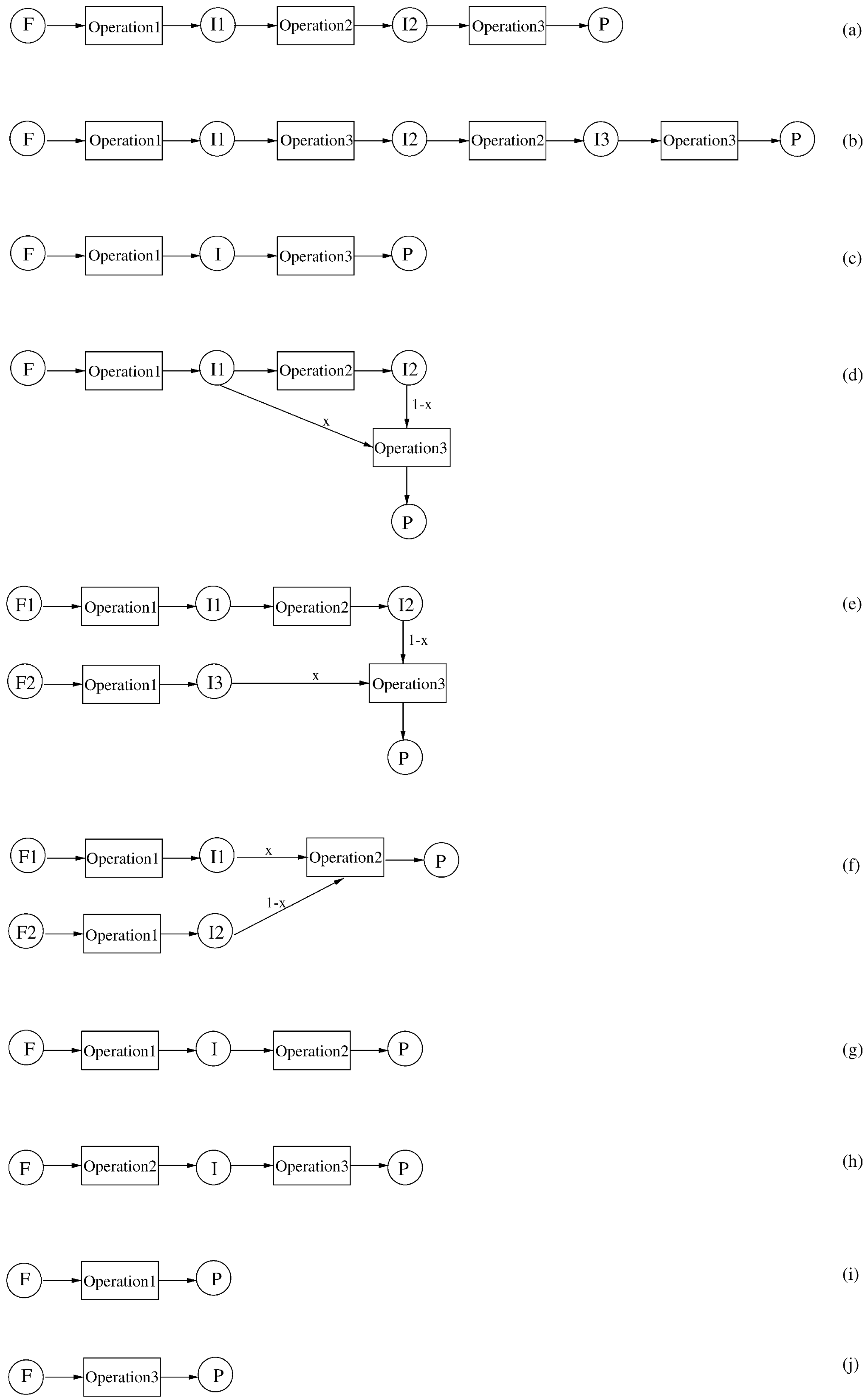

Fig. 9. State-Task Network of production recipes in the case study.

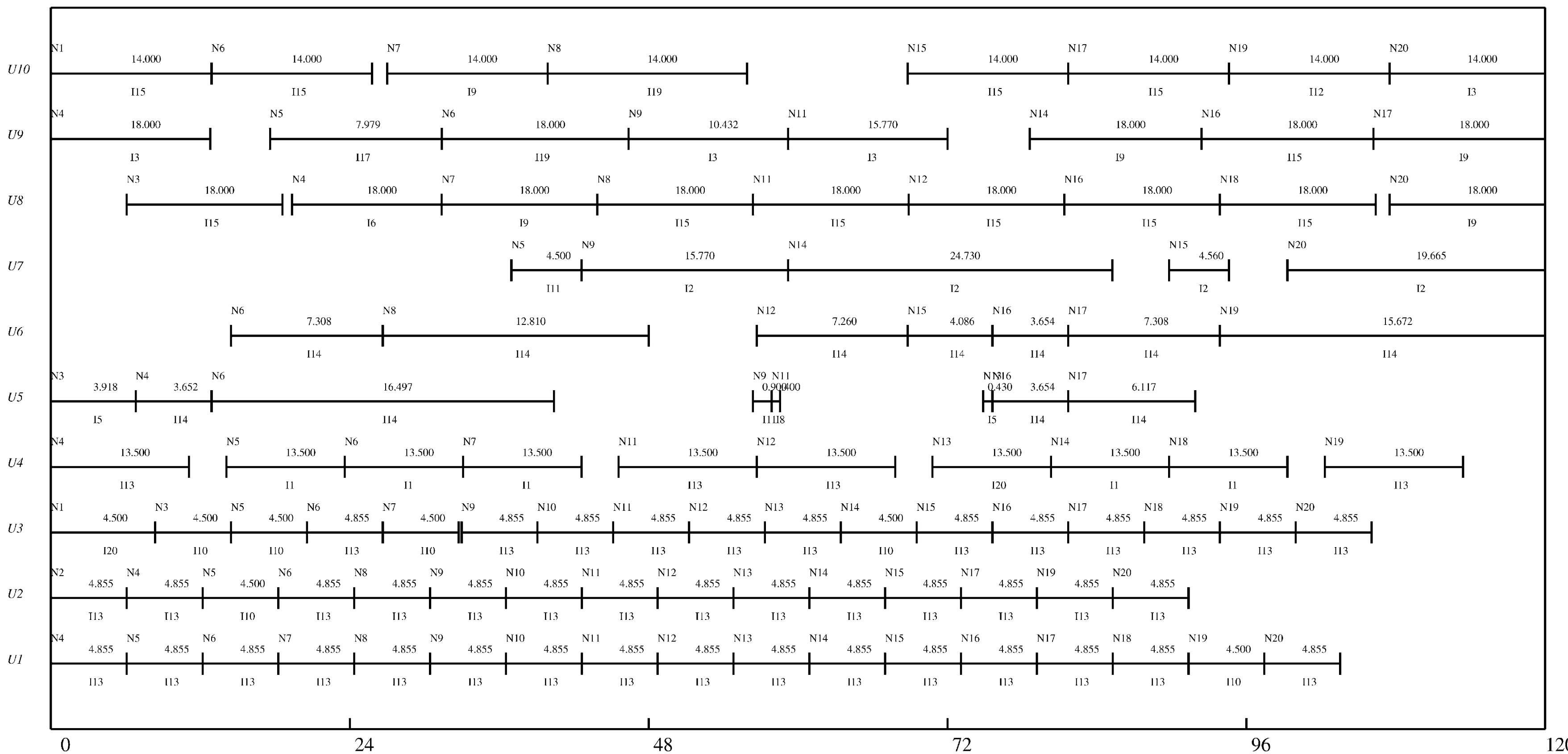

Fig. 10. Nominal schedule in the case study (Obj = 23.34; U1–10: units; I1–19: tasks).

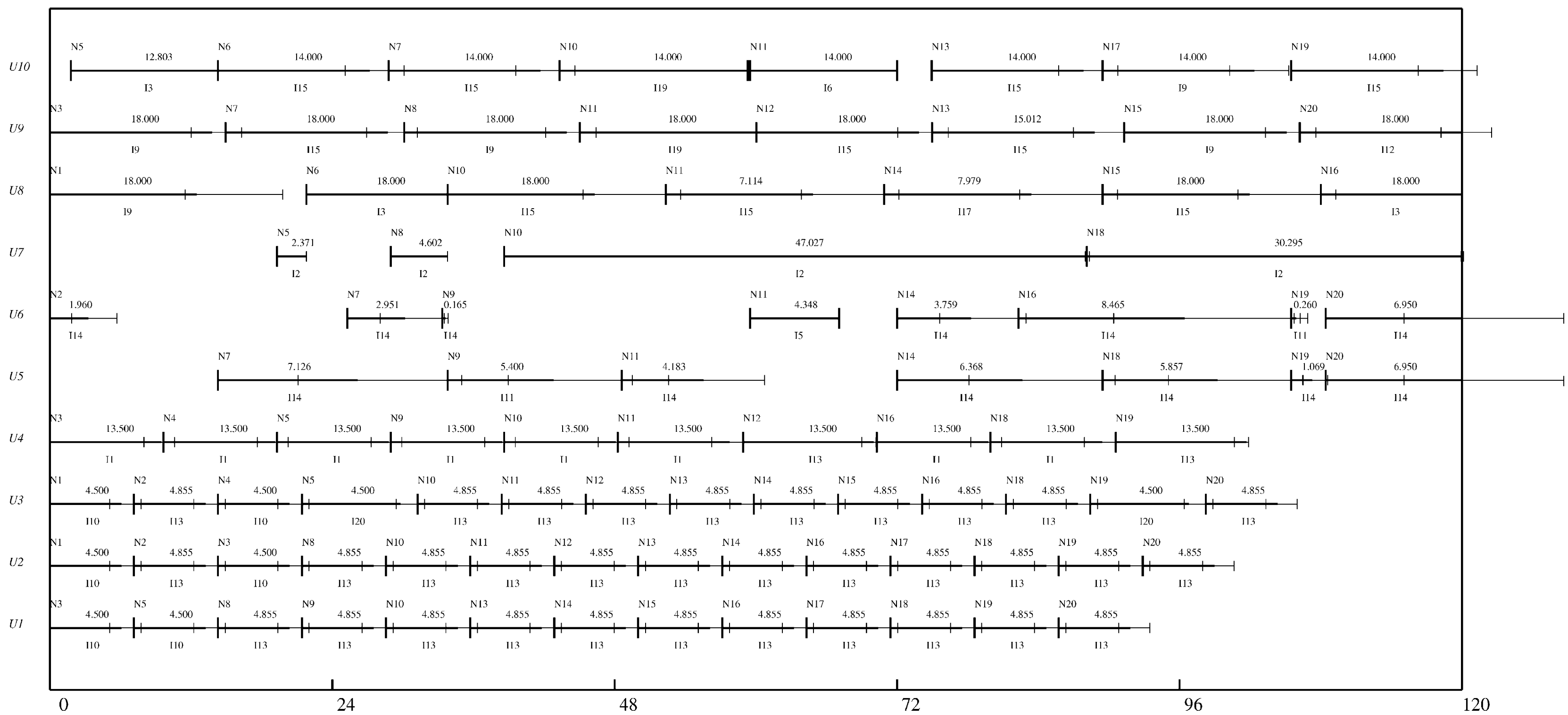

Fig. 11. Robust schedule in the case study (Obj = 21.99; δ = 10%; U1–10: units; I1–19: tasks. *Note*: the endpoint of each thick horizontal line indicates the finishing time of the task based on the nominal processing time while the thin vertical lines represent the range of the finishing time due to the bounded uncertainty present in the processing time).

Table 5
Bounded uncertainty in processing times/rates for the case study

Task	Unit	Nominal value	Range
1	4	9.5	8.00–10.6
7, 10, 13	1–3	6.09	5.09–7.75
7, 10, 13	4	11.1	10.1–11.3
20	3	8.38	8.00–10.42
2	7	0.95	0.947–0.991
8, 11, 14, 16	5–6	0.60	0.344–0.853
3, 6	9	12.8	10.5–19.3
9, 12, 15, 17	8	12.5	11.5–19.8
9, 12, 15, 17	9	13.8	12.0–16.3
9, 12, 15, 17	10	12.9	10.8–15.8

where $\delta 2$ is defined as a variable and correlates as follows with parameter δ that participates in the additional constraints corresponding to the basic sequencing constraints:

$$\delta + \delta 2 = \alpha^{\mathrm{U}} - \alpha^{\mathrm{L}} \quad \text{or} \quad \delta + \delta 2 = \beta^{\mathrm{U}} - \beta^{\mathrm{L}}. \tag{40}$$

The objective function for this problem is the maximization of production in terms of the relative value of all states minus a penalty term for not meeting demands at the intermediate due dates.

$$\gamma \sum_{s} \mathrm{vald}_s \mathrm{valp}_s \mathrm{valm}_s \mathrm{STF}(s) - \sum_{s} \sum_{n} \mathrm{pri}_{sn} \mathrm{SL}(s, n), \quad \forall s \in S,\ n \in N \tag{41}$$

where vald_s is the relative value of the corresponding product indicating its importance to fulfill future demands, valp_s is the relative value of the corresponding product indicating its priority, valm_s is is the relative value of state (s) in the sequence of materials for the corresponding product, $\mathrm{STF}(s)$ is the amount of state (s) at the end of the horizon, pri_{sn} is the priority of demand for state (s) at event point (n), $\mathrm{SL}(s, n)$ is a slack variable for the amount of state (s) not meeting the demand at event point (n), and γ is a constant coefficient used to balance the relative value of the two terms in the objective function.

6.3.2. Computational results and discussion

The nominal solution to this problem using the continuous-time formulation is shown in Fig. 10 and the objective function value is 23.34. At a (relative) infeasibility tolerance level (δ) of 10%, the solution to the interval robust counterpart problem when all twenty-seven uncertain parameters were considered is shown in Fig. 11 and the objective function value is 21.99. It can be seen that the processing time of each task is extended to ensure that the schedule is feasible within the specified uncertainty level and infeasibility tolerance; however, the objective function value has decreased. A closer examination of the terms involved in the objective function indicates that the objective function value for the robust solution decreased because the overall production decreased while the relative values of the violations of the intermediate dues dates also decreased, but by a smaller amount. A comparison of the model and solution statistics for the nominal and robust solutions of the industrial case study can be found in Table 6.

Table 6
Model and solution statistics for case study

	Nominal solution	Robust solution
Objective	23.34	21.99
CPU time (s)	4641.8	14721.8
Binary variables	1320	1320
Continuous variables	5036	6156
Constraints	21916	32444

7. Conclusions

In this work, we propose a new approach to address the scheduling under uncertainty problem based on a robust optimization methodology, which when applied to mixed-integer (MILP) problems produces “robust” solutions which are in a sense immune against uncertainties in both the coefficients in the objective function, the left-hand-side parameters and the right-hand-side parameters of the inequality constraints. A unique feature of the proposed approach is that it can address many uncertain parameters. The approach can be applied to address the problem of production scheduling with uncertain processing times, market demands, and/or prices of products and raw materials. Our computational results show that this approach provides an effective way to address scheduling problems under uncertainty, producing reliable schedules and generating helpful insights on the tradeoffs between conflicting objectives. Furthermore, due to its efficient transformation, the approach is capable of solving real-world problems with a large number of uncertain parameters.

Acknowledgements

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Appendix A. Continuous-time process scheduling formulation

This formulation was proposed by Floudas and coworkers (Ierapetritou & Floudas, 1998a,b; Ierapetritou et al., 1999; Lin & Floudas, 2001).

A.1. Nomenclature

$B(i, j, n)$	continuous, amount of material undertaking task (i) in unit (j) at event point (n)
dem_s	market demand for state (s) at the end of the time horizon
H	time horizon
$i \in I$	tasks
I_j	tasks which can be performed in unit (j)
I_s	tasks which produce or consume state (s)
$j \in J$	units
J_i	units which are suitable for performing task (i)
$n \in N$	event points representing the beginning of a task
p_s	price of state (s)
profit	continuous, overall profit
$s \in S$	states
S_p	states corresponding to final products
S_r	states corresponding to raw materials
$\mathrm{ST}(s, n)$	continuous, amount of state (s) at event point (n)
$\mathrm{ST}_s^{\max}$	available maximum storage capacity for state (s)
$\mathrm{STF}(s)$	continuous, final amount of state (s) at the end of the time horizon
$\mathrm{STI}(s)$	continuous, initial amount of state (s) at the beginning of the time horizon
$T^\mathrm{s}(i, j, n)$	continuous, time at which task (i) starts in unit (j) at event point (n)
$T^\mathrm{f}(i, j, n)$	continuous, time at which task (i) finishes in unit (j) while it starts at event point (n)
$V_{ij}^{\min}$	minimum amount of material processed by task (i) required to start operating unit (j)
$V_{ij}^{\max}$	maximum capacity of unit (j) when processing task (i)
$wv(i, n)$	binary, whether or not task (i) starts at event point (n)
$yv(j, n)$	binary, whether or not unit (j) is utilized at event point (n)
Greek letter	
α_{ij}	constant term of processing time of task (i) in unit (j)
β_{ij}	variable term of processing time of task (i) in unit (j) expressing the time required by the unit to process one unit of material performing task (i)
$\rho_{si}^\mathrm{p}, \rho_{si}^\mathrm{c}$	proportion of state (s) produced, consumed by task (i), respectively

A.2. Constraints

Allocation constraints

$$\sum_{i\in I_j} wv(i, n) = yv(j, n), \quad \forall j \in J,\ n \in N$$

Capacity constraints

$$V_{ij}^{\min} wv(i, n) \le B(i, j, n) \le V_{ij}^{\max} wv(i, n), \quad \forall i \in I,\ j \in J_i,\ n \in N$$

Storage constraints

$$\mathrm{ST}(s, n) \le \mathrm{ST}_s^{\max}, \quad \forall s \in S,\ n \in N$$

Material balances

$$\mathrm{ST}(s, n_{1st}) = \mathrm{STI}(s) + \sum_{i\in I_s} \rho_{si}^\mathrm{c} \sum_{j\in J_i} B(i, j, n_{1st}), \quad \forall s \in S$$

$$\mathrm{ST}(s, n) = \mathrm{ST}(s, n-1) + \sum_{i\in I_s} \rho_{si}^\mathrm{p} \sum_{j\in J_i} B(i, j, n-1) + \sum_{i\in I_s} \rho_{si}^\mathrm{c} \sum_{j\in J_i} B(i, j, n), \quad \forall s \in S,\ n \in N$$

$$\mathrm{STF}(s) = \mathrm{ST}(s, n_\mathrm{last}) + \sum_{i\in I_s} \rho_{si}^\mathrm{p} \sum_{j\in J_i} B(i, j, n_\mathrm{last}), \quad \forall s \in S$$

Demand constraints

$$\mathrm{STF}(s) \ge \mathrm{dem}_s, \quad \forall s \in S$$

Duration constraints

$$T^\mathrm{f}(i, j, n) = T^\mathrm{s}(i, j, n) + \alpha_{ij} wv(i, n) + \beta_{ij} B(i, j, n), \quad \forall i \in I,\ j \in J_i,\ n \in N$$

Sequence constraints: same task in the same unit

$$T^\mathrm{s}(i, j, n+1) \ge T^\mathrm{f}(i, j, n), \quad \forall i \in I,\ j \in J_i,\ n \in N,\ n \not\equiv N$$

Sequence constraints: different tasks in the same unit

$$T^\mathrm{s}(i, j, n+1) \ge T^\mathrm{f}(i', j, n) - H[1 - wv(i', n)], \quad \forall j \in J,\ i \in I_j,\ i' \in I_j,\ i \not\equiv i',\ n \in N,\ n \not\equiv N$$

Sequence constraints: different tasks in different units

$$T^\mathrm{s}(i, j, n+1) \ge T^\mathrm{f}(i', j', n) - H[1 - wv(i', n)], \quad \forall j, j' \in J,\ i \in I_j,\ i' \in I_{j'},\ i \not\equiv i',\ n \in N,\ n \not\equiv N$$

Time horizon constraints

$$T^\mathrm{f}(i, j, n) \le H, \quad \forall i \in I,\ j \in J_i,\ n \in N$$

$$T^\mathrm{s}(i, j, n) \le H, \quad \forall i \in I,\ j \in J_i,\ n \in N$$

A.3. Objective function

$$\text{Max} \quad \text{Profit} = \sum_{s \in S_{\mathrm{p}}} p_s \cdot \mathrm{STF}(s) - \sum_{s \in S_{\mathrm{r}}} p_s \cdot \mathrm{STI}(s)$$

References

Balasubramanian, J., & Grossmann, I. E. (2002). A novel branch and bound algorithm for scheduling flowshop plants with uncertain processing times. *Computers and Chemical Engineering, 26*, 41–57.

Bassett, M. H., Pekny, J. F., & Reklaitis, G. V. (1997). Using detailed scheduling to obtain realistic operating policies for a batch processing facility. *Industrial Engineering and Chemical Research, 36*, 1717–1726.

Ben-Tal, A., & Nemirovski, A. (2000). Robust solutions of linear programming problems contaminated with uncertain data. *Mathematical Programming, 88*, 411–424.

Brooke, A., Kendrick, D., & Meeraus, A. (1988). *GAMS: A User's Guide*. South San Franciso, CA.

Cott, B. J., & Macchietto, S. (1989). Minimizing the effects of batch process variability vsing online schedule modification. *Computers and Chemical Engineering, 13*, 105–113.

Floudas, C. A. & Lin, X. (2003). *Continuous-time versus discrete-time approaches for scheduling of chemical processes: A Review*. Computers and Chemical Engineering, in press.

Honkomp, S. J., Mockus, L., & Reklaitis, G. V. (1999). A framework for schedule evaluation with processing uncertainty. *Computers and Chemical Engineering, 23*, 595–609.

Ierapetritou, M. G., & Floudas, C. A. (1998a). Effective continuous-time formulation for short-term scheduling: 1. Multipurpose batch processes. *Industrial Engineering and Chemical Research, 37*, 4341–4359.

Ierapetritou, M. G., & Floudas, C. A. (1998b). Effective continuous-time formulation for short-term scheduling: 2. Continuous and semi-continuous processes. *Industrial Engineering and Chemical Research, 37*, 4360–4374.

Ierapetritou, M. G., Hené, T.S., & Floudas, C. A. (1999). Effective continuous-time formulation for short-term scheduling. 3. Multiple intermediate due dates. *Industrial Engineering and Chemical Research, 38*, 3446–3461.

Ierapetritou, M. G., & Pistikopoulos, E. N. (1996). Global optimization for stochastic planning, scheduling and design problems. In I. E. Grossmann (Ed.), *Global optimization in engineering design* (pp. 231–287), Dordretch: Kluwer Academic Publishers.

Kanakamedala, K. B., Reklaitis, G. V., & Venkatasubramanian, V. (1994). Reactive schedule modification in multipurpose batch chemical plants. *Industrial Engineering and Chemical Research, 33*, 77–90.

Kondili, E., Pantelides, C. C., & Sargent, R. W. H. (1993). A general algorithm for short-term scheduling of batch operations. I. MILP formulation. *Computers and Chemical Engineering, 17*, 211–227.

Lin, X., Chajakis, E. D., & Floudas, C. A. (2003). *Scheduling of tanker lightering via a novel continuous-time optimization framework*. Industrial Engineering and Chemical Research, in press.

Lin, X., & Floudas, C. A. (2001). Design, synthesis and scheduling of multipurpose batch plants via an effective continuous-time formulation. *Computers and Chemical Engineering, 25*, 665–674.

Lin, X., Floudas, C. A., Modi, S., & Juhasz, N. M. (2002). Continuous-time optimization approach for medium-range production scheduling of a multiproduct batch plant. *Industrial Engineering and Chemical Research, 41*, 3884–3906.

Orçun, S., Altinel, I. K., & Hortaçsu, Ö. (1996). Scheduling of batch processes with operational uncertainties. *Computers and Chemical Engineering, 20*, S1191–S1196.

Rodrigues, M. T. M., Gimeno, L., Passos, C. A. S., & Campos, M. D. (1996). Reactive scheduling approach for multipurpose chemical batch plants. *Computers and Chemical Engineering, 20*, S1215–S1220.

Sanmarti, E., Espuña, A., & Puigjaner, L. (1997). Batch production and preventive maintenance scheduling under equipment failure uncertainty. *Computers and Chemical Engineering, 21*, 1157–1168.

Sanmarti, E., Huercio, A., Espuña, A., & Puigjaner, L. (1996). A combined scheduling/reactive scheduling strategy to minimize the effect of process operations uncertainty in batch plants. *Computers and Chemical Engineering, 20*, S1263–S1268.

Schweiger, C. & Floudas, C. (1998). *MINOPT: A software package for mixed–integer nonlinear optimization, user's guide*. Computer-Aided Systems Laboratory, Department of Chemical Engineering, Princeton University, NJ.

Vin, J. P., & Ierapetritou, M. G. (2000). A new approach for efficient rescheduling of multiproduct batch plants. *Industrial Engineering and Chemical Research, 39*, 4228–4238.

Vin, J. P., & Ierapetritou, M. G. (2001). Robust short-term scheduling of multiproduct batch plants under demand uncertainty. *Industrial Engineering and Chemical Research, 40*, 4543–4554.

Viswanathan, J. & Grossmann, I. E. (1990). *DICOPT++: A program for mixed integer nonlinear optimization, user's guide*. Engineering Design Research Center, Carnegie Mellon University, Pittsburgh, Pennsylvania.