Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Number Theory 115 (2005) 215–229

JOURNAL OF Number Theory

www.elsevier.com/locate/jnt

A classification of (some) Pisot-Cyclotomic numbers

J.P. Bell[a,*], K.G. Hare[b,1]

[a]Mathematics Department, University of Michigan, East Hall, 525 East University, Ann Arbor, MI 48109-1109, USA

[b]Department of Pure Mathematics, University of Waterloo, Waterloo, Ont., Canada N2L 3G1

Received 28 March 2004; revised 15 September 2004

Available online 2 February 2005

Communicated by David Goss

Abstract

A complete classification of degree 2, 3 and degree 4 Pisot-Cyclotomic numbers is given. Some examples of higher degrees are also given. Pisot-Cyclotomic numbers have applications to quasicrystals and quasilattices.

Keywords: Pisot numbers; Quasilattices; Quasicrystals; Diophantine equation

1. Introduction

We begin by recalling the definition of a Pisot number:

Definition 1. A Pisot number is a real algebraic integer greater than 1, all of whose conjugates are of modulus strictly less than 1.

* Corresponding author.

E-mail addresses: belljp@umich.edu (J.P. Bell), kghare@math.uwaterloo.ca (K.G. Hare).

[1] Supported in part by NSERC of Canada and the Seggie Brown Fellowship, University of Edinburgh.

0022-314X/$ - see front matter
doi:10.1016/j.jnt.2004.11.009

An important subclass of Pisot numbers is the Pisot-Cyclotomic numbers (see for instance [1,3,5,7]). These have applications to the study of quasicrystals and quasilattices.

Definition 2. Let $\beta = 2\cos\left(\frac{2\pi}{n}\right)$. A Pisot-Cyclotomic number with symmetry of order n is a Pisot number q such that

$$\mathbb{Z}[q] = \mathbb{Z}[\beta].$$

It was asked in [5] whether there exists Pisot-Cyclotomic numbers with symmetry of order $\geqslant 16$. We answer this question in the affirmative, and give some examples. Further, we find all Pisot-Cyclotomic number of particular degrees. Until now all quadratic Pisot-Cyclotomic numbers were known, and some cubic examples were known. We extend this to the quartic case, and give some examples for higher degrees.

Results are listed in Table 1. It should be noted that if n is odd then the Pisot-Cyclotomic numbers with symmetry of order n and with symmetry of order $2n$ are exactly the same. For this reason, we do not list 10, 14, 18, 22, etc. All symmetries are proved to have a finite number of examples, except those with an (a) beside them. In these cases, all examples are listed. Those entries with an (a) beside them are not currently proved to have a finite number of examples, and in this case all known examples are listed. These examples were discovered experimentally, by searching for small solutions. Those polynomials with a $*$ beside them have been previously discovered, and can be found in [1,6,7]. Those polynomials with a $**$ beside them have been previously computed by David Boyd.

Section 2 gives some useful comments on solving particular types of Diophantine equations. Section 3 deals with the quadratic and cubic Pisot-Cyclotomic numbers. Section 4 deals with the quartic Pisot-Cyclotomic numbers. Section 5 gives some useful techniques for solving some of the Diophantine equations that arise. Section 6 lists some open questions.

2. Some notes on Diophantine equations

We will need the solutions to certain Diophantine equations in our analysis of cyclotomic Pisot numbers.

Equations of the form

$$X^3 - (n-1)X^2Y - (n+2)XY^2 - Y^3 = 1 \tag{1}$$

can be solved completely by the methods of [11].

Equations of the form

$$X(X-Y)(X+Y)(X-aY) + Y^4 = \pm 1 \tag{2}$$

can be solved completely by the methods of [9].

Table 1
Pisot-Cyclotomic number of degree at most 6

Symmetry	Pisot-Cyclotomic Number
5	$x^2 - x - 1$ *
	$x^2 - 3x + 1$ *
7	$x^3 - 2x^2 - x + 1$ *
	$x^3 - 20x^2 - 9x - 1$
	$x^3 - 23x^2 + 34x - 13$
	$x^3 - 3x^2 - 4x - 1$
	$x^3 - 6x^2 + 5x - 1$
8	$x^2 - 2x - 1$ *
	$x^2 - 4x + 2$ *
9	$x^3 - 3x^2 + 1$ *
	$x^3 - 6x^2 - 9x - 3$
	$x^3 - 9x^2 + 6x - 1$
11 (a)	$x^5 - 10x^4 - 15x^3 - 3x^2 + 3x + 1$ **
12	$x^2 - 2x - 2$ *
	$x^2 - 4x + 1$ *
13 (a)	$x^6 - 15x^5 - 20x^4 + 6x^3 + 18x^2 + 8x + 1$ **
15	$x^4 - 24x^3 + 26x^2 - 9x + 1$
	$x^4 - 20x^3 - 40x^2 - 25x - 5$
	$x^4 - 4x^3 - 4x^2 + x + 1$ **
16	None exist
20	$x^4 - 8x^3 - 11x^2 - 2x + 1$
21 (a)	None found
24	$x^4 - 12x^3 - 22x^2 - 12x - 2$
	$x^4 - 16x^3 + 20x^2 - 8x + 1$
32 (a)	None found
36 (a)	$x^6 - 30x^5 - 60x^4 - 32x^3 + 3x^2 + 6x + 1$

Integer solutions to equations of the form

$$X^2 - 2Y^2 = \pm 1, \tag{3}$$

where one of X and Y is a perfect square can be given (see sections A14.1, A14.4, A14.5, A16.6 of Ribenboim [10]).

Integer solutions to equations of the form

$$X^2 - 5Y^2 = \pm 1, \pm 4, \tag{4}$$

where one of X and Y is a perfect square can be given (see Table 16.1 of [10] and use the results of [8]).

Equations of the form

$$X^4 - aX^2Y^2 \pm Y^4 = \pm 1 \tag{5}$$

can be solved completely by the methods of [12].

3. Degree 2 and 3

It was proved in [2] that

Theorem 3. *Let r be an algebraic number of degree at most 3. Then there are finitely many Pisot numbers q such that $\mathbb{Z}[q] = \mathbb{Z}[r]$.*

Furthermore, a constructive method was given to find all such Pisot numbers.

We will demonstrate the basic idea of this method with a degree 3 example. Write $q = u + xr + yr^2$ and $q^2 = (u + xr + yr^2)^2 = u_1 + x_1 r + y_1 r^2$. Here u_1, x_1 and y_1 are completely determined by u, x, y and the minimal polynomial of r. We can now write

$$\begin{bmatrix} 1 \\ q \\ q^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ u & x & y \\ u_1 & x_1 & y_1 \end{bmatrix} \begin{bmatrix} 1 \\ r \\ r^2 \end{bmatrix}.$$

We see that $\mathbb{Z}[q] = \mathbb{Z}[r]$ if and only if the above matrix is invertible and the inverse has integer entries. Equivalently,

$$\det \begin{bmatrix} 1 & 0 & 0 \\ u & x & y \\ u_1 & x_1 & y_1 \end{bmatrix} = \pm 1.$$

It is worth observing that this determinant will be independent of u; moreover, since x_1 and y_1 are homogeneous of degree 2 in x and y, the determinant will be homogeneous of degree 3 in x and y. At this point it is a matter of solving a homogeneous Diophantine equation in two variables. After this, a quick check is done to see which of these solutions give rise to Pisot numbers.

This method was used to find all examples of degrees 2 and 3 (correspondingly symmetries of orders 5, 8, 12 and 7, 9). These are listed in Table 1.

It should be mentioned that for symmetries of orders 7 and 9 we needed to solve the two Thue equations

$$x^3 - 2x^2y - xy^2 + y^3 = \pm 1$$

and

$$x^3 - 3xy^2 + y^3 = \pm 1,$$

respectively. Notice that if $f(x, y)$ is a homogeneous polynomial of degree 3 (or really any odd degree) then $f(-x, -y) = -f(x, y)$. Thus it suffices to solve the equations when $+1$ is on the right-hand side and then add to the set of solutions to this equation, all elements of the form $(-a, -b)$, where (a, b) is in our set of solutions. Using the substitution $Y = -x$, $X = y$ in the first equation and $X = x$, $Y = -y$ in the second equation we get the two equations

$$X^3 + X^2Y - 2XY^2 - Y^3 = 1$$

and

$$X^3 - 3XY^2 - Y^3 = 1.$$

These are equations of type (1) with $n = 0$ and 1, respectively, and hence all solutions can be found.

4. Degree 4

The results of [2] can be used as a starting point to show

Theorem 4. *If* $\beta = 2\cos(2\pi/n)$, *and* $\deg(\beta) \leqslant 4$, *there are only a finite number of Pisot numbers* q *such that* $\mathbb{Z}[q] = \mathbb{Z}[\beta]$.

The proof of Theorem 4 actually has two components. The first component is proving that there are a finite number of solutions to a particular Diophantine equation or system of equations. The second component is to determine which Pisot numbers are associated with each of these solutions.

It is possible that for other related problems (e.g., finding Pisot numbers q such that $\mathbb{Z}[q]$ is equal to some other ring), that there are an infinite number of solutions to the corresponding Diophantine equations, and yet it may still be possible for there to be only finitely many, (or possibly no) Pisot numbers associated with these solutions.

So, although a finite number of solutions to the Diophantine equation implies that there are only a finite number of Pisot numbers associated with a ring, the converse is not necessarily true. It would actually be interesting to look at this problem further to see if there are examples where this is in fact the case.

In all cases, we are assuming that we can write the Pisot number q as $q = u + x\beta + y\beta^2 + z\beta^3$, where $\beta = 2\cos(2\pi/n)$ for the n in question.

From this, using the same method as was described in the paragraph after the statement of Theorem 3, we can determine a homogeneous polynomial, in terms of x, y and z, such that this polynomial must equal ± 1 at the integer point (x, y, z), as a necessary condition for $\mathbb{Z}[q] = \mathbb{Z}[\beta]$.

In all four cases, this resulting polynomial factors into two, or three distinct factors say P_1, P_2, P_3, (where P_3 may be 1). In this case we have that $P_1 = \pm 1$, $P_2 = \pm 1$ and $P_3 = \pm 1$. At this point, depending on the values of P_1 and P_2, we take the resultant of $P_1 \pm 1$ and $P_2 \pm 1$ with respect to either x or y, regarding x, y, and z as indeterminates for the moment. We consider the individual cases $\mathrm{res}_x(P_1 \pm 1, P_2 \pm 1)$. The resultant must be zero and by finding all integer solutions to this equation, we are able to determine all cyclotomic Pisot numbers when $\mathbb{Q}[2\cos(2\pi/n)]$ has degree at most 4 over $\mathbb{Q}$. The cases when $[\mathbb{Q}[2\cos(2\pi/n)] : \mathbb{Q}] < 4$ have been handled already and so we only need to look at the case when $[\mathbb{Q}[2\cos(2\pi/n)] : \mathbb{Q}] = 4$. We note that the degree of this extension is $\phi(n)/2$ and so, after exploiting symmetry, we only need to consider $n \in \{15, 16, 20, 24\}$.

4.1. Symmetry of order $n = 15$

We have

$$P_1 = -y^2 + xy + x^2 + 3yz + 6xz + 9z^2$$

and

$$\begin{aligned} P_2 = {} & y^4 - 26zy^3 - 7y^3x - 36y^2zx - 54z^2y^2 - 6y^2x^2 + 69yz^2x + 18yzx^2 \\ & +109z^3y + 2yx^3 + 233z^3x + x^4 + 17zx^3 + 181z^4 + 99z^2x^2. \end{aligned}$$

Suppose first that $P_1 = P_2 = 1$. Consider $\mathrm{res}_x(P_1 - 1, P_2 - 1)$, the resultant with respect to the variable x. The resultant has two factors, one of which must equal 0,

$$p_1 = y^4 + 2y^2 + 1 + 3zy^3 + 3yz - z^2y^2 - 3z^2 - 3z^3y + z^4$$

and

$$p_2 = y^4 + 3zy^3 - z^2y^2 - z^2 - 3z^3y + z^4.$$

We see that the equation $p_1 = 0$ has no solutions mod 3, and hence has no integer solutions.

Using the techniques of Section 5, we see that to solve $p_2 = 0$, it is sufficient to solve the Thue equation of type (2),

$$Y(Y + X)(Y - X)(Y \pm 3X) + X^4 = 1,$$

where $X = \gcd(y, z)$, $z = \pm X^2$, $y = XY$. Using the description of these solutions given by Mignotte et al. [9], combined with the fact that $P_1 = P_2 = 1$, we see that

$$(x, y, z) \in \pm\{(1, 0, 0), (\pm 1, 1, 0)\}.$$

From this we get the Pisot polynomial $q^4 - 4q^3 - 4q^2 + q + 1$.

Suppose that $P_1 = -P_2 = 1$. Consider $\mathrm{res}_x(P_1 - 1, P_2 + 1)$. We get that the resultant is equal to 4 mod 5, and hence there are no integer solutions in this case.

Suppose that $-P_1 = P_2 = 1$. Consider $\mathrm{res}_y(P_1 + 1, P_2 - 1)$, the resultant of the two polynomials with respect to y. We get that the resultant has two factors, one of which must equal 0,

$$p_1 = x^4 + 2x^2 + 1 + 8x^3z + 9xz + 14x^2z^2 + 6z^2 - 23xz^3 - 59z^4$$

and

$$p_2 = x^4 - x^2 + 8x^3z - 8xz + 14x^2z^2 - 16z^2 - 23xz^3 - 59z^4.$$

We see that the equation $p_1 = 0$ has no solutions mod 3, and hence no integer solutions.

We now consider the equation $p_2 = 0$. Let $u = x + 4z$. Then we can write this equation as

$$u^4 - 8u^3z + 14u^2z^2 - 7uz^3 + z^4 - u^2 = 0.$$

Using the techniques of Section 5, we see it suffices to Thue equation

$$U^4 - 8U^3Z + 14U^2Z^2 - 7UZ^3 + Z^4 = 1.$$

By using the substitution $Z = U + V$ we get an equation in U and V of type (2), namely

$$u(u - v)(u + v)(u + 3v) + v^4 = 1.$$

Thus there are only finitely many integer solutions in this case and they can be explicitly computed. The solutions to this Diophantine equation are

$$(U, V) \in \pm\{(0, 1), (3, -1), (-1, -3), (1, \pm 1), (-1, 0)\}.$$

From this we see that

$$(x, y, z) \in \pm\{(-1, 1, 1), (-3, 1, 1), (1, 2, 0)\}.$$

These solutions give rise to the two Pisot polynomials $q^4 - 20q^3 - 40q^2 - 25q - 5$ and $q^4 - 24q^3 + 26q^2 - 9q + 1$.

Suppose now that $P_1 = P_2 = -1$. Consider $\mathrm{res}_x(P_1 + 1, P_2 + 1)$. We get that the resultant is equal to $4 \bmod 5$, and hence there are no integer solutions in this case.

From this we see that there are exactly three Pisot numbers q satisfying $\mathbb{Z}[q] = \mathbb{Z}[2\cos(2\pi/15)]$. These Pisot numbers have corresponding Pisot polynomials $q^4 - 4q^3 - 4q^2 + q + 1$, $q^4 - 20q^3 - 40q^2 - 25q - 5$, and $q^4 - 24q^3 + 26q^2 - 9q + 1$.

4.2. Symmetry of order $n = 16$

We have

$$P_1 = x^2 + 2z^2 + 4xz$$

and

$$P_2 = 196z^4 + 224xz^3 + 92z^2x^2 - 80z^2y^2 - 48zxy^2 + 16x^3z - 8x^2y^2 + 8y^4 + x^4.$$

Observe that $P_2 \equiv x^4 \bmod 4$ and hence we cannot possibly have $P_2 = -1$. We conclude that $P_2 = 1$ and so we have only two cases to consider.

Suppose first that $P_1 = 1$. Consider $\mathrm{res}_x(P_1 - 1, P_2 - 1)$. We get that the resultant has two factors, one of which must equal 0,

$$p_1 = y^4 - 4z^2y^2 - z^2 + 2z^4$$

and

$$p_2 = y^4 - 2y^2 + 1 - 4z^2y^2 + 3z^2 + 2z^4.$$

For the equation $p_1 = 0$, using the techniques of Section 5 we see that other than the trivial solution $y = z = 0$, all solutions to $p_1 = 0$ can be found by solving the Thue equation

$$(Y^2 - 2X^2)^2 - 2X^4 = 1,$$

where $z = \pm X^2$ and $y = YX$. which is an equation of form (3), which can be explicitly done. The only solutions to this equation are $X = 0$, $Y = \pm 1$. Hence we see that $(x, y, z) = (\pm 1, 0, 0)$. Thus from the equation $p_1 = 0$ we get no Pisot polynomials.

Now consider the equation $p_2 = 0$. First write $u = y^2$ and $v = z^2$. Then the equation $p_2 = 0$ is equivalent to

$$u^2 - 2u + 1 - 4vu + 3v + 2v^2 = 0.$$

Solving for u in terms of v, we see that

$$u = 2v + 1 \pm \sqrt{(2v+1)^2 - (2v^2 + 3v + 1)} = \sqrt{2v+1}\big(\sqrt{2v+1} \pm \sqrt{v}\big).$$

Since v is a perfect square and u is an integer, we deduce that $2v + 1$ must also be a perfect square. Furthermore, observe that

$$2(\sqrt{2v+1} + \sqrt{v})(\sqrt{2v+1} - \sqrt{v}) - (2v + 1) = 1.$$

Consequently $\gcd(\sqrt{2v+1}, \sqrt{v} \pm \sqrt{2v+1}) = 1$, and since u is a perfect square, we conclude that $\sqrt{2v+1}$ must be a perfect square. Hence $2v+1 = t^4$ for some integer t. Since $v = z^2$ we obtain the Diophantine equation $t^4 - 2z^2 = 1$. By the work of Ribenboim [10], the only integer solutions to this equation occur when $z = \pm 1, \pm 2$. By assumption $P_1 = 1$. Since $z = 0$, we have $x = \pm 1$. Since $P_2 = 1$, $y = 0, \pm 1$. These values do not give rise to any Pisot numbers.

Suppose that $P_1 = -1$ and consider $\operatorname{res}_x(P_1 + 1, P_2 - 1)$. We get two factors, one of which must equal zero

$$p_1 = y^4 - 4z^2y^2 + z^2 + 2z^4$$

and

$$p_2 = y^4 + 2y^2 + 1 - 4z^2y^2 - 3z^2 + 2z^4.$$

For the equation $p_1 = 0$, using the techniques of Section 5, we see that we need to solve the equation

$$(Y^2 - 2X^2)^2 - 2X^4 = -1,$$

where $y = YX$, $z = \pm X^2$. This is an equation of form (3), and hence we can find all integer solutions to this equation. The solutions to this equation are $(X, Y) \in \pm\{(1, \pm 1)\}$. Thus the only solutions we obtain when $p_1 = 0$ are $(x, y, z) \in \pm\{(-3, \pm 1, 1)\}$. Thus from the equation $p_1 = 0$ we get no Pisot polynomials.

We now find all integer solutions to the equation $p_2 = 0$. First we substitute $u = y^2$ and $v = z^2$ to get

$$u^2 - 4uv + 2v^2 + 2u - 3v + 1 = 0.$$

Regarding the left-hand side as a quadratic in u and solving for u we get

$$u = 2v - 1 \pm \sqrt{(2v-1)^2 - (2v^2 - 3v + 1)} = \sqrt{2v-1}(\sqrt{2v-1} \pm \sqrt{v}).$$

Arguing as we did in Case 16.1, we see that $\sqrt{2v-1}$ is a perfect square. Thus $2v - 1 = t^4$ for some integer t. Since $v = z^2$, we see that

$$t^4 - 2z^2 = -1.$$

From Ribenboim [10], the only integer solutions to this equation occur when $z = \pm 1$. By assumption $P_1 = -1$ and so $x^2 + 2z^2 + 4xz = -1$. Thus $(x, z) \in \pm\{(-1, 1), (-3, 1)\}$.

Solving for y we see that the only solutions in integers are

$$(x, y, z) = \pm\{(-3, 0, 1), (-3, \pm 1, 1)\}.$$

None of these values of x, y, z give rise to Pisot numbers.

Hence there are no Pisot numbers q satisfying $\mathbb{Z}[q] = \mathbb{Z}[2\cos(2\pi/16)]$.

4.3. Symmetry of order $n = 20$

We have two factors,

$$P_1 = x^2 + 5z^2 + 5xz$$

and

$$P_2 = 400z^4 + 400xz^3 + 140z^2x^2 - 100z^2y^2 - 60zxy^2 + 20x^3z - 10x^2y^2 + 5y^4 + x^4.$$

Notice that $P_2 \equiv x^4 \bmod 5$. Since $x^4 \equiv 0, 1 \bmod 5$, we see that P_2 must be equal to 1. This leaves us with two cases to consider, namely $P_1 = 1$ and $P_1 = -1$.

Suppose that $P_1 = 1$. Consider $\mathrm{res}_x(P_1 - 1, P_2 - 1)$. This has two factors, one of which must be zero,

$$p_1 = y^4 - 5z^2y^2 - z^2 + 5z^4,$$

$$p_2 = y^4 - 4y^2 + 4 - 5z^2y^2 + 9z^2 + 5z^4.$$

Using the techniques of Section 5, we see that the equation $p_1 = 0$ can be transformed into the Thue equation

$$Y^4 - 5Y^2X^2 + 5X^4 = 1,$$

where $z = \pm X^2$, $y = YX$. Factoring this over $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, we see that

$$(Y^2 - 5X^2/2 - \sqrt{5}X^2/2)(Y^2 - 5X^2/2 + \sqrt{5}X^2/2) = 1.$$

Since the unit group of $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ is generated by $\rho = \frac{1+\sqrt{5}}{2}$ and -1, we see

$$Y^2 - 5X^2/2 - \sqrt{5}X^2/2 = (-1)^k\rho^d,$$

$$Y^2 - 5X^2/2 + \sqrt{5}X^2/2 = (-1)^k\rho^{-d}.$$

Subtracting these two equations gives

$$X^2 = \pm F_d$$

for some d, where F_d is the dth Fibonacci number. Since the only square Fibonacci numbers are $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_{12} = 144$ (see [4]), we see that $z \in \{0, \pm 1, \pm 144\}$. From this and the assumption that $p_1 = 0$ we get the solutions

$$(x, y, z) \in \pm\{(1, 0, 0), (-1, \pm 2, 1), (-4, \pm 1, 1)\}.$$

This leads to the Pisot polynomial $q^4 - 8q^3 - 11q^2 - 2q + 1$.

We now solve the equation $p_2 = 0$. First we write $u = y^2$ and $v = z^2$ to obtain the equation

$$u^2 - 4u + 4 - 5vu + 9v + 5v^2 = 0.$$

Solving for u in terms of v, we get

$$u = \frac{5v + 4 \pm \sqrt{v}\sqrt{5v + 4}}{2}.$$

Since u is an integer and v is a perfect square we see that $5v + 4$ must also be a perfect square. Write

$$u = \sqrt{5v + 4}\left(\frac{\sqrt{5v + 4} \pm \sqrt{v}}{2}\right).$$

Then it is easy to see that $\sqrt{5v + 4}$ and $\sqrt{v}$ have the same parity and that $\sqrt{5v + 4}$ and $(\sqrt{5v + 4} \pm \sqrt{v})/2$ are relatively prime integers. Since u is a square, $\sqrt{5v + 4}$ is a perfect square. Thus $5v + 4 = t^4$ for some integer t. Since $v = z^2$, we obtain the Diophantine equation

$$t^4 - 5z^2 = 4.$$

From McDaniel and Ribenboim [8], we see that this equation has no integer solutions.

Consider the case that $P_1 = -1$ and look at $\mathrm{res}_x(P_1 + 1, P_2 - 1)$. The resultant has two factors, one of which must equal zero

$$p_1 = y^4 - 5z^2y^2 + z^2 + 5z^4$$

and

$$p_2 = y^4 + 4y^2 + 4 - 5z^2y^2 - 9z^2 + 5z^4.$$

Using the techniques of Section 5 for the equation $p_1 = 0$ we see that we need to solve the Thue equation

$$Y^4 - 5Y^2X^2 + 5X^4 = -1,$$

with $z = \pm X^2$, $y = YX$. But this has no solutions mod 5, and hence $p_1 = 0$ has only the trivial solution $z = y = 0$. If $z = 0$, since $P_1 = -1$ we get that $x^2 + 1 = 0$ and hence there are no integer solutions.

We now consider the equation $p_2 = 0$. First let $u = y^2$ and $v = z^2$ to get:

$$u^2 + 4u + 4 - 5vu - 9v + 5v^2.$$

Solving for u in terms of v, we see that

$$u = \frac{5v-4}{2} \pm \frac{\sqrt{(5v-4)^2 - 4(4-9v+5v^2)}}{2} = \frac{5v-4}{2} \pm \frac{\sqrt{v}\sqrt{5v-4}}{2}.$$

Since v is a perfect square and u is an integer, we have that $5v-4$ is a perfect square. Write

$$u = \sqrt{5v-4}\left(\frac{\sqrt{5v-4} \pm \sqrt{v}}{2}\right).$$

Then, just as in case 20.1, it is easy to see that $\sqrt{5v-4}$ and $\sqrt{v}$ have the same parity and that $\sqrt{5v-4}$ and $(\sqrt{5v-4} \pm \sqrt{v})/2$ are relatively prime integers. Since u is a square, $\sqrt{5v-4}$ is a perfect square. Thus $5v - 4 = t^4$ for some integer t. Writing $v = z^2$, we obtain the Diophantine equation $t^4 - 5z^2 = -4$. By the work of McDaniel and Ribenboim [8], this has only solutions if $z = \pm 1, \pm 2$. Solving for y and then x using the fact that $P_1 = -1$, we see that

$$(x, y, z) \in \pm \left\{ \begin{array}{c} (-2,0,1), (-3,0,1), (-2,1,1), (-3,1,1), (2,1,-1), \\ (3,1,-1), (-3,2,2), (-7,2,2), (3,2,-2), (7,2,-2) \end{array} \right\}.$$

Analyzing this set, we quickly deduce that in this case there are no Pisot numbers q which satisfy $\mathbb{Z}[q] = \mathbb{Z}[2\cos(2\pi/20)]$.

Thus there is only one Pisot number q satisfying $\mathbb{Z}[q] = \mathbb{Z}[2\cos(2\pi/20)]$. It has minimal polynomial $q^4 - 8q^3 - 11q^2 - 2q + 1$.

4.4. Symmetry of order $n = 24$

We have three factors in this case

$$P_1 = x^2 + z^2 + 4xz,$$

$$P_2 = -x^2 + 2y^2 - 6xz - 9z^2,$$

$$P_3 = -x^2 + 6y^2 - 10xz - 25z^2.$$

Notice that $P_1 \equiv x^2 + z^2 \bmod 4$ and hence $P_1 \neq -1$ when x, y, z are integers. We conclude if we have integer solutions, then P_1 must be equal to 1. Notice also that $P_3 - 3P_2 - 2P_1 = 0$. Since $P_1 = 1$ we have $P_3 - 3P_2 = 2$. Thus $P_3 = -1$ and $P_2 = -1$. Consider $\mathrm{res}_x(P_1 - 1, P_2 + 1)$. This resultant gives us the factor

$$p_1 = -z^2 - 4z^2y^2 + z^4 + y^4.$$

This can be solved by the techniques of Section 5. This requires a solution to the Thue equation of form (5),

$$Y^4 - 4Y^2X^2 + X^4 = 1,$$

where $y = YX, z = \pm X^2$. This can be solved explicitly and we find that

$$(x, y, z) \in \pm\{(1, 0, 0), (0, \pm 2, 1), (-4, 0, 1)\}.$$

These solutions give two Pisot polynomials $q^4 - 12q^3 - 22q^2 - 12q - 2$ and $q^4 - 16q^3 + 20q^2 - 8q + 1$.

5. Technique for a homogeneous polynomials minus z^2

Let

$$P(y, z) = y^4 + a_1y^3z + a_2y^2z^2 + a_3yz^3 + a_4z^4 - bz^2,$$

where $b = \pm 1$. We solve $P(y, z) = 0$.

Write $X = \gcd(y, z)$ and $y = YX$ and $z = ZX$. Then we have

$$X^2(Y^4 + a_1Y^3Z + a_2Y^2Z^2 + a_3YZ^3 + a_4Z^4) = bZ^2.$$

As $|b| = 1$ we have that $X^2 | Z^2$. So we can write $Z = XZ_0$. Thus we get

$$(Y^4 + a_1Y^3XZ_0 + a_2Y^2X^2Z_0^2 + a_3YX^3Z_0^3 + a_4X^4Z_0^4) = bZ_0^2.$$

Now, if p is a prime with $p|Z_0$, we see that $p|Y^4$ and hence $p|Y$. By construction $\gcd(Z_0, Y) = 1$ since Z and Y are relatively prime and so we conclude that $Z_0 \in \{\pm 1, 0\}$. If $Z_0 = 0$ we get either the trivial solutions $(y, z) = (\pm 1, 0)$ or no solutions. If $Z_0 = 1$, then we get the Thue equation

$$Y^4 + a_1Y^3X + a_2Y^2X^2 + a_3YX^3 + a_4X^4 = b.$$

If $Z_0 = -1$, then we get the Thue equation

$$Y^4 - a_1 Y^3 X + a_2 Y^2 X^2 - a_3 Y X^3 + a_4 X^4 = b.$$

For the purposes of this paper, these Thue equations always turn out to be of form (2), (3) or (5). Once we solve these equations, we have

$$y = XY \quad \text{and} \quad z = \pm X^2 \tag{6}$$

and so we see that other than the trivial solution, we can find all integer solutions to our original equation by looking at Thue equations.

6. Conclusions and open questions

All Pisot-Cyclotomic numbers up to degree 4 are determined by this paper. It may be possible to extend these results to higher degrees, but it will require much more difficult techniques. Unfortunately, when trying to find examples for degree n, using these techniques, one ends up with a polynomial P such that there are integer solutions to $P = \pm 1$. Here P has $n - 1$ variables, and is of degree $\frac{n(n-1)}{2}$. Even computing these polynomials is expensive in terms of computer power.

Without lucky factorization (as was the case with degree 4), it is unclear how you would find all integer solutions. It is worth mentioning that for degree 5, ($n = 11$) then the polynomial does factor.

That being said, given the severe restrictions on the values of these variables, first that they must satisfy $P = \pm 1$, and second, that they must give rise to a Pisot number, it is still plausible that there are only finitely many Pisot-Cyclotomic numbers for every symmetry. It is even plausible that there are only finitely many Pisot-Cyclotomic numbers, but a lot more work would have to be done before any reasonable conjecture could be made in this direction.

So, the main open questions are

- Is there an effective way to compute the Pisot-Cyclotomic numbers of degrees higher than 4?
- Are there finitely many Pisot-Cyclotomic numbers of any particular symmetry?
- Are there finitely many Pisot-Cyclotomic numbers?
- Under what conditions does the polynomial P (as computed by the determinant) factor?
- Are there examples where the Diophantine equation associated with the ring allows for an infinite number of solutions, and yet the number of Pisot numbers associated with the ring is still finite?

Acknowledgments

We would like to thank Chris Smyth for some useful suggestions on how to deal with the degree 4 case. We would also like to thank Christine Frougny for directing us to Refs. [1,6,7]. Finally, we thank the referee for many helpful comments.

References

[1] D. Barache, B. Champagne, J.-P. Gazeau, Pisot-cyclotomic quasilattices and their symmetry semigroups, in: Quasicrystals and Discrete Geometry (Toronto, ON, 1995), American Mathematical Society, Providence, RI, 1998, pp. 15–66.
[2] J.P. Bell, K.G. Hare, On $\mathbb{Z}$-modules of algebraic numbers, manuscript.
[3] Č. Burdík, C. Frougny, J.P. Gazeau, R. Krejcar, Beta-integers as natural counting systems for quasicrystals, J. Phys. A 31 (30) (1998) 6449–6472.
[4] J.H.E. Cohn, On square Fibonacci numbers, J. London Math. Soc. 39 (1964) 537–540.
[5] A. Elkharrat, C. Frougny, J.-P. Gazeau, J.-L. Verger-Gaugry, Symmetry groups for beta-lattices, Theoret. Comput. Sci. 319 (2004) 281–305.
[6] C. Frougny, J.-P. Gazeau, R. Krejcar, Additive and multiplicative properties of point sets based on beta-integers, Theoret. Comput. Sci. 303 (2–3) (2003) 491–516 tilings of the plane.
[7] J.-P. Gazeau, Pisot-cyclotomic integers for quasilattices, in: The mathematics of long-range aperiodic order (Waterloo, ON, 1995), Kluwer Academic Publishers, Dordrecht, 1997, pp. 175–198.
[8] W.L. McDaniel, P. Ribenboim, The square terms in Lucas sequences, J. Number Theory 58 (1) (1996) 104–123.
[9] M. Mignotte, A. Pethő, R. Roth, Complete solutions of a family of quartic Thue and index form equations, Math. Comp. 65 (213) (1996) 341–354.
[10] P. Ribenboim, Catalan's Conjecture: are 8 and 9 the only Consecutive Powers?, Academic Press Inc., Boston, MA, 1994.
[11] E. Thomas, Complete solutions to a family of cubic Diophantine equations, J. Number Theory 34 (2) (1990) 235–250.
[12] I. Wakabayashi, On a family of quartic Thue inequalities. I, J. Number Theory 66 (1) (1997) 70–84.