ON INVARIANTS OF THE ELASTICITY TENSOR FOR ORTHOTROPIC MATERIALS

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Summary The fourth order elasticity tensor for linear elastic material of orthotropic symmetry is studied. The set of nine invariants of this tensor is proposed allowing uniquely specify the form of the tensor in any basis. The set of invariants is composed of six Kelvin moduli and three invariants of the selected orthogonal projectors. In order to derive the presented results spectral theorem for elasticity tensor as well as harmonic decomposition for the orthogonal projectors of the orthotropic stiffness tensor are used.

INTRODUCTION

Materials of new generation already during their projecting are supposed to be characterized by some prescribed properties. These properties can refer to their strength, way of deformation, resistance to external forces. Therefore, it is demanded to know appropriate criteria for the assessment of these properties. It requires profound theoretical knowledge and working-out new technological solutions. At the same time in many cases computer Finite Element Method simulations are conducted to determine elastic constants for the considered biological materials such as bones and other tissues or the current properties of damaged engineering materials. It is necessary to formulate the criteria for identification of symmetry type of the material.

The vast number of composites materials and textured metals exhibit at least orthotropic symmetry. Also for biological tissues as well as in damage analysis orthotropic approximation of elastic properties is considered to be admissible.

The form of stiffness tensor for orthotropic material and its spectral analysis is widely known [5] [2], [6]. As it was shown, such tensor is specified by nine independent parameters: six Kelvin moduli and three stiffness distributors defining three out of six eigen-states. However, definitions of three stiffness distributors with use of invariants of eigen-states was not yet proposed. Below, we derived such definitions that allow to specify uniquely elasticity tensor for any orthotropic material.

SPECTRAL THEOREM FOR ORTHOTROPIC SYMMETRY

Any orthotropic linear elastic material can be described by the fourth order tensor that in general has six mutually different positive eigenvalues called Kelvin moduli λ_K and six orthonormal eigen-states $\mathbf{w}_K$ among which three are pure shears with common direction of shearing for every pair among them. Let us number them subsequently by $I = 4, 5, 6$. Moreover, three direction of shearing obtained by these definition are mutually orthogonal and form the orthonormal basis of main orthotropy directions [1]. Remaining three eigen-states are coaxial and their eigenvectors are align along orthotropy directions. We may derive diads of orthotropy directions $\mathbf{m}_k \otimes \mathbf{m}_k$ where $k = 1, 2, 3$ (no summation) in following way

$$\mathbf{m}_1 \otimes \mathbf{m}_1 = \mathbf{I} - 2\mathbf{w}_4^2, \quad \mathbf{m}_2 \otimes \mathbf{m}_2 = \mathbf{I} - 2\mathbf{w}_5^2, \quad \mathbf{m}_3 \otimes \mathbf{m}_3 = \mathbf{I} - 2\mathbf{w}_6^2, \tag{1}$$

It should be noted that as far as ordering rule for $\mathbf{w}_I$, $i = 4, 5, 6$ has not been yet uniquely specified also *numbering* of orthotropy directions has not been established at this stage. We will introduce such rule later.

It was shown by use of spectral theorem that orthotropy can be described by 9 values independent on the basis selection that is six Kelvin moduli and three stiffness distributors specifying orientation of three orthonormal coaxial eigen-states in three dimensional space of the second order tensors with common eigenvectors. These three stiffness distributors may be defined with use of the polynomial invariants of considered eigen-states.

Six Kelvin moduli are derived from the characteristic equations for $\mathbf{C}$ where coefficients can be expressed by six invariants of powers of stiffness tensors $\mathbf{C}$, that is

$$\mathrm{Tr}\mathbf{C}, \quad \mathrm{Tr}\mathbf{C}^2, \quad \mathrm{Tr}\mathbf{C}^3, \quad \mathrm{Tr}\mathbf{C}^4, \quad \mathrm{Tr}\mathbf{C}^5, \quad \mathrm{Det}\mathbf{C} \tag{2}$$

where for $\mathbf{A}$ being the fourth order tensor with components A_{ijkl}, operation $\mathrm{Tr}\mathbf{A}$ gives A_{ijij}, $\mathrm{Det}\mathbf{A}$ denotes determinant of matrix with components $A_{KL} = \mathbf{a}_K : \mathbf{A} : \mathbf{a}_L$ where second order tensors $\mathbf{a}_K$ constitute orthonormal basis in the space of second order symmetric tensors.

STIFFNESS DISTRIBUTORS

Definition

First, let us order coaxial eigen-states according to the following rule

$$K > L \Longleftrightarrow (\mathrm{tr}\mathbf{w}_K)^2 > (\mathrm{tr}\mathbf{w}_L)^2 \quad K, L = 1, 2, 3 \tag{3}$$

If above squeres of traces are equal to each other we number the eigen-states by deacresing values of $(\det \mathbf{w}_K)^2$.

It can be prooved that the following set of three invariants allow uniquely specify diads of coaxial eigen-states called orthogonal projectors

$$\eta_1 = \mathrm{tr}\mathbf{h}_1^2, \quad \eta_2 = \frac{\det \mathbf{h}_1}{(\mathrm{tr}\mathbf{w}_1)^3} \tag{4}$$

where by $\mathbf{h}_K$ we denote deviator of $\mathbf{w}_K$. These two distributors one may expressed by following relations

$$\eta_1 = 1 - \frac{1}{3}(\mathrm{tr}\mathbf{w}_1)^2, \quad \eta_2 = \frac{1}{(\mathrm{tr}\mathbf{w}_1)^3}\left[\det\mathbf{w}_1 - \frac{5}{54}(\mathrm{tr}\mathbf{w}_1)^3 + \frac{1}{6}\mathrm{tr}\mathbf{w}_1\right] \tag{5}$$

Third distributor is defined in the general case as a following function of common invariants of $\mathbf{w}_1$ i $\mathbf{w}_2$

$$\eta_3 = \frac{\mathrm{tr}(\mathbf{w_1^2 w_2})}{\mathrm{tr}\mathbf{w}_2}. \tag{6}$$

The above definition must be modified in the case when $\eta_1 = 0$ or two eigenvalues of $\mathbf{w}_1$ are equal to each other correspondingly in the form

$$\eta_3^* = (\det\mathbf{h}_2)^2, \quad \eta_3^{**} = \frac{\det\mathbf{h}_2}{(\mathrm{tr}\mathbf{w}_2)^3}. \tag{7}$$

First two distributors allow to derive two sets of eigenvalues corresponding to $\mathbf{w}_1$ and $-\mathbf{w}_1$. According to the decreasing absolute value of these eigenvalues orthotropy axes are numbered. Let us underlined that the eigenstate $\mathbf{w}_3$ is specified within its sign by orthonormality conditions.

Let us note that $\eta_1 = 0$ implies immediately $\eta_2 = 0$ and specifies $\mathbf{w}_1$ as the normalized hydrostatic state $\frac{1}{\sqrt{3}}\mathbf{I}$. Remaining eigenstates are then deviatoric and orthotropy axes are numbered in view of decreasing absolute value of eigenvalues of $\pm\mathbf{w}_2$.

If two eigenvalues of $\mathbf{w}_1$ have the same absolute value then corresponding orthotropy axes are numbered in view of appropriate rule applied for $\pm\mathbf{w}_2$.

Finally, let us show that the invariants of eigenstates specified by equations (4) and (6) are also the invariants of the corresponding orthogonal projectors $\mathbf{P}_K = \mathbf{w}_K \otimes \mathbf{w}_K$ ($K = 1, 2, 3$, no summation). Let us apply for them invariant harmonic decomposition [3]: $\mathbf{P}_K \Leftrightarrow \{h_P^{(K)}, \quad h_D^{(K)}, \quad \mathbf{n}^{(K)}, \quad \mathbf{m}^{(K)}, \quad \mathbf{D}^{(K)}\}$. We find that

$$\eta_1 = 1 - h_P^{(1)}, \quad \eta_2 = \frac{\det\mathbf{m}^{(1)}}{(3h_P^{(1)})^3}, \quad \eta_3 = \frac{1}{3h_P^{(2)}}\left(\mathrm{tr}(\mathbf{m}^{(2)}\mathbf{n}^{(1)}) + h_P^{(2)}\right) \tag{8}$$

where $h_P^{(K)} = \frac{1}{3}\mathbf{I} : \mathbf{P}_K : \mathbf{I}$ and second order tensors $\mathbf{m}^{(K)}$ and $\mathbf{n}^{(K)}$ are deviators of the following tensors

$$\widetilde{\mathbf{m}}^{(K)} = \mathbf{P}_K : \mathbf{I}, \quad \widetilde{\mathbf{n}}^{(K)} = \mathrm{tr}_{2,5}\mathrm{tr}_{3,6}(\mathbf{P}_K \otimes \mathbf{I}). \tag{9}$$

Reduction for special cases of orthotropy

Let us now shortly discuss reduction of invariants for special cases of orthotropy such as volume-isotropic material, transversally isotropic material and material of cubic symmetry.

In the case of the volume-isotropic material the stiffness tensor is specified by six Kelvin moduli and one distributor $\eta_3^\star$. In this case $\mathbf{w}_I$ is hydrostatic and $\eta_1 = \eta_2 = 0$.

In the case of transversally isotropic material we have to do only with four essentially different Kelvin moduli. More details concerning spectral decomposition one may find in [4]. Furthermore it can be shown that out of three distributors only one is necessary to establish $\mathbf{w}_1$ and $\mathbf{w}_2$ that is η_2, so only five invariants uniquely define stiffness tensor for any transversally isotropic material.

For the material of cubic symmetry three Kelvin moduli are essentially different. For three pure shears $\mathbf{w}_K$, $K = 4, 5, 6$ we have one Kelvin modulus. For $\mathbf{w}_2$ and $\mathbf{w}_3$ we have the second Kelvin modulus and the third Kelvin modulus corresponds to $\mathbf{w}_1$ where $\eta_1 = \eta_2 = 0$. Distributor η_3 is then insignificant.

CONCLUSIONS

Nine invariants of the orthotropic elasticity tensor has been derived that allow uniquely specify this tensor independently on the basis selection. It allows to investigate apparent orthotropy of the material as well as to compare two orthotropic materials. It seems useful in projecting materials as well as in identifying material orthotropy for biological and damaged material. The proposed approach that is based on the spectral theorem can be applied for the stiffness tensors of lower symmetry.

References

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