ICCM2014
28-30th July, Cambridge, England

Geometrically exact beam dynamics, with and without rotational degree of freedom

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Abstract

Non-linear rod dynamics is the focus of research in many engineering areas such as structural, aerospace and petroleum engineering as well as multibody dynamics. Also in non-classical areas such as biomechanics, micro- and nano-mechanics, geometrically exact formulations for rod dynamics are of importance. Rod formulations can be distinguished in regard to the basic kinematic assumption underlying the formulation. In the so-called Timoshenko-type beams, shear effects are taken into account and so rotational degrees of freedom describing the rotation of the cross section are considered. These are highly non-linear in nature. In contrast the Euler-Bernoulli assumption of zero shear deformation can be carried over into the non-linear regime resulting in displacement-only formulation but with highly non-linear expressions for the strain tensor incorporating higher gradients. In either formulation, the integration of the time dependent equations is challenging. It has been recognised that energy conservation is key for stable integration in long term dynamics. The so-called energy-momentum methods is a class of integrators, which, by design, conserve the momentum, angular momentum and the energy in the discrete case, if the same conservation properties are present in the continuous case. While for the Timoshenko beam some progress has been made and specific energy-momentum methods are known in the literature, the same is not true far the higher-gradient beam formulation of the Bernoulli beam.

In this paper, we are going to develop a unified formulation of an energy-momentum integration scheme for both geometrically exact Bernoulli and Timoshenko beams. We will show that the stable integration in either case is achievable with excellent results. Further important novel aspect of the models are the full incorporation of the rotational inertia. A range of applications from structural dynamics to flexible multibody dynamics will show the excellent performance of the new energy-momentum integration scheme.

Keywords: Non-linear dynamics, Computational method, Euler-Bernoulli rod, Timoshenko rod, Energy-momentum method, Multi-body dynamics.

Introduction

Dynamics of beam as well as in many new emerging areas of applications such as nano, bio mechanics, remains a very active topic of research. In the geometrically exact beam theories, a popular approach is the one so-called Timoshenko kinematics which still makes use of the assumption of planar cross sections in the deformed configurations but allows for shear to be considered by dropping the assumption that the vector normal to the centre line remains normal after the deformation [Ibrahimbegovic and Fray (1993); Iura and Iwakuma (1991)]. An example is the formulation of Simo and Vu Quoc [Simo and Vu-Quoc (1986)] which is based on a previous intrinsic formulation by Reissner [Reissner (1972)]. The Timoshenko type beam is well known to be suitable for short beams, sandwich composite beams and high-frequency excitation beams. On the contrary, in many applications, we desire to have a displacement-only formulation for example in mechanisms, nano and bio mechanics where the Euler-Bernoulli model is the best choice dispite of the complexities involved due to the kinematics assumptions.

Beside the kinematics descriptions and the strain measures, the time integration of the dynamical equations has also been the focus of research for decades. It is now generally accepted that classical time integration methods such as Newmark, standard midpoint rule do suffer severe shortcomings [Newmark (1959); Chung and Hulbert (1993)]. Especially the lack of stability is a major issue. It has been soon recognized that the conservation of energy is the key for the stability of the time integration scheme. Moreover, an efficient time integrator so-called energy-momentum method has been developed which conserves not only the energy but also the momentum and the angular momentum of the system. This method can provide good accuracy and stability in long-term dynamics. The first attempt to an energy-momentum method was proposed by Simo and Tarrow [Simo and Tarnow (1993)] but this algorithm is only valid for quadratic-nonlinearities. The method has many applications in Timoshenko beams but the treatment of rotation is anything but trivial, especially when it comes to incorporate the inertial term. Due to the highly complex non-linearity as a result of the kinematics assumptions, such this formulation is not as common for Euler-Bernoulli beam model.

In this paper, we aim to develop an energy-momentum integration scheme for geometrically exact Bernoulli and Timoshenko beams. Numerical examples will be provided to show the excellent performance of the method.

Kinematics, dynamics equation and finite discretization

Euler-Bernoulli beam model and kinematics description

Let $\mathcal{B} \subset \mathbb{R}^3$, with $\mathbb{R}$ denoting the real numbers, define a reference configuration of the body. Without loss of generality we want to identify the reference configuration with the body itself. The actual configuration is denoted by $\mathcal{B}_t \in \mathbb{R}^3$. We assume that our body is thin in two dimensions such that it is rod-like with a cross section A at the reference configuration. The material particles are identified by their position vectors $\in \mathcal{B}$, the corresponding placement at the actual configuration by $\mathbf{x} \in \mathcal{B}_t$. A deformation is a map $\mathbf{x} = \varphi(\mathbf{X})$, the gradient of which defines the deformation gradient $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$. We want to restrict ourselves to plane deformations and assume that the deformation lies in the $\mathbf{e}_1 - \mathbf{e}_2$ plane. For any material point in the cross section a suitable curvilinear coordinate system which we consider to be convected, is then given by the triple $\mathrm{s}, \mathrm{z}, \mathbf{e}_3$. z is the coordinate in the direction of the normal vector in the cross section. The relation holds

$$\mathbf{X}(\mathrm{s},\mathrm{z}) = \mathbf{X}_0(\mathrm{s},\mathrm{z}) + \mathrm{z}\mathbf{N}(\mathrm{s}), \tag{1}$$

where $\mathbf{X}_0(\mathrm{s},\mathrm{z})$ is the placement of the central line at the reference configuration. Correspondingly, $\mathbf{G}_0 = \frac{\partial \mathbf{X}_0}{\partial \mathrm{s}}$, is a tangent vector. Similarly, we can introduce $\mathbf{G} = \frac{\partial \mathbf{X}}{\partial \mathrm{s}}$, $\mathbf{G}_0 = \left.\frac{\partial \mathbf{X}}{\partial \mathrm{s}}\right|_{\mathrm{z}=0}$, $\mathbf{N} = \frac{\partial \mathbf{X}}{\partial \mathrm{z}}$. The triple $(\mathbf{G}, \mathbf{N}, \mathrm{e}_3)$ defines a local curvilinear bases. The relations also hold

$$\mathbf{G}_0 \cdot \mathbf{N} = 0, \qquad |\mathbf{N}| = 1, \qquad \mathbf{N} = \mathbf{e}_3 \times \frac{\mathbf{G}}{|\mathbf{G}|}, \tag{2}$$

where $\times$ denotes the cross product of vectors, a dot denotes the scalar product of vectors. The corresponding tangent vectors at the deformed configuration are defined as $(\mathbf{g}, \mathbf{n}, \mathbf{e}_3)$ with $\mathbf{g}_0 = \mathbf{g}|_{\mathrm{z}=0}$, $\mathbf{n}$ is the normal vector in the deformed configuration and

$$\mathbf{g} = \frac{\partial \mathbf{x}}{\partial \mathrm{s}}, \qquad \mathbf{n} = \mathbf{e}_3 \times \frac{\mathbf{g}}{|\mathbf{g}|} = \mathbf{e}_3 \times \frac{\mathbf{X}_{0,s} + \mathbf{u}_{,s}}{|\mathbf{X}_{0,s} + \mathbf{u}_{,s}|}. \tag{3}$$

To derive the rod theory we adopt the Bernoulli hypothesis which assumes rigid cross sections and that the deformation can be completely characterized by the assumption

$$\mathbf{x} = \mathbf{X}(\mathrm{s}) - z\mathbf{N}(\mathrm{s}) + \mathbf{u}(\mathrm{s}) + \mathrm{z}\mathbf{n}(\mathrm{s}) = \mathbf{X}_0(\mathrm{s}) + \mathbf{u}(\mathrm{s}) + z\mathbf{n}(\mathrm{s}), \tag{4}$$

where $\mathrm{u}(\mathrm{s})$ is the displacement at the curvilinear coordinate s.
In the context of an in-plane Bernouilli beam, the right Cauchy deformation tensor has only one single non-trivial component which is C_{11} which reads

$$C_{11} = \left(\mathbf{X}_{0_{,s}} + \mathbf{u}_{,s} + z\mathbf{n}_{,s}\right)\cdot\left(\mathbf{X}_{0_{,s}} + \mathbf{u}_{,s} + z\mathbf{n}_{,s}\right), \tag{5}$$

where a comma denotes a derivative. With $\mathbf{E} = \frac{1}{2}(\mathbf{C} - 1)$ as the Green strain tensor, one is then left with one sigle non-trivial component E_{11} which is given by (the term in z^2 can be neglected since our thickness of the beam is small compared to its length)

$$E_{11} = \mathbf{u}_{,s}\cdot\mathbf{X}_{0_{,s}} + \frac{1}{2}\mathbf{u}_{,s}\cdot\mathbf{u}_{,s} + z\left(\mathbf{n}_{,s}\cdot\left(\mathbf{X}_{0_{,s}} + \mathbf{u}_{,s}\right) - \mathbf{N}_{,s}\cdot\mathbf{X}_{0_{,s}}\right). \tag{6}$$

By defining ε_{11} as the axial strain, κ as the change of curvature, their expressions read

$$\varepsilon_{11} = \mathbf{u}_{,s}\cdot\mathbf{X}_{0_{,s}} + \frac{1}{2}\left(\mathbf{u}_{,s}\cdot\mathbf{u}_{,s}\right) \tag{7}$$

$$\kappa = \mathbf{n}_{,s}\cdot\left(\mathbf{X}_{0_{,s}} + \mathbf{u}_{,s}\right) - \mathbf{N}_{,s}\cdot\mathbf{X}_{0_{,s}}. \tag{8}$$

Timoshenko beam model and kinematics description

Timoshenko beam model and Bernoulli one differ only in the assumption about the cross section which is still rigid but no longer perpendicular to the central line. Therefore, the kinematics should be described differently. The triple $(\mathbf{G}, \mathbf{N}, \mathbf{e}_3)$ defines the local curvilinear bases, $\mathbf{N}$ is the normal vector to the tangent space of the rod. $\mathbf{N}$ is given by the relation $\mathbf{N} = \mathbf{G}\times\mathbf{e}_3$. Altogether, the relations hold

$$\mathbf{G} = -\sin\alpha\ \mathbf{e}_1 + \cos\alpha\ \mathbf{e}_2, \qquad \mathbf{N} = \cos\alpha\ \mathrm{e}_1 + \sin\alpha\ \mathrm{e}_2, \tag{9}$$

Where α is a function of s and determines the angle closed between e_1 and $\mathbf{N}$. Accordingly**, t**he displacement field is defined by

$$\mathbf{u} = \mathbf{x} - \mathbf{X}.$$

Furthermore, one has $\mathbf{g} = \mathbf{G} + \mathbf{u}_{,s}$ and correspondingly we obtain $\mathbf{F} = (\mathbf{G} + \mathbf{u}_{,s})\otimes\mathbf{G}$.
We consider now a rotational field $\mathbf{R} \in \mathrm{SO}(3)$, where $\mathrm{SO}(3)$ is a group of orthogonal tensors with positive determinant. Since we remain in plane $(\mathbf{e}_1 - \mathbf{e}_2)$, the rotation vector is fixed to vector $\mathbf{e}_3$. Therefore, we obtain the following expression of the rotation tensor

$$\mathbf{R} = \cos\omega\,(\mathbf{e}_1\otimes\mathbf{e}_1 + \mathbf{e}_2\otimes\mathbf{e}_2) - \sin\omega\,(\mathbf{e}_1\otimes\mathbf{e}_2 - \mathbf{e}_2\otimes\mathbf{e}_1) + \mathbf{e}_3\otimes\mathbf{e}_3.$$

We denote the corresponding axial vector by $\mathbf{k}$. To get the strain measures, we apply the direct method of a Cosserat line. Accordingly, we get the first Cosserat deformation tensor (the stretch tensor) $\mathbf{U} := \mathbf{R}^{\mathrm{T}}\mathbf{F}$, the second Cosserat strain tensor $\mathbf{K} := -\mathbf{k}\otimes\mathbf{G}$. Because the Cosserat is assumed to be one-dimensional and in-plane deformation, we can write down $\mathbf{u}, \mathbf{U}, \mathbf{K}$ as follows $\mathbf{u} = \mathrm{u}_1\mathbf{e}_1 + \mathrm{u}_2\mathbf{e}_2$, $\mathbf{U} = \mathrm{U}_{11}\mathbf{G}\otimes\mathbf{G} + \mathrm{U}_{13}\mathbf{G}\otimes\mathbf{N}$, $\mathbf{K} = \kappa\,\mathbf{e}_3\otimes\mathbf{e}_3$, where the expressions of the components are defined as

$$U_{11} = cos\,\omega - sin(\alpha+\omega)\,u_{1,s} + cos(\alpha+\omega)\,u_{2,s}, \tag{10}$$

$$U_{13} = sin\,\omega + cos(\alpha+\omega)\,u_{1,s} + sin(\alpha+\omega)\,u_{2,s}, \tag{11}$$

$$\kappa = \omega_{,s}. \tag{12}$$

In this model, the shear deformation is included in this model and is not assumed to be zero.

Dynamics equations

Starting from Hamilton Principle for our conservative mechanical system, the dynamics equation for our Bernoulli beam is written down as follows

$$\rho A \int_L (\ddot{\mathbf{u}} \cdot \delta\mathbf{u}) ds + \rho I \int_L \ddot{\mathbf{n}} \cdot \delta\mathbf{n} ds + \int_0^L (EA\varepsilon_{11}\delta\varepsilon_{11} + EI\kappa\delta\kappa) ds - \int_0^L (\mathbf{p} \cdot \delta\mathbf{u} + \mathrm{M}\delta\omega) ds + (\mathbf{p} \cdot \delta\mathbf{u} + \mathrm{M}\delta\omega)|_0^L = 0, \tag{13}$$

where E is Young's Modulus of the material, **p** is external force and M is external moments, I is the moment of inertia of the section and L is the length of the beam, ρ is the material density.
The dynamics equation for Timoshenko beam has the following form

$$\rho A \int_L (\ddot{\mathbf{u}} \cdot \delta\mathbf{u}) ds + \rho I \int_L \ddot{\omega}\delta\omega ds - \int_L \left[\delta\mathbf{n} \cdot \boldsymbol{\epsilon}(\boldsymbol{n}, \omega) + \mathbf{n} \cdot \frac{\partial \boldsymbol{\epsilon}(\boldsymbol{u},\omega)}{\partial \mathbf{u}} \delta\mathbf{u} + \mathbf{n} \cdot \frac{\partial \boldsymbol{\epsilon}(\boldsymbol{u},\omega)}{\partial \omega} \delta\omega - \frac{\partial \tilde{\psi}(\mathbf{n})}{\partial \mathbf{n}} \delta\mathbf{n} + \frac{\partial \psi_2(\omega)}{\partial \omega} \delta\omega\right] ds + \int_0^L (\mathbf{p} \cdot \delta\mathbf{u} + \mathrm{M}\delta\omega) ds + (\mathbf{p} \cdot \delta\mathbf{u} + \mathrm{M}\delta\omega)|_0^L = 0, \tag{14}$$

where $\mathbf{n}$ and $\boldsymbol{\epsilon}$ is the force and strain vectors respectively, ψ is the stored energy, $\tilde{\psi}$ is the complementary energy related to the strains $\boldsymbol{\epsilon}$ using Legendre transformation in order to avoid locking phenomena and construct robust finite elements.

Regarding to finite element approach in the Bernoulli case, given the fact that second derivatives are present in the equations (a result of the Bernoulli hypothesis), the finite element formulation must exhibit continuous first derivatives. Hence, we resort, within a finite element context to interpolation functions defined by cubic Hermite polynomials. For Timoshenko model, the finite element will be of hybrid type, a two-node elements with linear kinematical fields and constant force **n** (normal and shear components) is considered.

Energy-momentum time integration scheme

After the spatial discretisation via the finite element method, the numerical approach is completed by devising a step-by-step time integration scheme for the time dependent equations. Classical implicite schemes like the Midpoint rule or Newmark method have been very popular in the structural dynamics community. However while these are stable integration methods in the linear regime, they proved less so in the highly non-linear one, especially in long-term dynamics. They suffer from numerical instabilities like blow-ups as well documented in the literature [Bathe (1997); Sansour et al. (1997); Sansour et al. (2004)]. Energy-momentum methods proved to provide here the necessary stability. In what follows we will develop such a method tailored to our rod formulation. However, so far no such formulation was attempted for the Bernoulli beam because of the complexities involved in the kinematic assumptions. In the following we want to develop for the first time such an Energy-momentum method. In doing so, we resort to an idea developed in [Sansour et al. (1997); Sansour et al. (2004)]. The method described there is attractive because it is independent of the involved non-linearity, the source of problem in the presently considered beam. The starting point, however is the standard midpoint rule. From step n, where all kinematical fields and velocities are known, we need to find these quantities at time step n+1, Consider ξ to be a scalar which defines any position within the time interval ΔT, with $0 \leq \xi \leq 1$. We start with the following expressions:

$$\mathbf{x_{n+\xi}} = \mathbf{x_{n+1}} + (\mathbf{1} - \boldsymbol{\xi})\mathbf{x_n}, \tag{15}$$

$$\dot{\mathbf{x}}_{\mathbf{n+\eta}} = \frac{\mathbf{x_{n+1}} - \mathbf{x_n}}{\Delta\mathbf{T}}, \tag{16}$$

$$\ddot{\mathbf{x}}_{\mathbf{n+\eta}} = \frac{\mathbf{x_{n+1}} - \mathbf{x_n}}{\Delta\mathbf{T}}, \tag{17}$$

Where $\boldsymbol{\eta}$ is an open parameter. The first defines a convex set, the following two are true for some value of $\boldsymbol{\eta}$. The midpoint rule corresponding to $\boldsymbol{\eta} = 0.5$.

The key step is to employ strain velocity fields to define the strain fields. Let us consider the following velocity fields:

$$\dot{\varepsilon} = \dot{\boldsymbol{u}}_{,s} \cdot \boldsymbol{X}_{0_{,s}} + \boldsymbol{u}_{,s} \cdot \dot{\boldsymbol{u}}_{,s}. \tag{18}$$

$$\dot{\kappa} = \left(\frac{\partial\kappa}{\partial\boldsymbol{u}_{,s}} \cdot \dot{\boldsymbol{u}}_{,s} + \frac{\partial\kappa}{\partial\boldsymbol{u}_{,ss}} \cdot \dot{\boldsymbol{u}}_{,ss}\right) \tag{19}$$

Given the strain field defined at time n, the strain field at step n+1 then defined as following:

$$\varepsilon_{n+\xi} = \varepsilon_n + \xi\Delta T\dot{\varepsilon}_{n+\frac{1}{2}} \tag{20}$$

$$\kappa_{n+\xi} = \kappa_n + \xi\Delta T\dot{\kappa}_{n+\frac{1}{2}} \tag{21}$$

Specifically for ξ=1, the relations hold

$$\varepsilon_{n+1} = \varepsilon_n + \Delta T\dot{\varepsilon}_{n+\frac{1}{2}} \tag{22}$$

$$\kappa_{n+1} = \kappa_n + \Delta T\dot{\kappa}_{n+\frac{1}{2}} \tag{23}$$

This time integration scheme is proved formally and numerically to be stable and which conserve the energy, the momentum and the angular-momentum for a dynamic nonlinear system in long-term, some example are provided in the next section.

Numerical example

Example 1: Flying beam

In the first example, to investigate not only the conservation of energy but also of momentum and angular momentum, we consider a flying beam without support, the beam is depicted in Fig. 1 The loading increases linearly to a peak and decreases at the same rate to zero, Fig. 2. We run the calculation for one million time steps with $\Delta T = 1\mathrm{E} - 4\mathrm{s}$.

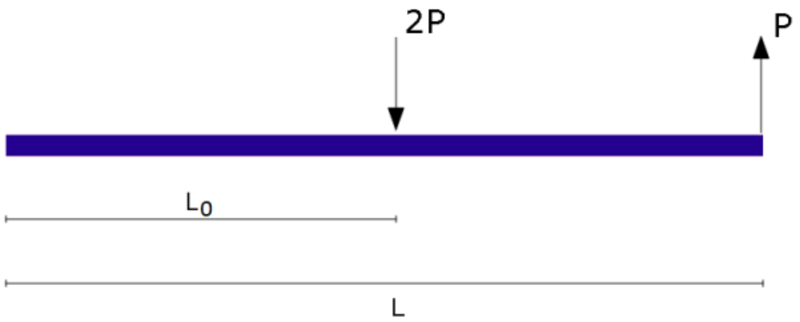

Figure 1: Beam figure

Table 1. Parameters

Parameters	symbol	value
Beam length	L	3m
Length	L_0	1.5m
Cross section area	A	200 cm^2
Cross section inertia	I	66.67 cm^4
Young's modulus	E	0.2E12 Pa
Density	ρ	48831Kg/m^3
Number of elements		4

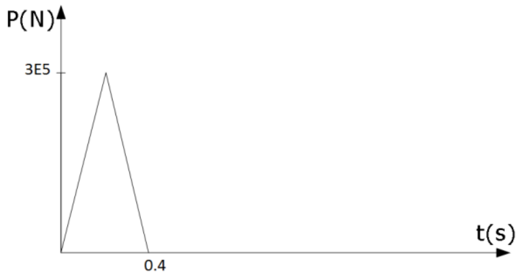

Figure 2: Loading history

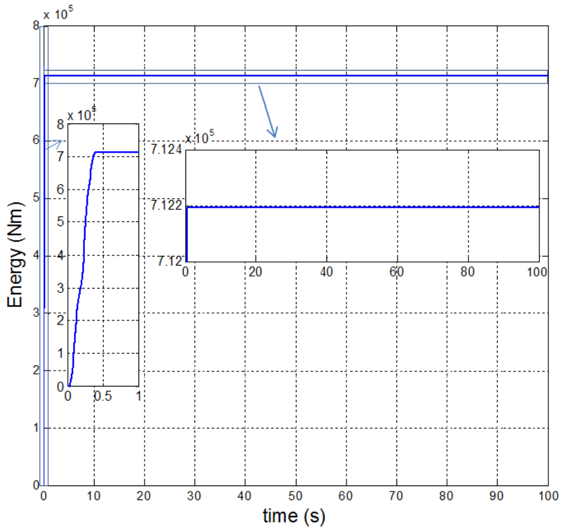

Figure 3: Energy history

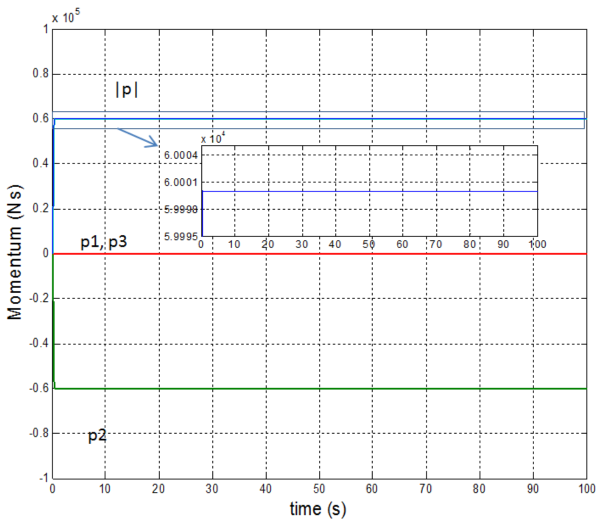

Figure 4: Momentum history

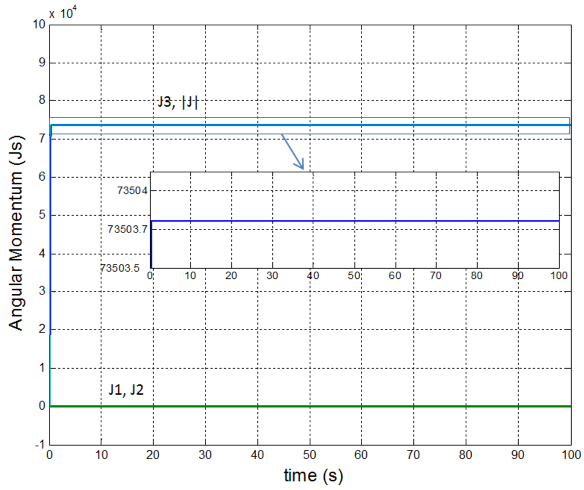

Figure 5: Angular momentum history

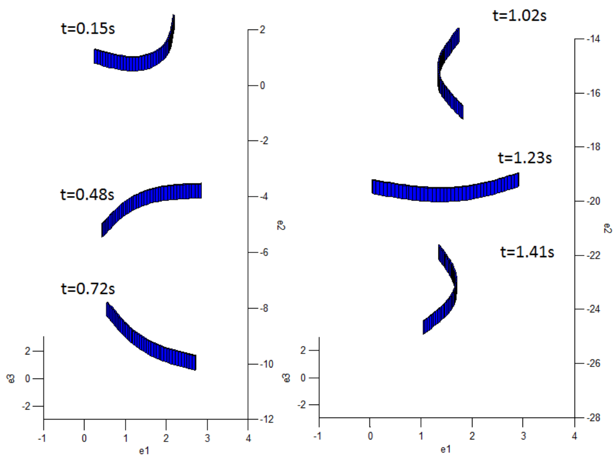

Figure 6: Beam motion snapshots

The Energy history is depicted in Fig. 3, while Fig. 4 and Fig. 5 reflect the linear momentum and angular momentum, respectively. In both figures not only the absolute value but also the components of the mentioned quantities are considered. Conservation is valid for the momentum and the angular momentum vector. Some deformations of the beam in space are captured in Fig. 6 which shows not only that the beam experiencing high deformation but also large overall displacement (24m at t≈1.4s).

Example 2: Chaotic motion of shallow arch

In this example, we investigate a chaotic motion of an arch. The arch configuration is given in Fig.~\ref{shaar}. From the configuration it can be seen that the arch is shallow and indeed can undergo a snap-through phenomena.

We consider here a system with a velocity-dependent damping with damping parameter D=2.5E-3 Ns/m. The system is subjected to a time-dependent concentrated force at its center of the form $P = F\cos(\omega t + \Pi)$, $\omega = 1000$ Hz , F = 800N. The excitation can be modified either by changing its amplitude or its frequency.

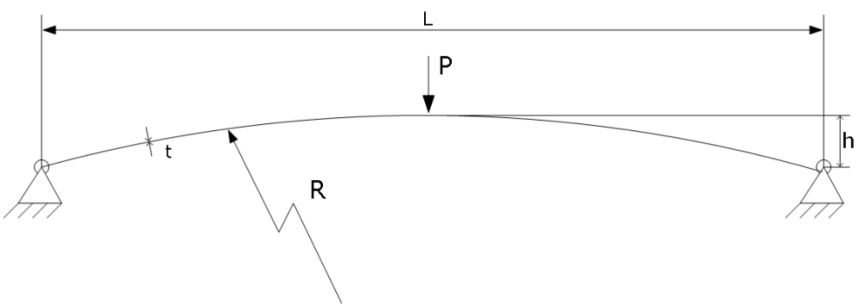

Figure 7: Shallow arch

Parameters:

Length $L = 70\text{cm}$
Height h = 1.53 cm
Density $\rho = 7.5E3\ \text{kg/m}^3$
Time increment: $\Delta T = 1\text{E} - 4\text{s}$

Thickness t= 1 cm
Young's modulus E=0.2E12 Pa
Number of elements = 10
Number of time steps = 3E6

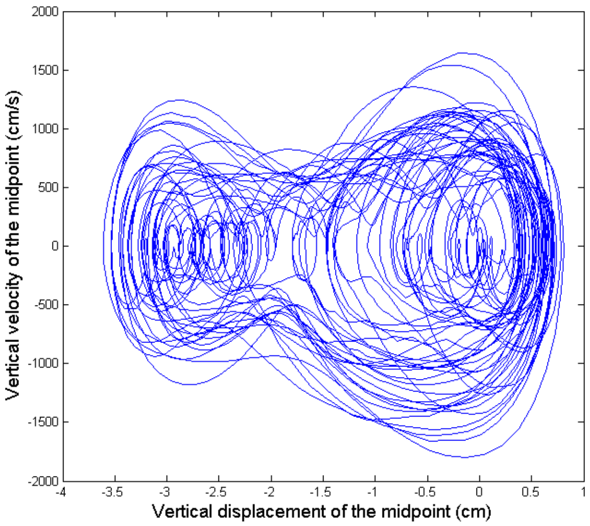

Figure 8: Phase space

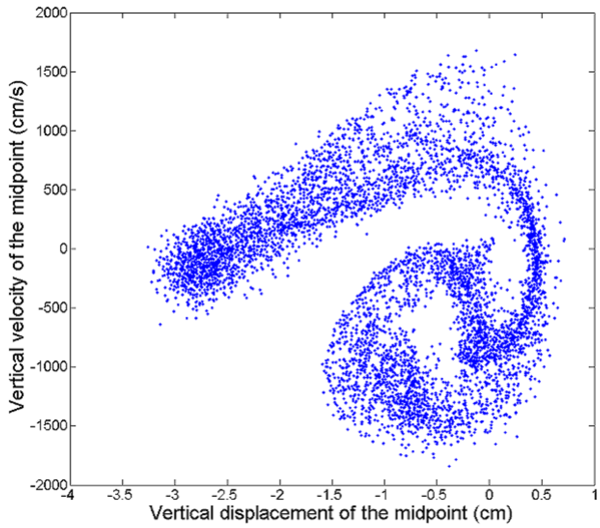

Figure 9: Poincaré section

Fig. 8 shows the phase space which plots the displacement against the velocity of the midpoint on the arch at each time step. The Poincaré section is presented in Fig.9. Those graphs show that the motion is chaotic which means long-term calculation is applicable.

Conclusion

A new time integration scheme has been presented for in-plane geometrically exact beam with/without rotational degree of freedom. The results showed an excellent performance of the method in term of accuracy and stability.

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