

# Local existence and uniqueness of solutions of the weak electrolyte model describing electro-convection in nematic liquid crystals

WOLF-PATRICK DÜLL, GUIDO SCHNEIDER,  
Institut für Analysis, Dynamik und Modellierung,  
Universität Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart

HANNES UECKER  
Institut für Mathematik, Universität Oldenburg, D-26111 Oldenburg

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## Abstract

We show the local existence and uniqueness of solutions of the most advanced model for the description of electro-convection in nematic liquid crystals, namely the weak electrolyte model (WEM), which is a mixture of quasilinear parabolic equations and balance laws. We do this by bringing the WEM in a form where a standard iteration scheme can be applied.

## 1 Introduction

Electro-convection in nematic liquid crystals is a paradigm for pattern formation in non-isotropic media. Experimentally, a thin layer of such a material is contained in between two spatially extended electrode plates. When an alternating current is applied to the electrodes an electro-hydrodynamic instability occurs if the voltage is above a certain threshold. The trivial spatially homogeneous solution becomes unstable and bifurcates into non-trivial pattern [Cha77, PB98].

There are essentially two models for the mathematical description of electro-convection in nematic liquid crystals. These are the standard model ([ZK85] and the references therein) and the weak electrolyte model (WEM). The latter has been introduced by Kramer and Treiber in [Tre96, TK98] to overcome a number of insufficiencies of the standard model. In particular,

the WEM has a number of pattern forming instabilities which agree well with experimental results [Tre96], see also [SU07].

The local existence and uniqueness of solutions of the WEM is a nontrivial task since the governing equations are a relatively complicated mixture of quasilinear parabolic equations and balance laws. Therefore, in [SU07] a regularized semilinear parabolic WEM was considered. Here, we solve the problem for the original WEM by combining optimal regularity theory for quasilinear parabolic systems and Kato's method for quasilinear hyperbolic systems. As a consequence, the justification results for the approximation of a regularized WEM by Ginzburg–Landau equations from [SU07] also hold for the original WEM.

The following presentation and non-dimensionalization of the WEM follows [DO04, SU07]. We consider a layer of nematic liquid crystals in between two infinitely extended horizontal plates of height  $\pi$ , i.e. in the following  $(x, y, z) \in \Omega = \mathbb{R}^2 \times (0, \pi)$ . In the WEM the average molecular axis of the nematic liquid crystals is described locally by a director field  $n$  of unit vectors. The Leslie-Erickson equations for  $n$  and the generalized Navier-Stokes equations for the fluid velocity  $v$  and the pressure  $p$  in the presence of an electric field  $E$  are given by

$$(\partial_t + v \cdot \nabla)n = \omega \times n + \delta^\perp(\lambda An - h), \quad (1)$$

$$P_2(\partial_t + v \cdot \nabla)v = -\nabla p - \nabla \cdot (T^{visc} + \Pi) + \pi^2 \rho E, \quad (2)$$

$$\nabla \cdot v = 0, \quad (3)$$

for  $(x, y, z) \in \Omega$ . These equations turn out to be a quasilinear parabolic system. The meaning of the quantities is as follows. The vorticity is

$$\omega = \frac{1}{2}(\nabla \times v), \quad (4)$$

and the molecular field  $h$  is given by

$$h = 2 \left( \frac{\partial f}{\partial n} - \nabla \cdot \frac{\partial f}{\partial \nabla n} \right) - \varepsilon_a \pi^2 (n \cdot E) E \quad (5)$$

where

$$2f = (\nabla \cdot n)^2 + K_2 [n \times (\nabla \times n)]^2 + K_3 [n \cdot (\nabla \times n)]^2, \quad (6)$$

is the elastic energy density describing splay, twist ( $K_2$ ), and bend ( $K_3$ ) deformations, and where

$$\left( \frac{\partial f}{\partial \nabla n} \right)_{ij} := \frac{\partial f}{\partial n_{i,j}} \text{ with } n_{i,j} = \partial_{x_j} n_i.$$

We refer to [DO04] for a physical interpretation of the constants  $P_2$ ,  $\lambda$ ,  $K_2$ ,  $K_3$ , and  $\varepsilon_a$ . The electric field  $E = E(x, y, z, t) \in \mathbb{R}^3$  is considered to be quasistationary, i.e.  $\text{rot } E = 0$ . It is then split into an external forcing and some potential part, i.e.

$$E = E_p(t)(0, 0, 1)^T - \nabla \phi, \quad (7)$$

where usually in the experiments

$$E_p(t) = E_0 \cos \omega_0 t \quad (8)$$

with an  $E_0 > 0$  and  $\omega_0 > 0$ .

The tensors  $A$ , and  $T^{visc}$  are, respectively, the shear flow tensor

$$A_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i), \quad (9)$$

and the viscous stress tensor

$$\begin{aligned} -T_{ij}^{visc} = & \sum_{k=1}^3 \left( \alpha_1 n_i n_j n_k \left( \sum_{l=1}^3 n_l A_{kl} \right) + \alpha_5 n_j n_k A_{ki} + \alpha_6 n_i n_k A_{kj} \right) \\ & + \alpha_2 n_j m_i + \alpha_3 n_i m_j + \alpha_4 A_{ij} \end{aligned} \quad (10)$$

where

$$m = \delta^\perp (\lambda A n - h) \quad (11)$$

and with constant coefficients  $\alpha_1, \dots, \alpha_6$ . The tensor  $\Pi$  with

$$\Pi_{ij} = \sum_{k=1}^3 \frac{\partial f}{\partial n_{k,j}} n_{k,i} \quad (12)$$

is called the nonlinear Ericksen stress tensor. The projection tensor

$$\delta_{ij}^\perp = \delta_{ij} - n_i n_j \quad (13)$$

in (1) guarantees that  $|n| = 1$  as long as the solution exists. See [Tre96].

The second part of the WEM comes from the quasi-static Maxwell equations. In the WEM [Tre96, TK98] there are two species of oppositely charged mobile ions. Under the assumption of a linear recombination and zero diffusivity, the WEM consists of (1)-(3) and two balance equations for the charge density  $\rho$  and the deviation  $\sigma$  of the local conductivity from 1, namely

$$P_1(\partial_t + v \cdot \nabla)\rho = -\nabla \cdot (\mu E \sigma), \quad (14)$$

$$(\partial_t + v \cdot \nabla)\sigma = -\alpha^2 \pi^2 \nabla \cdot (\mu E \rho) - \frac{r}{2} (2\sigma + \sigma^2 - P_1 \pi^2 \alpha \rho^2). \quad (15)$$

Finally the system is closed by Poisson's law

$$\rho = \nabla \cdot (\varepsilon E). \quad (16)$$

The dielectric tensor  $\varepsilon$  and the conductivity tensor  $\mu$  are given by

$$\varepsilon_{ij} = \delta_{ij} + \varepsilon_a n_i n_j \quad \text{and} \quad \mu_{ij} = \delta_{ij} + \sigma_a n_i n_j.$$

Similar to  $P_2$ , the parameter  $P_1$  is a Prandtl–type time scale ratio. For a physical interpretation of the constants  $P_1$ ,  $\sigma_a$ ,  $\alpha$ , and  $r$  we again refer to [DO04]. The WEM will be considered with the boundary conditions

$$\partial_z n_2 = n_3 = \partial_z v_1 = \partial_z v_2 = v_3 = \phi = 0 \quad (17)$$

at  $z = 0, \pi$ .

Using Poisson’s law  $E$ , resp.  $\phi$ , can be expressed in terms of  $\rho$  and so (1)-(3) and (14)-(15) can be rewritten as a system of dynamical equations for  $V = (n_2, n_3, v_1, v_2, v_3, \rho, \sigma)$ . Thus, (1)-(3), (14), (15) is abbreviated as

$$\partial_t V = M(t)V + \tilde{N}(t, V) \quad (18)$$

where  $M(t)V$  stands for the linear and  $\tilde{N}(t, V)$  for the nonlinear terms with respect to  $V$ .

The WEM equations are invariant under arbitrary translations in  $x$  and  $y$  and under the reflections

$$\mathcal{S}_1 : (x, n_2, n_3, v_1) \rightarrow -(x, n_2, n_3, v_1), \quad (19)$$

$$\mathcal{S}_2 : (y, n_2, v_2) \rightarrow -(y, n_2, v_2), \quad (20)$$

$$\mathcal{S}_3 : (z, n_3, v_3, \phi) \rightarrow -(z, n_3, v_3, \phi). \quad (21)$$

The local existence and uniqueness of solutions is nontrivial due to the relatively complicated mixture of quasilinear parabolic equations and balance laws. To our knowledge no local existence and uniqueness result is documented so far in the literature. Thus, here we make a first step and prove the local existence and uniqueness for initial conditions in a neighborhood of the trivial solution  $V = 0$ . We do this by bringing the WEM in a form where a standard iteration scheme can be applied. In order to do so the regularity of the components of  $V$  has to be chosen properly, i.e. for instance  $n$  has to be chosen one time more regular than  $v$ .

**Notation.** The Sobolev space  $H^m(\Omega)$  is the space of  $m$ -times weakly differentiable functions  $\Omega \rightarrow \mathbb{R}$  equipped with the norm

$$\|u\|_{H^m(\Omega)} = \sum_{|j|=0}^m \|\partial_x^j u\|_{L^2(\Omega)} \quad \text{with} \quad \|u\|_{L^2(\Omega)}^2 = \int_{\Omega} |u(x)|^2 dx.$$

We shall also need fractional order Sobolev spaces and interpolation spaces. The symmetry  $\mathcal{S}_3$  allows to extend the WEM periodically into the bounded direction and to expand the variables in Fourier series with respect to  $z$ , i.e we write for instance

$$n_2(x, y, z) = \int \int \sum_{k_3 \in \mathbb{Z}} \hat{n}_2(k_1, k_2, k_3) e^{ik_1 x + ik_2 y + ik_3 z} dk_1 dk_2.$$

and similar for  $n_3, \dots, \sigma$ . Here the extensions  $n_2, v_1, v_2, \sigma, \rho$  will be even and  $n_3, v_3, \phi$  odd w.r.t. the third variable. We define

$$\|\hat{u}\|_{L^2(s)}^2 = \int \int \sum_{k_3 \in \mathbb{Z}} |\hat{u}(k_1, k_2, k_3)|^2 (1 + |k_1|^2 + |k_2|^2 + |k_3|^2)^s dk_1 dk_2.$$

Due to Parseval's identity there is a one to one relation and norm equivalence between physical and Fourier space, i.e., for all  $m \in \mathbb{N}$  there is a constant  $C > 0$  such that

$$C^{-1} \|\hat{u}\|_{l^2(m)} \leq \|u\|_{H^m} \leq C \|\hat{u}\|_{l^2(m)}. \quad (22)$$

In the following we use  $H^m$  as abbreviation for  $H^m(\mathbb{R}^2 \times [0, 2\pi])$  with periodic boundary conditions in the third variable.

Using (22) we also define  $H^s$  for every  $s \geq 0$  as functions in  $L^2$  whose Fourier transform is in  $l^2(s)$ . It is equipped with the norm

$$\|u\|_{H^s} = \|\hat{u}\|_{l^2(s)}.$$

Finally we note that as a consequence of the periodic boundary conditions interpolation spaces ([LM72]) are trivially characterized. For instance for  $\Delta : H^{s+2} \rightarrow H^s$  we have

$$\begin{aligned} [H^s, H^{s+2}]_\theta &= \{u : (1 + (-\Delta)^\theta)u \in H^s\} = \{u : (1 + |k|^{2\theta})\hat{u} \in l^2(s)\} \\ &= \{u : \hat{u} \in l^2(s + 2\theta)\} = H^{s+2\theta}. \end{aligned}$$

**Theorem 1.1.** *Let  $\theta \in (0, 1)$  and  $m \in \mathbb{N}$ . There exists a  $C_1 > 0$  such that for all initial conditions*

$$V_0 = (n_2, n_3, v_1, v_2, v_3, \sigma, \rho)|_{t=0} \in [H^{m+3+2\theta}]^2 \times [H^{m+2+2\theta}]^3 \times [H^{m+2}]^2$$

*with  $\nabla \cdot v = 0$  and  $\|V_0\|_{[H^{m+3+2\theta}]^2 \times [H^{m+2+2\theta}]^3 \times [H^{m+2}]^2} \leq C_1$  there exists a  $T_0 > 0$  such that (18) has a unique mild solution*

$$V \in C([0, T_0], [H^{m+3}]^2 \times [H^{m+2}]^3 \times [H^{m+2}]^2) \cap C^1([0, T_0], [H^{m+1}]^2 \times [H^m]^3 \times [H^{m+1}]^2)$$

*with  $V|_{t=0} = V_0$ .*

The additional regularity for  $(n_2, n_3)|_{t=0}$  and  $v|_{t=0}$  described by  $\theta$  is needed to fulfill some compatibility conditions at  $t = 0$  to apply maximal regularity to the quasilinear parabolic subsystem for  $(n_2, n_3)$  and  $v$ . As consequence,  $(n_1, n_2)$  and  $v$  enjoy further regularity properties, e.g., they are Hölder continuous in time with values in  $[H^{m+3}]^2 \times [H^{m+2}]^3$ , while further regularity for the charge density  $\rho$  and the local conductivity  $\sigma$  are unclear, and we restrict to the simple formulation of Theorem 1.1.

The plan of the proof is as follows. In §2.1 we explain that the WEM is an evolutionary system for the variables collected in  $V$ , i.e. we eliminate the pressure term  $\nabla p$  and express  $E$  in terms of  $V$ . In §2.2–§2.3 we extract the leading terms in the  $(n, v)$ -part, rewrite the balance laws as quasilinear hyperbolic systems in the sense of [Kat75], prove a number of a priori estimates and give the estimates for the remaining quasilinear and semilinear terms. Then in §3 we formulate an iteration scheme and prove the convergence of the sequence constructed by the iteration scheme.

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## 2 The structure of the WEM

### 2.1 The WEM as a dynamical system

To write the WEM as an evolutionary system in  $V = (n_2, n_3, v_1, v_2, v_3, \rho, \sigma)$  we proceed as in [SU07], where also the three Lemmas below are proved (Lemmas A.2, A.3, A.4 in [SU07]). Essentially the proofs follow by explicit calculation from the above Fourier representation and (22). First we need to express  $E$  in terms of  $V$ . Therefore we have to solve

$$\rho = \sum_{k=1}^3 \partial_k (\varepsilon_{km} E_m) = \sum_{k=1}^3 \sum_{m=1}^3 \partial_k [(\delta_{km} + \varepsilon_a n_k n_m) (E_0 \cos(\omega_0 t) \delta_{m3} - \partial_m \phi)]$$

with respect to  $\phi$  under the boundary conditions  $\phi|_{z=0,\pi} = 0$ . We find

$$(M + G)\phi = F(n, \rho, E_0)$$

where

$$F(n, \rho, E_0) = -\rho + \cos(\omega_0 t) \sum_{k=1}^3 \sum_{m=1}^3 \partial_k ((\delta_{km} + \varepsilon_a n_k n_m) E_0 \delta_{m3}),$$

$$M\phi = \Delta\phi + \varepsilon_a \partial_1^2 \phi, \quad G\phi = \varepsilon_a \sum_{k=1}^3 \sum_{m=1}^3 \partial_k (n_k n_m \partial_m \phi) - \varepsilon_a \partial_1^2 \phi.$$

**Lemma 2.1.** *The linear operator  $M^{-1}$  is bounded from  $H^s$  into  $\{\phi \in H^{s+2} : \phi = 0 \text{ at } z = 0, \pi\}$ .*

Hence the electric potential  $\phi$  satisfies  $(1 + GM^{-1})M\phi = F(n, \rho, E_0)$ , where  $GM^{-1}$  is small for  $\tilde{n} = n - (1, 0, 0)^T$  small. By using Neumann's series we formally obtain

$$\phi = M^{-1}(1 + GM^{-1})^{-1}F(n, \rho, E_0). \quad (23)$$

**Lemma 2.2.** *Let  $s \geq 2$ , and let  $\|V\|_{H^s}$  be sufficiently small. Then the operator  $M^{-1}(1 + GM^{-1})^{-1}$  is bounded from  $H^s$  into  $H^{s+2}$ .*

Next we focus on the hydrodynamic part of (18) and define the projection  $Q$  on the divergence free vector fields by  $w = Qf$ , where  $w$  solves

$$w - \nabla p = f, \quad \nabla \cdot w = 0, \quad \partial_z w_1 = \partial_z w_2 = w_3 = 0 \text{ at } z = 0, \pi. \quad (24)$$

**Lemma 2.3.** *The projection  $Q$  is continuous from  $[H^m]^3$  into  $\{v \in [H^m]^3 : \nabla \cdot v = 0\}$ .*

Since  $n_1^2 + n_2^2 + n_3^2 = 1$  for our purposes it is sufficient to consider  $n_2$  and  $n_3$ . Hence we finally consider

$$\partial_t n_2 = \langle e_2, -(v \cdot \nabla)n + \omega \times n + \delta^\perp(\lambda A n - h) \rangle, \quad (25)$$

$$\partial_t n_3 = \langle e_3, -(v \cdot \nabla)n + \omega \times n + \delta^\perp(\lambda A n - h) \rangle, \quad (26)$$

$$\partial_t v = P_2^{-1}Q(-(v \cdot \nabla)v - \nabla \cdot (T^{visc} + \Pi) + \pi^2 \rho E), \quad (27)$$

$$\partial_t \rho = -v \cdot \nabla \rho - P_1^{-1} \nabla \cdot (\mu E \sigma), \quad (28)$$

$$\partial_t \sigma = -v \cdot \nabla \sigma - \alpha^2 \pi^2 \nabla \cdot (\mu E \rho) - \frac{r}{2} (2\sigma + \sigma^2 - P_1 \pi^2 \alpha \rho^2), \quad (29)$$

under the boundary conditions (17).

## 2.2 The quasilinear parabolic part

We start with the computation of the highest order derivative part in the  $v$  and  $n$  parts of the system. Here and in the following  $\star$  stands for terms with less derivatives or terms in which the highest derivative occurs nonlinearly.

We introduce the derivation  $\tilde{n}$  of the director from the planar alignment by

$$n = (1 + \tilde{n}_1, \tilde{n}_2, \tilde{n}_3)^T .$$

From  $(1 + \tilde{n}_1)^2 + \tilde{n}_2^2 + \tilde{n}_3^2 = 1$  we find  $\tilde{n}_1 = \mathcal{O}(\tilde{n}_2^2 + \tilde{n}_3^2)$ . Therefore

$$\begin{aligned} (\nabla \cdot n)^2 &= (\partial_{x_2} \tilde{n}_2 + \partial_{x_3} \tilde{n}_3)^2 + \star, & \nabla \times n &= (\partial_{x_2} \tilde{n}_3 - \partial_{x_3} \tilde{n}_2, -\partial_{x_1} \tilde{n}_3, \partial_{x_1} \tilde{n}_2)^T + \star, \\ n \cdot (\nabla \times n) &= (\partial_{x_2} \tilde{n}_3 - \partial_{x_3} \tilde{n}_2) + \star & \text{and} & \quad n \times (\nabla \times n) = (0, -\partial_{x_1} \tilde{n}_2, -\partial_{x_1} \tilde{n}_3)^T + \star. \end{aligned}$$

Thus

$$2f = (\partial_{x_2} \tilde{n}_2 + \partial_{x_3} \tilde{n}_3)^2 + K_2((\partial_{x_1} \tilde{n}_2)^2 + (\partial_{x_1} \tilde{n}_3)^2) + K_3(\partial_{x_2} \tilde{n}_3 - \partial_{x_3} \tilde{n}_2)^2 + \star .$$

Moreover

$$\delta^\perp = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \star,$$

and therefore to calculate  $h$  in (5) we only need to calculate rows 2 and 3 of  $\frac{\partial f}{\partial(\nabla n)}$ , i.e.,

$$2 \frac{\partial f}{\partial(\nabla n)} = \begin{pmatrix} \star & \star & \star \\ 2K_2 \partial_{x_1} n_2 & 2(\partial_{x_2} n_2 + \partial_{x_3} n_3) & -2K_3(\partial_{x_2} n_3 - \partial_{x_3} n_2) \\ 2K_2 \partial_{x_1} n_3 & 2K_3(\partial_{x_2} n_3 - \partial_{x_3} n_2) & 2(\partial_{x_2} n_2 + \partial_{x_3} n_3) \end{pmatrix} + \star .$$

Thus,

$$2\nabla \cdot \frac{\partial f}{\partial(\nabla n)} = \begin{pmatrix} \star \\ 2K_2 \partial_{x_1}^2 \tilde{n}_2 + 2\partial_{x_2}^2 \tilde{n}_2 + 2\partial_{x_3} \partial_{x_2} \tilde{n}_3 + 2K_3 \partial_{x_3}^2 \tilde{n}_2 - 2K_3 \partial_{x_2} \partial_{x_3} \tilde{n}_3 \\ 2K_2 \partial_{x_1}^2 \tilde{n}_3 + 2K_3 \partial_{x_2}^2 \tilde{n}_3 - 2K_3 \partial_{x_2} \partial_{x_3} \tilde{n}_2 + 2\partial_{x_2} \partial_{x_3} \tilde{n}_2 + 2\partial_{x_3}^2 \tilde{n}_3 \end{pmatrix} + \star .$$

Using this expansion and  $\langle e_j, \omega \times n + \delta^\perp(\lambda A n) \rangle = \star$  we find for the equations for  $\partial_t \tilde{n}_2$  and  $\partial_t \tilde{n}_3$  in Fourier space that

$$\partial_t \begin{pmatrix} \tilde{n}_2 \\ \tilde{n}_3 \end{pmatrix} = -2 \begin{pmatrix} K_2 k_1^2 + k_2^2 + K_3 k_3^2 & k_2 k_3 - K_3 k_2 k_3 \\ k_2 k_3 - K_3 k_2 k_3 & K_2 k_1^2 + K_3 k_2^2 + k_3^2 \end{pmatrix} \begin{pmatrix} \tilde{n}_2 \\ \tilde{n}_3 \end{pmatrix} + \star .$$

This matrix turns out to be negative definite if  $K_2 > 0$  and  $K_3 > 0$ .

Next we come to the equation for  $v$ . We proceed as above and compute the terms with highest derivatives which are linear. Since in the  $\partial_t n$ -equation no  $v$  terms played any role

we have some lower triangular block structure and so in the  $\partial_t v$  equation it is sufficient to consider the linear terms with highest derivative of  $v$ . All the rest will be denoted as above with  $\star$ . Hence, it is sufficient to analyse  $T^{visc}$  and in  $T^{visc}$  the  $A_{kl}$  terms. We find

$$-T_{ij}^{visc} = (\alpha_1 + \alpha_5 + \alpha_6)\delta_{i1}\delta_{j1}A_{ij} + \alpha_4 A_{ij} + \star$$

so that

$$\nabla \cdot T^{visc} = \alpha_4 \Delta v + (\alpha_1 + \alpha_5 + \alpha_6) \begin{pmatrix} \partial_1^2 v_1 \\ 0 \\ 0 \end{pmatrix} + \star$$

where we used  $\nabla \cdot v = 0$ . Therefore the  $(n, v)$ -part is of the form

$$\begin{aligned} \partial_t \begin{pmatrix} \tilde{n}_2 \\ \tilde{n}_3 \end{pmatrix} &= L_n \begin{pmatrix} \tilde{n}_2 \\ \tilde{n}_3 \end{pmatrix} + G_n, \\ \partial_t v &= L_v v + G_v, \end{aligned}$$

where  $L_n$  is defined by its symbol in Fourier space

$$-\hat{L}_n = 2 \begin{pmatrix} K_2 k_1^2 + k_2^2 + K_3 k_3^2 & k_2 k_3 - K_3 k_2 k_3 \\ k_2 k_3 - K_3 k_2 k_3 & K_2 k_1^2 + K_3 k_2^2 + k_3^2 \end{pmatrix},$$

where

$$L_v v = P_2^{-1} Q \left[ \alpha_4 \Delta v + (\alpha_1 + \alpha_5 + \alpha_6) \begin{pmatrix} \partial_1^2 v_1 \\ 0 \\ 0 \end{pmatrix} \right], \quad (30)$$

and where  $G_n$  and  $G_v$  stand for the remaining terms.  $L_n$  and  $L_v$  generate analytic semigroups which later allow to control  $G_n$  and  $G_v$  by optimal regularity results.

**Lemma 2.4.** *Let  $\theta \geq 0$  and  $m \in \mathbb{N}$ . a) The operator  $L_n : [H^{m+2}]^2 \rightarrow [H^m]^2$  defines an analytic semigroup  $e^{tL_n}$  in  $[H^m]^2$  satisfying*

$$\|e^{tL_n} u\|_{[H^{m+2\theta}]^2} \leq C(1 + t^{-\theta}) \|u\|_{[H^m]^2}.$$

*b) The operator  $L_v : Q[[H^{m+2}]^3] \rightarrow Q[[H^m]^3]$  defines an analytic semigroup  $e^{tL_v}$  in  $Q[[H^m]^3]$  satisfying*

$$\|e^{L_v t} u\|_{[H^{m+2\theta}]^3} \leq C(1 + t^{-\theta}) \|u\|_{[H^m]^3}.$$

**Proof.** The result follows from the fact that under the chosen boundary conditions the problem can be extended periodically into the bounded  $z$ -direction such that the estimate is a consequence of the representations of  $L_n$  and  $L_v$  in Fourier space and (22). Since  $\|e^{t\hat{L}_n}\|_{\mathbb{R}^{2 \times 2}} \leq e^{-\tilde{C}|k|^2}$  for a  $\tilde{C} \geq 0$ , we have that

$$\begin{aligned} \|e^{tL_n} u\|_{[H^{m+2\theta}]^2} &\leq C \|e^{t\hat{L}_n} \hat{u}\|_{[l^2(m+2\theta)]^2} \leq C \|e^{-\tilde{C}|k|^2} |\hat{u}|\|_{l^2(m+2\theta)} \\ &\leq C \sup_k |e^{-\tilde{C}|k|^2} (1+k^2)^\theta| \|\hat{u}\|_{(l^2(m))} \leq C(1+t^{-\theta}) \|\hat{u}\|_{(l^2(m))} \leq C(1+t^{-\theta}) \|u\|_{(H^m)^2}. \end{aligned}$$



Similarly the estimate for  $e^{tL_v}$  follows.  $\square$

In order to apply an iteration scheme to solve the quasilinear problem for  $(n_2, n_3)$  and  $v$  coupled to the hyperbolic problem for  $(\rho, \sigma)$  we shall need maximal regularity results. Therefore we first study the linear inhomogeneous problems

$$\partial_t(n_2, n_3) = L_n(n_2, n_3) + f_n \quad (31)$$

and

$$\partial_t v = L_v v + f_v. \quad (32)$$

Given, e.g.,  $f_v \in C^{0,\theta}([0, T_0], X)$  with  $0 < \theta < 1$ , where  $X$  is some Banach space, maximal regularity means that  $\partial_t v$  and  $L_v v$  enjoy the same regularity as  $f_v$ . Additional to the natural assumption that  $v_0 = v|_{t=0} \in D(L_v)$ , the crucial point to obtain such maximal regularity results are compatibility conditions at  $t = 0$ , namely

$$L_v v_0 + f_v(0) \in D_{L_v}(\theta, \infty). \quad (33)$$

Here the real interpolation space  $D_{L_v}(\theta, \infty)$  is the set of all  $v \in X$  such that  $t^{1-\theta} \|L_v e^{tL_v} v\|_X$  is bounded as  $t \rightarrow 0$ , see, e.g., [Lun95]. From Lemma 2.4a) or the fact that we have periodic boundary conditions in  $z$ -direction we see that for  $X = [H^m]^3$  we obtain  $D_{L_v}(\theta, \infty) = [H^{m+2\theta}]^3$ . Since later in the iteration scheme for the nonlinear problem we have  $f_v(t) = G_v(V(t))$  it will be sufficient to require sufficient regularity for the initial data  $V(0)$  of the nonlinear problem. The problem for  $\partial_t(n_2, n_3)$  can be analyzed in the same manner, and for later reference we note the following Lemma.

**Lemma 2.5.** *For all  $\theta \in (0, 1)$ ,  $m \geq 0$  and  $T_0 > 0$  there exists a  $C_2 > 0$  such that the following holds.*

*a) If  $f_n \in C^{0,\theta}([0, T_0], [H^{m+1}]^2)$  and  $L_n(n_2, n_3)|_{t=0} + f_n(0) \in [H^{m+3+2\theta}]^2$ , then there exists a unique solution  $(n_1, n_2) \in C^{0,\theta}([0, T_0], [H^{m+3}]^2) \cap C^{1,\theta}([0, T_0], [H^{m+1}]^2)$  of (31) which is bounded in this space by  $C_2(\|L_n(n_2, n_3)|_{t=0} + f_n(0)\|_{[H^{m+3+2\theta}]^2} + \|f_n\|_{C^{0,\theta}([0, T_0], [H^{m+1}]^2)} + \|(n_2, n_3)|_{t=0}\|_{[H^{m+3+2\theta}]^2})$ .*

*b) If  $f_v \in C^{0,\theta}([0, T_0], Q[[H^m]^3])$  and  $L_v v|_{t=0} + f_v(0) \in [H^{m+2+2\theta}]^3$ , then there exists a unique solution  $v \in C^{0,\theta}([0, T_0], Q[[H^{m+2}]^3]) \cap C^{1,\theta}([0, T_0], Q[[H^m]^3])$  of (32) with norm bounded in this space by  $C_2(\|L_v v|_{t=0} + f_v(0)\|_{[H^{m+2+2\theta}]^3} + \|f_v\|_{C^{0,\theta}([0, T_0], [H^m]^3)} + \|v|_{t=0}\|_{[H^{m+2+2\theta}]^3})$ .*

**Proof.** These are consequences of Lemma 2.4 and optimal regularity theory. See, e.g., [Lun95, Theorem 4.3.1] or [Sin85].  $\square$

## 2.3 The balance laws

The equations (28), (29) for  $\partial_t \varrho$  and  $\partial_t \sigma$  are of different type than (25)–(27). They are balance laws, hence quasilinear hyperbolic and not quasilinear parabolic. Nevertheless there is some damping in the  $(\rho, \sigma)$  part due to the  $-2\sigma$  term in the  $\sigma$  equation.

Again we only concentrate on the terms with highest derivatives, i.e.

$$\partial_t \rho = -v_j \partial_j \rho - P_1^{-1} \partial_j (a_j \sigma) \quad (34)$$

$$\partial_t \sigma = -v_j \partial_j \sigma - \alpha^2 \pi^2 \partial_j (a_j \rho) + G_\sigma . \quad (35)$$

where we used Einstein's sum convention ( $a_i b_i = \sum_i a_i b_i$ ) and the abbreviation

$$a = \mu E,$$

and where  $G_\sigma$  stands for the remaining terms, which are semilinear. Again, we first treat  $G_\sigma$  as an inhomogeneity and assume that  $G_\sigma$  and the coefficients  $v_j$  and  $a_j$  are sufficiently smooth and later relate this to the smoothness of  $V$ .

To apply [Kat75] we need some a priori estimates. For  $s \geq 0$  and with  $\int \cdot = \int \cdot dx$  we find

$$\begin{aligned} \frac{1}{2} \partial_t \int (\partial_k^s \rho)^2 &= - \int (\partial_k^s \rho) \partial_k^s (v_j \partial_j \rho) - \int (\partial_k^s \rho) P_1^{-1} \partial_k^s \partial_j (a_j \sigma) \\ &= - \int \frac{1}{2} \partial_j ((\partial_k^s \rho)^2) v_j + s.t. - \int P_1^{-1} a_j (\partial_k^s \rho) (\partial_k^s \partial_j \sigma) + s.t. \\ &= + \int \frac{1}{2} (\partial_k^s \rho)^2 (\partial_j v_j) + s.t. - \int P_1^{-1} a_j (\partial_k^s \rho) (\partial_k^s \partial_j \sigma) + s.t. \\ &= - \int P_1^{-1} a_j (\partial_k^s \rho) (\partial_k^s \partial_j \sigma) + s.t. , \end{aligned}$$

where  $k = x, y, z$  and s.t. stands here and in the following for semilinear terms, i.e. for terms with  $s$  or less derivatives acting on  $\rho, \sigma$ . Similarly, we find

$$\begin{aligned} \frac{1}{2} \partial_t \int (\partial_k^s \sigma)^2 &= - \int (\partial_k^s \sigma) \partial_k^s (v_j \partial_j \sigma) - \int (\partial_k^s \sigma) \partial_k^s (\alpha^2 \pi^2 \partial_j (a_j \rho)) + \int (\partial_k^s \sigma) (\partial_k^s G_\sigma) \\ &= - \int \alpha^2 \pi^2 a_j (\partial_k^s \sigma) (\partial_k^s \partial_j \rho) + s.t. = \int \alpha^2 \pi^2 a_j (\partial_k^s \partial_j \sigma) (\partial_k^s \rho) + s.t. . \end{aligned}$$

Thus we have

$$\frac{1}{2} \partial_t \left[ \alpha^2 \pi^2 \int (\partial_k^s \rho)^2 + P_1^{-1} \int (\partial_k^s \sigma)^2 \right] = s.t. .$$

Using Gronwall's inequality gives the following result.

**Lemma 2.6.** *Let  $s \geq 1$  and  $T_0 > 0$ . Then for all  $C_1 > 0$  there exists a  $C_2 > 0$  such that the following holds. Let  $(\rho, \sigma)|_{t=0} \in [H^s]^2$ ,  $v \in C([0, T_0], Q[[H^s]^3])$ ,  $a \in C([0, T_0], [H^{s+1}]^3)$  and  $G_\sigma \in C([0, T_0], H^s)$ , with norms bounded in these spaces by  $C_1$ . Then there exists a unique solution*

$$(\rho, \sigma) \in C([0, T_0], [H^s]^2) \cap C^1([0, T_0], [H^{s-1}]^2)$$

*of (34) and (35) with norm bounded in this space by  $C_2 \|(\rho, \sigma)|_{t=0}\|_{[H^s]^2}$ .*

### 3 The nonlinear terms and the iteration scheme

In order to prove Theorem 1.1 we now combine the optimal regularity theory of [Sin85, Lun95] for quasilinear parabolic equations and the existence theory of [Kat74, Kat75] for quasilinear hyperbolic systems. The combination of the two methods is nontrivial since, in contrast to solutions of quasilinear parabolic systems, solutions of quasilinear hyperbolic systems in general are not Hölder–continuous in time, cf. [Kat74, Sec.5.3].

The idea is to find solutions by the iteration scheme

$$\begin{aligned}
\partial_t(n_2, n_3)_{i+1} &= L_n(n_2, n_3)_{i+1} + G_n(V_i), \\
\partial_t v_{i+1} &= L_v v_{i+1} + G_v(V_i), \\
\partial_t \rho_{i+1} &= - \sum_{j=1}^3 (v_j)_i \partial_j \rho_{i+1} - P_1^{-1} \sum_{j=1}^3 \partial_j((a_j)_i \sigma_{i+1}) \\
\partial_t \sigma_{i+1} &= - \sum_{j=1}^3 (v_j)_i \partial_j \sigma_{i+1} - \alpha^2 \pi^2 \sum_{j=1}^3 \partial_j((a_j)_i \rho_{i+1}) + G_\sigma(V_i).
\end{aligned} \tag{36}$$

Thus it remains to choose the space for  $V$  in such a way that given  $V_i$  we have  $G_n(V_i), G_v(V_i)$  fulfill the assumptions of Lemma 2.5 and  $v_i, a_i$  and  $G_\sigma(V_i)$  fulfill the assumptions of Lemma 2.6. Therefore we note the following Lemma, where we add the parameter  $\alpha$  to deal with the compatibility conditions in Lemma 2.5.

**Lemma 3.1.** *For  $m \geq 1$  and  $\theta \in [0, 1)$  the nonlinearity  $G = (G_n, G_v, 0, G_\sigma)$  is locally Lipschitz continuous from  $[H^{m+3+2\theta}]^2 \times [H^{m+2+2\theta}]^3 \times [H^{m+2}]^2$  into  $[H^{m+1+2\theta}]^2 \times [H^{m+2\theta}]^3 \times [H^{m+2}]^2$ .*

**Proof.** We have  $\omega \in H^{m+1+2\theta}$  by (4),  $f \in H^{m+2+2\theta}$  by (6),  $E \in H^{m+3}$  by (7) and Lemma 2.2,  $h \in H^{m+1+2\theta}$  by (5),  $\delta_{ij}^\perp - \delta_{ij} \in H^{m+3+2\theta}$  by (13),  $A \in H^{m+1+2\theta}$  by (9),  $m \in H^{m+1+2\theta}$  by (11),  $T^{visc} \in H^{m+1+2\theta}$  by (10), and  $\Pi \in H^{m+2+2\theta}$  by (12). Therefore the right hand side  $G_n$  of (1) is in  $H^{m+1+2\theta}$  and the right hand side  $G_v$  of (2) is in  $H^{m+2\theta}$ . We have  $\varepsilon_{ij} - \delta_{ij} \in H^{m+3+2\theta}$  and  $\mu_{ij} - \delta_{ij} \in H^{m+3+2\theta}$ . From  $\rho \in H^{m+2}$  we immediately find that the terms collected in  $G_\sigma$  are in  $H^{m+2}$ .  $\square$

**Proof of Theorem 1.1.** To use the iteration scheme (36) we need to satisfy, in each step  $i \mapsto i + 1$ ,

- a) the regularity of the initial data, and the compatibility conditions

$$L_n(n_2, n_3)|_{t=0} + f_n(0) \in [H^{m+3+2\theta}]^2 \text{ and } L_v v|_{t=0} + f_v(0) \in [H^{m+3+2\theta}]^3.$$

- b) the conditions on  $v, a$  and  $G_\sigma$  in Lemma 2.6;

- c) the conditions  $G_n \in C^{0,\theta}([0, T_0], [H^{m+1}]^2)$  and  $G_v \in C^{0,\theta}([0, T_0], Q[[H^{m+1}]^3])$  in Lemma 2.5.

Since the initial data are always the same, a) follows from Lemma 3.1 with  $\theta > 0$  chosen in Theorem 1.1.

It is clear that Lemma 3.1 also holds for functions continuous resp. Hölder continuous in time with values in the respective Sobolev spaces. Thus, we also obtain b).

Finally we need to check that the lack of Hölder continuity in time of the solutions  $(\rho_i, \sigma_i)$  does not cause problems for  $G_n(V_i), G_v(V_i)$ . The idea is to trade some spatial differentiability of  $(\rho, \sigma)$  for Lipschitz continuity in time. In detail, from (34) we find that

$$\begin{aligned} \|\rho(t + \delta) - \rho(t)\|_{H^m} &= \left\| \int_t^{t+\delta} \partial_t \rho \, d\tau \right\|_{H^m} = \left\| \int_t^{t+\delta} - \sum_{j=1}^3 v_j \partial_j \rho - P_1^{-1} \sum_{j=1}^3 \partial_j (a_j \sigma) \, d\tau \right\|_{H^m} \\ &\leq C\delta \left( \|v\|_{C([0, T_0], [H^m]^3)} \|\rho\|_{C([0, T_0], H^{m+1})} \right. \\ &\quad \left. + (\|\rho\|_{C([0, T_0], H^{m+1})} + \|n\|_{C([0, T_0], [H^{m+2}]^2)}) \|\sigma\|_{C([0, T_0], H^{m+1})} \right) \end{aligned}$$

and similarly for  $\sigma$ . Hence

$$V \in C^{0,\theta}([0, T_0], [H^{m+3}]^2) \times C^{0,\theta}([0, T_0], [H^{m+2}]^3) \times C([0, T_0], [H^{m+2}]^2)$$

implies  $\sigma, \rho \in C^{0,1}([0, T_0], H^m)$ , where  $C^{0,1}([0, T_0], X) \subset C^{0,\beta}([0, T_0], X)$  denotes Lipschitz continuity in time with values in  $X$ . Thus we obtain c).

For small  $C_1, T_0 > 0$  from Theorem 1.1 we obtain a small Lipschitz constant in Lemma 3.1 which by using Lemma 2.5 and Lemma 2.6 implies the convergence of the iteration scheme. Therefore, we are done.  $\square$

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