Algebraic Quantum Field Theory

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Literature:

1. R. Haag: *Local Quantum Physics*, Springer 1992/1996
2. H. Araki: *Mathematical Theory of Quantum Fields*, Oxford University Press 2000.
3. D. Buchholz: *Introduction to Algebraic QFT*, lectures, University of Goettingen, winter semester 2007. (Main source for sections 1,2,5 below).

Programme of the lectures:

1. Algebraic structure of quantum theory
   (a) quantum mechanics: Heisenberg, Weyl and resolvent algebra.
   (b) infinite quantum systems.
2. Operator algebras and local (relativistic) quantum physics
   (a) abstract algebras, representations
   (b) locality, covariance
   (c) vacuum
3. Construction of models
   (a) free theories, conformal field theories
   (b) wedge-local theories and Rieffel deformations
4. Scattering theory
   (a) Scattering matrix
   (b) Asymptotic completeness
   (c) Infrared problems
5. Superselection structure and statistics
   (a) DHR analysis (charges, statistics etc.)
   (b) charged fields, gauge groups
   (c) Infrared problems

1 Algebraic structure of quantum theory

1.1 Quantum systems with a finite number of degrees of freedom

- *Observables* describe properties of measuring devices (possible measured values, commensurability properties).
- *States* describe properties of prepared ensembles (probability distributions of measured values, correlations between observables)

Mathematical description based on Hilbert space formalism, Hilbert space $\mathcal{H}$.

- Observables: self-adjoint operators A on $\mathcal{H}$.
- States: density matrices ρ on $\mathcal{H}$ (i.e. $\rho \geq 0$, $\mathrm{Tr}\,\rho = 1$).
- Expectation values $A, \rho \mapsto \mathrm{Tr}\rho A$.

Remark 1.1 *pure states ('optimal information')= rays $e^{i\phi}\phi \in \mathcal{H}$, $\|\phi\| = 1$ = orthogonal projections $\rho^2 = \rho$. (*Question: Why equivalent? *Express in a basis, there can be just one eigenvalue with multiplicity one).*

- *Usual framework*: fixed by specifying $\mathcal{H}$. E.g. for spin $\mathcal{H} = \mathbb{C}^2$, for particle $L^2(\mathbb{R}^3)$. *Question: What is the Hilbert space for a particle with spin?* $L^2(\mathbb{R}^3; \mathbb{C}^2)$.
- *Question:* Does every s.a. operator A correspond to some measurement? Does every density matrix ρ correspond to some ensamble which can be prepared? In general no. Superselection rules. For example, you cannot superpose two states with different charges.
- *New point of view:* Observables are primary objects (we specify the family of measuring devices). The rest of the theory follows.

1.1.1 Heisenberg algebra

Quantum Mechanics. Observables:
$Q_j, j = 1, \dots, n$ and $P_k, k = 1, \dots, n$.
($n = Nd$, N-number of particles, d-dimension of space).

We demand that observables form (generate) an algebra.

Definition 1.2 *The "free (polynomial) $*$-algebra $\mathcal{P}$" is a complex vector space whose basis vectors are monomials ("words") in Q_j, P_k (denoted $Q_{j_1} \dots P_{k_1} \dots Q_{j_n} \dots P_{k_n}$).*

1. *Sums: Elements of $\mathcal{P}$ have the form*

$$\sum c_{j_1 \dots k_n} Q_{j_1} \dots P_{k_n}. \tag{1}$$

2. *Products: The product operation is defined on monomials by*

$$(Q_{j_1} \dots P_{k_1} \dots Q_{j_n} \dots P_{k_n}) \cdot (Q_{j'_1} \dots P_{k'_1} \dots Q_{j'_n} \dots P_{k'_n})$$
$$= Q_{j_1} \dots P_{k_1} \dots Q_{j_n} \dots P_{k_n} Q_{j'_1} \dots P_{k'_1} \dots Q_{j'_n} \dots P_{k'_n}$$

3. *Adjoints:* $Q_j^* = Q_j$, $P_k^* = P_k$,

$$\left(\sum c_{j_1 \dots k_n} Q_{j_1} \dots P_{k_n} \right)^* = \sum \overline{c_{j_1 \dots k_n}} P_{k_n} \dots Q_{j_1}. \quad (2)$$

4. *Unit:* 1.

The operations $(+, \cdot, ^*)$ *are subject to standard rules (associativity, distributivity, antilinearity etc.) but not commutativity.*

- Quantum Mechanics requires the following *relations*:

$$[Q_j, Q_k] = [P_j, P_k] = 0, \quad \big([Q_j, P_k] - i\delta_{j,k}1\big) = 0. \quad (3)$$

- Consider a two-sided ideal $\mathcal{J}$ generated by all linear combinations of

$$A[Q_j, Q_k]B, \quad A[P_j, P_k]B, \quad A\big([Q_j, P_k] - i\delta_{j,k}1\big)B \quad (4)$$

  for all $A, B \in \mathcal{P}$.

Definition 1.3 *Quotient* $\mathcal{P}\backslash\mathcal{J}$ *is again a* $*$*-algebra, since* $\mathcal{J}$ *is a two-sided ideal and* $\mathcal{J}^* = \mathcal{J}$*. We will call it "Heisenberg algebra". This is the free algebra 'modulo relations' (3).*

1.1.2 Weyl algebra

The elements of polynomial algebra are intrinsically unbounded (values of position and momentum can be arbitrarily large). This causes technical problems. A way out is to consider their bounded functions. For $z = u + iv \in \mathbb{C}^n$ we would like to set $W(z) \approx \exp(i \sum_k (u_k P_k + v_k Q_k))$. We cannot do it directly, because exp is undefined for 'symbols' P_k, Q_k. But we can consider abstract symbols $W(z)$ satisfying the expected relations keeping in mind the formal Baker-Campbell-Hausdorff (BCH) relation. The BCH formula gives

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \quad (5)$$

We have $z = u + iv, z' = u' + iv'$, $W(z) = e^A$, $W(z') = e^B$, $A = i(uP + vQ)$, $B = i(u'P + v'Q)$ and $[Q, P] = i$. Thus we have

$$[A, B] = (-1)[uP + vQ, u'P + v'Q] = (-1)(ivu' - iuv') = i(uv' - vu'). \quad (6)$$

On the other hand:

$$\mathrm{Im}\langle z, z' \rangle = \mathrm{Im}\langle u + iv, u' + iv' \rangle = \mathrm{Im}(-ivu' + iuv') = uv' - vu'. \quad (7)$$

Hence

$$W(z)W(z') = e^{\frac{i}{2}\mathrm{Im}\langle z, z' \rangle} W(z + z'). \quad (8)$$

Definition 1.4 *The (pre-)Weyl algebra $\mathcal{W}$ is the free polynomial $*$-algebra generated by the symbols $W(z)$, $z \in \mathbb{C}^n$ modulo the relations*

$$W(z)W(z') - e^{\frac{i}{2}\mathrm{Im}\langle z|z'\rangle}W(z+z') = 0, \quad W(z)^* - W(-z) = 0, \tag{9}$$

where $\langle z|z'\rangle = \sum_k \bar{z}_k z'_k$ is the canonical scalar product in $\mathbb{C}^n$.

The Weyl algebra has the following properties:

1. We have $W(0) = 1$ (by the uniqueness of unity).
2. By the above $W(z)W(z)^* = W(z)^*W(z) = 1$ i.e. Weyl operators are unitary.
3. We have

$$\big(\sum_z a_z W(z)\big)\big(\sum_{z'} b_{z'} W(z')\big) = \sum_{z,z'} a_z b_{z'} e^{\frac{i}{2}\mathrm{Im}\langle z,z'\rangle}W(z+z'). \tag{10}$$

   Thus elements of $\mathcal{W}$ are linear combinations of Weyl operators $W(z)$.

1.1.3 Representations of the Weyl algebra

Definition 1.5 *A $*$-representation $\pi : \mathcal{W} \mapsto B(\mathcal{H})$ is a homomorphism i.e. a map which preserves the algebraic structure. That is for $W, W_1, W_2 \in \mathcal{W}$:*

1. linearity $\pi(c_1W_1 + c_2W_2) = c_1\pi(W_1) + c_2\pi(W_2)$,
2. multiplicativity $\pi(W_1W_2) = \pi(W_1)\pi(W_2)$,
3. symmetry $\pi(W^*) = \pi(W)^*$.

If in addition $\pi(1) = I$, we say that the representation is unital. (In these lectures we consider unital representations unless specified otherwise).

Example 1.6 *Let $\mathcal{H}_1 = L^2(\mathbb{R}^n)$ with scalar products $\langle f, g\rangle = \int d^n x\, \overline{f}(x)g(x)$. One defines*

$$\big(\pi_1(W(z))f\big)(x) = e^{\frac{i}{2}uv}e^{ivx}f(x+u), \quad z = u + iv. \tag{11}$$

(Note that for $u = 0$ $\pi_1(W(z)$ is a multiplication operator and for $v = 0$ it is a shift). This is Schrödinger representation in configuration space.

Remark 1.7 *Heuristics: Recall that $W(z) = e^{(i\sum_k(u_kP_k+v_kQ_k))}$ and Baker-Campbell-Hausdorff*

$$\begin{aligned} (e^{i(uP+vQ)}f)(x) &= e^{\frac{i}{2}uv}(e^{ivQ}e^{i\sum uP}f)(x) && (12)\\ &= e^{\frac{i}{2}uv}e^{ivx}(e^{iuP}f)(x) = e^{\frac{i}{2}uv}e^{ivx}(f)(x+u) && (13) \end{aligned}$$

For the last step note $(e^{iuP}f)(x) = (e^{iu\frac{1}{i}\partial_x}f)(x) = (\sum_n \frac{u^n}{n!}\partial_x^n f)(x) = f(x+u)$.

Example 1.8 *Let* $\mathcal{H}_2 = L^2(\mathbb{R}^n)$ *with scalar products* $\langle f, g\rangle = \int d^n x\, \overline{f}(x) g(x)$*. One defines*

$$\big(\pi_2(W(z))f\big)(x) = e^{-\frac{i}{2}uv} e^{iux} f(x - v), \quad z = u + iv. \tag{14}$$

This is Schrödinger representation in momentum space.

Relation between $(\pi_1, \mathcal{H}_1)$, $(\pi_2, \mathcal{H}_2)$ is provided by the Fourier transform

$$(\mathcal{F}f)(y) := (2\pi)^{-n/2} \int d^n x\, e^{-ixy} f(x), \tag{15}$$

$$(\mathcal{F}^{-1}f)(y) := (2\pi)^{-n/2} \int d^n x\, e^{ixy} f(x). \tag{16}$$

$\mathcal{F}$ is isometric, i.e. $\langle \mathcal{F}f, \mathcal{F}f\rangle = \langle f, f\rangle$, (Plancherel theorem) and invertible (Fourier theorem). Hence it is unitary. We have

$$\pi_2(W) = \mathcal{F}\pi_1(W)\mathcal{F}^{-1}, \quad W \in \mathcal{W}. \tag{17}$$

Definition 1.9 *Let* $(\pi_a, \mathcal{H}_a)$, $(\pi_b, \mathcal{H}_b)$ *be two representations. If there exists an invertible isometry* $U : \mathcal{H}_a \to \mathcal{H}_b$ *(a unitary) s.t.*

$$\pi_b(\,\cdot\,) = U\pi_a(\,\cdot\,)U^{-1} \tag{18}$$

the two representations are said to be (unitarily) equivalent (denoted $(\pi_a, \mathcal{H}_a) \simeq (\pi_b, \mathcal{H}_b)$*). As we will see, equivalent representations describe the same set of states.*

Is any representation of $\mathcal{W}$ unitarily equivalent to the Schrödinger representation π_1? Certainly not, because we can form direct sums e.g. $\pi = \pi_1 \oplus \pi_1$ is not unitarily equivalent to π_1. We have to restrict attention to representations which cannot be decomposed into "smaller" ones.

Definition 1.10 *Irreducibility of representations: We say that a closed subspace* $\mathcal{K} \subset \mathcal{H}$ *is invariant (under the action of* $\pi(\mathcal{W})$*) if* $\pi(\mathcal{W})\mathcal{K} \subset \mathcal{K}$*. We say that a representation of* $(\pi, \mathcal{H})$ *of* $\mathcal{W}$ *is irreducible, if the only closed invariant subspaces are* $\mathcal{H}$ *and* $\{0\}$.

Remark 1.11 *The Schroedinger representation* π_1 *is irreducible (Homework).*

Lemma 1.12 *Irreducibility of* $(\pi, \mathcal{H})$ *is equivalent to any of the two conditions below:*

1. *For any non-zero* $\Psi \in \mathcal{H}$

$$\overline{\{\pi(W)\Psi \,|\, W \in \mathcal{W}\}} = \mathcal{H} \tag{19}$$

(i.e. if every non-zero vector is cyclic).

2. *Given $A \in B(\mathcal{H})$,*

$$[A, \pi(W)] = 0 \quad \textit{for all} \quad W \in \mathcal{W} \tag{20}$$

implies that $A \in \mathbb{C}I$ ("Schur lemma")
(i.e. the commutant of $\pi(\mathcal{W})$ is trivial).

Remark 1.13 *Recall that the commutant of $\pi(\mathcal{W})$ is defined as*

$$\pi(\mathcal{W})' = \{\, A \in B(\mathcal{H}) \; | [A, \pi(W)] = 0 \;\; \textit{for all}\; W \in \mathcal{W}\}. \tag{21}$$

Proof. For complete proof see e.g. Proposition 2.3.8 in [1]. We will show here only that 1. $\Rightarrow$ 2.: By contradiction, we assume that there is $A \notin \mathbb{C}I$ in $\pi(\mathcal{W})'$. If $A \in \pi(\mathcal{W})'$ then also $A^* \in \pi(\mathcal{W})'$ hence also s.a. operators $\frac{A+A^*}{2}$ and $\frac{A-A^*}{2i}$ are in $\pi(\mathcal{W})'$. Thus, we can in fact assume that there is a s.a. operator $B \in \pi(\mathcal{W})'$, $B \notin \mathbb{C}1$. Then also bounded Borel functions of B are in $\pi(\mathcal{W})'$. In particular characteristic functions $\chi_\Delta(B)$, $\Delta \subset \mathbb{R}$ (spectral projections of B) are in $\pi(\mathcal{W})'$. Since $B \notin \mathbb{C}1$, we can find $0 \neq \chi_\Delta(B) \neq I$. Let $\Psi \in \mathrm{Ran}\, \chi_\Delta(B)$ i.e. $\Psi = \chi_\Delta(B)\Psi$. Then for any $W \in \mathcal{W}$

$$\pi(W)\Psi = \pi(W)\chi_\Delta(B)\Psi = \chi_\Delta(B)\pi(W)\Psi, \tag{22}$$

hence Ψ cannot be cyclic because $\chi_\Delta(B)$ projects on a subspace which is strictly smaller than $\mathcal{H}$. $\square$

Question: Are any two irreducible representations of the Weyl algebra unitarily equivalent?

Answer: In general, no. After excluding pathologies yes.

Example 1.14 *Let $\mathcal{H}_3$ be a non-separable Hilbert space with a basis $e_p, p \in \mathbb{R}^n$. Elements of $\mathcal{H}_3$:*

$$f = \sum_p c_p e_p, \quad \text{with} \quad \sum_p |c_p|^2 < \infty \tag{23}$$

(i.e. all $c_p = 0$ apart from some countable set). $\langle f|f'\rangle = \sum_p \bar{c}_p c'_p$. We define

$$\pi_3(W(z))e_p = e^{-\frac{i}{2}uv} e^{iup} e_{p+v}. \tag{24}$$

This representation is irreducible but not unitarily equivalent to $(\pi_1, \mathcal{H}_1) \simeq (\pi_2, \mathcal{H}_2)$ because $\mathcal{H}_{1,2}$ and $\mathcal{H}_3$ have different dimension.

Criterion: Representation $(\pi, \mathcal{H})$ of $\mathcal{W}$ is of "physical interest" if for any $f \in \mathcal{H}$ the expectation values

$$z \mapsto \langle f, \pi(W(z))f\rangle \tag{25}$$

depend continuously on z.

Physical meaning of the Criterion: Set $v = 0$. Then $u \mapsto \pi(W(u))$ is an n-parameter unitary representation of translations on $\mathcal{H}$. Hence, by the Criterion and Stone's theorem

$$\pi(W(u)) = e^{i(u_1 P_{\pi,1} + \cdots + u_n P_{\pi,n})}, \tag{26}$$

where $P_{\pi,i}$ is a family of commuting s.a operators on (a domain in) $\mathcal{H}$. They can be interpreted as momentum operators in this representation. Analogously, we obtain the position operators $Q_{\pi,i}$. By taking derivatives of the Weyl relations w.r.t, u_l, v_k one obtains $[Q_{\pi,j}, P_{\pi,k}] = i\delta_{j,k}1$ on a certain domain (on which the derivatives exist).

Theorem 1.15 *(Stone-von Neumann uniqueness theorem) Any irreducible representation of $\mathcal{W}$, satisfying the Criterion, is unitarily equivalent to the Schrödinger representation.*

For a proof see Theorem 4.34 and Theorem 8.15 in [2].

Remark 1.16 *This theorem does not generalize to systems with infinitely many degrees of freedom ($n = \infty$). In particular, it does not hold in QFT. This is one reason why charges, internal ('gauge') symmetries, and groups play much more prominent role in QFT than in QM. As we will see in Section 5, they will be needed to keep track of all these inequivalent representations.*

1.1.4 States

Definition 1.17 *A state ω of a physical system is described by*

1. *specifying a representation $(\pi, \mathcal{H})$ of $\mathcal{W}$,*

2. *specifying a density matrix ρ on $\mathcal{H}$.*

Then $\omega(W) = \mathrm{Tr}\rho\pi(W)$.

Lemma 1.18 *A state is a map $\omega : \mathcal{W} \mapsto \mathbb{C}$ which satisfies*

1. linearity $\omega(c_1 W_1 + c_2 W_2) = c_1\omega(W_1) + c_2\omega(W_2)$.

2. normalization $\omega(1) = 1$.

3. positivity $\omega(W^*W) \geq 0$ *for all* $W \in \mathcal{W}$.

Proof. The only non-trivial fact is positivity: Write $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$, $p_i \geq 0$, $\|\Psi_i\| = 1$. Then, if the sum is finite, we can write

$$\begin{aligned} \omega(W^*W) &= \sum_i p_i \mathrm{Tr}(|\Psi_i\rangle\langle\Psi_i|\pi(W^*W)) \\ &= \sum_i p_i \langle\Psi_i|\pi(W^*W)\Psi_i\rangle = \sum_i p_i \|W\Psi_i\|^2, \end{aligned} \tag{27}$$

by completing Ψ_i to orthonormal bases.

In the general case we can use cyclicity of the trace

$$\begin{aligned} \mathrm{Tr}\,\rho\pi(W^*W) &= \mathrm{Tr}\,\rho\pi(W)^*\pi(W) = \mathrm{Tr}\,\pi(W)\rho\pi(W)^* && (28) \\ &= \sum_i \sum_j p_j |\langle e_i, \pi(W)\Psi_j\rangle|^2. && (29) \end{aligned}$$

The result is finite (because $\rho\pi(W^*W)$ is trace-class) and manifestly positive. □

Definition 1.19 *We say that a representation $(\pi, \mathcal{H})$ is cyclic, if $\mathcal{H}$ contains a cyclic vector Ω. (Cf. Lemma 1.12). Such representations will be denoted $(\pi, \mathcal{H}, \Omega)$. For example, any irreducible representation is cyclic.*

Theorem 1.20 *Any linear functional $\omega : \mathcal{W} \to \mathbb{C}$, which is positive and normalized, is a state in the sense of Definition 1.17 above. More precisely, it induces a unique (up to unitary equivalence) cyclic representation $(\pi, \mathcal{H}, \Omega)$ s.t.*

$$\omega(W) = \langle \Omega, \pi(W)\Omega \rangle, \quad W \in \mathcal{W}. \tag{30}$$

Proof. GNS construction (we will come to that). □

Lemma 1.21 *If $(\pi_1, \mathcal{H}_1) \simeq (\pi_2, \mathcal{H}_2)$ then the corresponding sets of states coincide.*

Proof. Let ρ_1 be a density matrix in representation π_1 and $W \in \mathcal{W}$. Then

$$\mathrm{Tr}\,\rho_1\pi_1(W) = \mathrm{Tr}\,\rho_1 U\pi_2(W)U^{-1} = \mathrm{Tr}\,U^{-1}\rho_1 U\pi_2(W). \tag{31}$$

Hence it does not matter if we measure W in representation π_1 on ρ_1 or in π_2 on $\rho_2 = U^{-1}\rho_1 U$.□

1.1.5 Weyl C^*-algebra

Definition 1.22 *We define a seminorm on $\mathcal{W}$:*

$$\|W\| := \sup_{\pi} \|\pi(W)\|, \quad W \in \mathcal{W}, \tag{32}$$

where the supremum extends over all cyclic representations. The completion of $\mathcal{W}/\ker \|\cdot\|$ is the Weyl C^*-algebra which we denote $\tilde{\mathcal{W}}$.

A few remarks about this definition:

1. The supremum is finite because for any representation π we have

$$\|\pi(W(z))\|^2 = \|\pi(W(z))^*\pi(W(z))\| = \|\pi(1)\| = 1 \tag{33}$$

   and thus $\|\pi(W)\|$ for any $W \in \mathcal{W}$ is finite.

2. We cannot take supremum over all representations because this is not a set. In fact, take the direct sum of all the representations which do not have themselves as a direct summand and call this representation Π. Then we get the Russel's paradox:

$$\Pi := \bigoplus\{\pi \,|\, \pi \notin \pi\} \text{ then } \Pi \in \Pi \Leftrightarrow \Pi \notin \Pi, \tag{34}$$

where $\pi_1 \in \pi_2$ means here that π_1 is contained in π_2 as a direct summand.

3. Using the GNS theorem one can show that

$$\|W\| = \sup_\omega \omega(W^*W)^{1/2}. \tag{35}$$

Here the supremum extends over the *set* of states. Indeed:

$$\sup_\omega \omega(W^*W)^{1/2} = \sup_{(\pi,\Omega)} \langle \Omega, \pi(W^*W)\Omega\rangle \leq \sup_\pi \|\pi(W)\|. \tag{36}$$

On the other hand

$$\begin{aligned} \sup_\pi \|\pi(W)\| = \sup_\pi \sup_{\|\Psi\|=1} \|\pi(W)\Psi\| &= \sup_\pi \sup_{\|\Psi\|=1} \langle \Psi, \pi(W^*W)\Psi\rangle^{1/2} \\ &\leq \sup_\omega \omega(W^*W)^{1/2}. \end{aligned} \tag{37}$$

4. In the case of the Weyl algebra $\ker \|\cdot\| = 0$ so the seminorm (32) is actually a norm. [5]

Apart from standard properties of the norm, it satisfies

$$\|W_1W_2\| \leq \|W_1\|\,\|W_2\| \text{ Banach algebra property} \tag{38}$$

$$\|WW^*\| = \|W\|^2 \; C^*\text{-property} \tag{39}$$

This is adventageous from the point of view of functional calculus: For $W \in \mathcal{W}$ we have $f(W) \in \mathcal{W}$ for polynomials f, but for more complicated functions there is no guarantee. For $W \in \tilde{\mathcal{W}}$ we have $f(W) \in \tilde{\mathcal{W}}$ for any continuous function f.

Nevertheless, in the next few subsections we will still work with the pre-Weyl algebra $\mathcal{W}$.

1.1.6 Symmetries

Postulate: Symmetry transformations are described by automorphisms (invertible homomorphisms) of $\mathcal{W}$.

Definition 1.23 *We say that a map $\alpha : \mathcal{W} \to \mathcal{W}$ is an automorphism if it is a bijection and satisfies*

1. $\alpha(c_1W_1 + c_2W_2) = c_1\alpha(W_1) + c_2\alpha(W_2)$

2. $\alpha(W_1W_2) = \alpha(W_1)\alpha(W_2)$

3. $\alpha(W)^* = \alpha(W^*)$

4. $\alpha(1) = 1$.

Automorphisms of $\mathcal{W}$ *form a group which we denote* $\mathrm{Aut}\,\mathcal{W}$*.*

Example 1.24 *If* $U \in \mathcal{W}$ *is a unitary, then* $\alpha_U(W) = UWU^{-1}$ *is called an inner automorphism. Inner automorphisms form a group* $\mathrm{In}\,\mathcal{W}$*. For example, for* $U = W(u_0)$ *we have*

$$\alpha_{u_0}(W(z)) = W(u_0)W(z)W(u_0)^{-1} = e^{i\langle u_0,v\rangle}W(z) \tag{40}$$

This is translation of coordinates, as one can see in the Schroedinger representation π_1*:*

$$\pi_1(\alpha_{u_0}(W(z))) = e^{i\langle u_0,v\rangle}e^{i(uP+vQ)} = e^{i(uP+v(Q+u_0))}. \tag{41}$$

Similarly, for $v_0 \in \mathbb{R}^n$

$$\alpha_{iv_0}(W(z)) = W(iv_0)W(z)W(iv_0)^{-1} = e^{-i\langle v_0,u\rangle}W(z) \tag{42}$$

is a translation in momentum space.

Example 1.25 *Let* $R \in SO(n)$*. Then*

$$\alpha_R(W(z)) = W(Rz) \tag{43}$$

is an automorphism which is not inner. (Set $n = 3$ *and let* R *be a rotation around the* z *axis by angle* θ*. Then, in the Schrödinger representation*

$$\pi_1(\alpha_R(W(z))) = U\pi_1(W(z))U^{-1} \tag{44}$$

$U = e^{i\theta L_z}$*, where* $L_z = Q_xP_y - Q_yP_x$*. Clearly,* U *is not an element of* $\mathcal{W}$*). Automorphisms which are not inner are called outer automorphisms. They form a set* $\mathrm{Out}\mathcal{W}$ *which is not a group.*

As we have seen above, even if an automorphism is not inner, it can be implemented by a unitary in some given representation.

Definition 1.26 *Let* $(\pi, \mathcal{H})$ *be a representation of* $\mathcal{W}$*. Then* $\alpha \in \mathrm{Aut}\mathcal{W}$ *is said to be unitarily implementable on* $\mathcal{H}$ *if there exists some unitary* $U \in B(\mathcal{H})$ *s.t.*

$$\pi(\alpha(W)) = U\pi(W)U^{-1}, \quad W \in \mathcal{W}. \tag{45}$$

Example 1.27 *A large class of automorphisms is obtained as follows*

$$\alpha(W(z)) = c(z)W(Sz) \tag{46}$$

where $c(z) \in \mathbb{C}\backslash\{0\}$ *and* $S : \mathbb{C}^n \to \mathbb{C}^n$ *a continuous bijection. Weyl relations impose restrictions on* c, S:

$$c(z+z') = c(z)c(z'), \quad c(-z) = \overline{c(z)}, \quad |c(z)| = 1, \tag{47}$$

$$S(z+z') = S(z) + S(z'), \quad S(-z) = -S(z), \quad \mathrm{Im}\langle Sz, Sz'\rangle = \mathrm{Im}\langle z|z'\rangle. \tag{48}$$

The latter property means that S *is a real-linear symplectic transformation.*

For continuous c *and* S *such automorphisms are unitarily implementable in all irreducible representations satisfying the Criterion (consequence of the v.N. uniqueness theorem). See Homeworks.*

Remark 1.28 $\omega(z_1, z_2) := \mathrm{Im}\langle z|z'\rangle$ *is an example of a* symplectic form. *In general, we say that a bilinear form* ω *is symplectic if it is:*

1. *Antisymmetric:* $\omega(z_1, z_2) = -\omega(z_2, z_1)$
2. *Non-degenerate: If* $\omega(z_1, z_2) = 0$ *for all* z_2, *then* $z_1 = 0$.

1.1.7 Dynamics

Definition 1.29 *A dynamics on* $\mathcal{W}$ *is a one-parameter group of automorphisms on* $\mathcal{W}$ *i.e.* $\mathbb{R} \ni t \mapsto \alpha_t$ *s.t.* $\alpha_0 = \mathrm{id}$, $\alpha_{t+s} = \alpha_t \circ \alpha_s$.

Proposition 1.30 *Suppose that the dynamics is unitarily implemented in an irreducible representation* π *i.e. there exists a family of unitaries s.t.*

$$\pi(\alpha_t(W)) = U(t)\pi(W)U(t)^{-1}, \quad W \in \mathcal{W}. \tag{49}$$

Suppose in addition that $t \mapsto U(t)$ *continuous (in the sense of matrix elements) and differentiable (i.e. for some* $0 \neq \Psi \in \mathcal{H}$, $\partial_t U(t)\Psi$ *exists in norm).*

Then there exists a continuous group *of unitaries* $t \mapsto V(t)$ *(i.e.* $V(0) = 1$, $V(s+t) = V(s)V(t)$*) s.t.*

$$\pi(\alpha_t(W)) = V(t)\pi(W)V(t)^{-1}. \tag{50}$$

Remark 1.31 *By the Stone's theorem we have* $V(t) = e^{itH}$ *for some self-adjoint operator* H *on (a domain in)* $\mathcal{H}$ *(the Hamiltonian). Whereas* α_t *is intrinsic, the Hamiltonian is not. Its properties (spectrum etc.) depend in general on representation.*

Proof. We have $\alpha_s \circ \alpha_t = \alpha_{s+t}$. Hence

$$U(s)U(t)\pi(W)U(t)^{-1}U(s)^{-1} = U(s+t)\pi(W)U(s+t)^{-1}, \tag{51}$$

$$U(s+t)^{-1}U(s)U(t)\pi(W) = \pi(W)U(s+t)^{-1}U(s)U(t). \tag{52}$$

By irreducibility of π

$$U(s+t) = \eta(s,t)U(s)U(t), \text{ where } |\eta(s,t)| = 1. \tag{53}$$

By multiplying U by a constant phase $e^{i\phi_0}$ we can assume that $U(0) = I$, hence

$$\eta(0,t) = \eta(s,0) = 1. \tag{54}$$

Now consider a new family of unitaries $V(s) = \xi(s)U(s)$, $|\xi(s)| = 1$. We have

$$\begin{aligned} V(s+t) &= \eta'(s,t)V(s)V(t) = \xi(s+t)U(s+t) \\ &= \xi(s+t)\eta(s,t)U(s)U(t) = \xi(s+t)\eta(s,t)\xi(s)^{-1}\xi(t)^{-1}V(s)V(t). \end{aligned} \tag{55}$$

Hence

$$\eta'(s,t) = \frac{\xi(s+t)}{\xi(s)\xi(t)}\eta(s,t). \tag{56}$$

The task is to obtain $\eta'(s,t) = 1$ for all s, t for a suitable choice of ξ (depending on η). The key observation is that associativity of addition in $\mathbb{R}$ imposes a constraint on η: In fact, we can write

$$U(r+s+t) = \eta(r,s+t)U(r)U(s+t) = \eta(r,s+t)\eta(s,t)U(r)U(s)U(t), \tag{57}$$
$$U(r+s+t) = \eta(r+s,t)U(r+s)U(t) = \eta(r+s,t)\eta(r,s)U(r)U(s)U(t). \tag{58}$$

Hence we get the "cocycle relation" (cohomology theory)

$$\eta(r,s+t)\eta(s,t) = \eta(r+s,t)\eta(r,s). \tag{59}$$

Using this relation one can show that given η one can find such ξ that $\eta' = 1$. "cocycle is a coboundary" (Howework). Important intermediate step is to show, using the cocycle relation that

$$\eta(s,t) = \eta(t,s). \tag{60}$$

To express ξ as a function of η we will have to differentiate η. By assumption, there is $\Psi \in \mathcal{H}$, $\|\Psi\|$=1 s.t. $\partial_t U(t)\Psi$ exists. By (53), we have

$$\begin{aligned} \eta(s,t) &= U(t)^*U(s)^*U(s+t) = \langle\Psi, U(t)^*U(s)^*U(s+t)\Psi\rangle \\ &= \langle U(t)\Psi, U(s)^*U(s+t)\Psi\rangle. \end{aligned} \tag{61}$$

Hence $\partial_t\eta(s,t)$ exists and by (60) also $\partial_s\eta(s,t)$. □

Example 1.32 Isotropic harmonic oscillator*: In the framework of the polynomial algebra $\mathcal{P}$ we have (heuristically)*

$$\alpha_t(Q_i) = \cos(\omega_0 t)Q_i - \sin(\omega_0 t)P_i, \tag{62}$$
$$\alpha_t(P_i) = \cos(\omega_0 t)P_i + \sin(\omega_0 t)Q_i. \tag{63}$$

In the Weyl setting $\alpha_t(W(z)) = W(e^{it\omega_0}z)$. This defines a group of automorphisms from Example 1.27 with $S_t z = e^{it\omega_0}z$, $c(z) = 1$. (S_t is complex-linear). This dynamics is unitarily implemented in the Schrödinger representation:

$$\pi_1(\alpha_t(W)) = U(t)\pi_1(W)U(t)^{-1}, \quad W \in \mathcal{W}, \tag{64}$$

$U(t) = e^{itH}$, $H = \sum_i \left(\frac{P_i^2}{2m} + \frac{kQ_i^2}{2}\right)$, $\omega_0 = \sqrt{\frac{k}{m}}$.

Example 1.33 *Free motion in the framework of $\mathcal{P}$:*

$$\alpha_t(Q_j) = Q_j + \frac{t}{m}P_j, \tag{65}$$

$$\alpha_t(P_k) = P_k. \tag{66}$$

In the framework of $\mathcal{W}$:

$$\alpha_t(W(z)) = W(\mathrm{Re} z + (t/m + i)\mathrm{Im}\, z) \tag{67}$$

We have that $S_t(z) = \mathrm{Re} z + (t/m+i)\mathrm{Im}\, z$ is a symplectic transformation, but only real linear. This dynamics is unitarily implemented in the Schrödinger representation:

$$\pi_1(\alpha_t(W)) = U(t)\pi_1(W)U(t)^{-1}, \quad W \in \mathcal{W}, \tag{68}$$

$U(t) = e^{itH}$, $H = \sum_i \frac{P_i^2}{2m}$.

By generalizing the above discussion, one can show that dynamics governed by Hamiltonians which are quadratic in P_i, Q_j correspond to groups of automorphisms of $\mathcal{W}$. But there are many other interesting Hamiltonians, for example:

$$H = \frac{P^2}{2m} + V(Q) \tag{69}$$

where $n = 1$, $V \in C_0^\infty(\mathbb{R})_{\mathbb{R}}$ (smooth, compactly supported, real).

Theorem 1.34 *(No-go theorem) Let $H = \frac{P^2}{2m} + V(Q)$, $V \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $U(t) = e^{itH}$. Then*

$$U(t)\pi_1(W)U(t)^{-1} \in \pi_1(\widetilde{\mathcal{W}}), \quad W \in \widetilde{\mathcal{W}}, t \in \mathbb{R}. \tag{70}$$

implies that $V = 0$.

Proof. See [3]. □

Thus Aut$\tilde{\mathcal{W}}$ does not contain dynamics corresponding to Hamiltonians (69). A recently proposed solution to this problem is to pass from exponentials $W(z) = e^{i(uP+vQ)}$ to resolvents $R(\lambda, z) = (i\lambda - uP - vQ)^{-1}$ and work with an algebra generated by these resolvents [4].

1.1.8 Resolvent algebra

Definition 1.35 *The pre-resolvent algebra $\mathcal{R}$ is the free polynomial $*$-algebra generated by symbols $R(\lambda, z)$, $\lambda \in \mathbb{R}\backslash\{0\}, z \in \mathbb{C}^n$ modulo the relations*

$$
\begin{aligned}
R(\lambda, z) - R(\mu, z) &= i(\mu - \lambda) R(\lambda, z) R(\mu, z), && (71)\\
R(\lambda, z)^* &= R(-\lambda, z), && (72)\\
[R(\lambda, z), R(\mu, z')] &= i\mathrm{Im}\langle z, z'\rangle R(\lambda, z) R(\mu, z')^2 R(\lambda, z), && (73)\\
\nu R(\nu\lambda, \nu z) &= R(\lambda, z), && (74)\\
R(\lambda, z) R(\mu, z') &= R(\lambda + \mu, z + z')(R(\lambda, z) + R(\mu, z') \\
&\quad + i\mathrm{Im}\langle z, z'\rangle R(\lambda, z)^2 R(\mu, z')), && (75)\\
R(\lambda, 0) &= \frac{1}{i\lambda}, && (76)
\end{aligned}
$$

where $\lambda, \mu, \nu \in \mathbb{R}\backslash\{0\}$ and in (75) we require $\lambda + \mu \neq 0$.

Remark 1.36 *Heuristically $R(\lambda, z) = (i\lambda - uP - vQ)^{-1}$. Realtions (71), (72) encode the algebraic properties of the resolvent of some self-adjoint operator. (73) encodes the canonical commutation relations. (74), (75), (76) encode linearity of the map $(u, v) \mapsto uP + vQ$.*

Definition 1.37 *The Schrödinger representation of $\mathcal{R}$ is defined as follows: Let $(\pi_1, \mathcal{H}_1)$ be the Schrödinger representation of $\mathcal{W}$. Since it satisfies the Criterion (i.e. it is "regular") we have P_i, Q_j as self-adjoint operators on $L^2(\mathbb{R}^n)$. Thus we can define*

$$\pi_1(R(\lambda, z)) = (i\lambda - uP - vQ)^{-1}. \tag{77}$$

One can check that this prescription defines a representation of $\mathcal{R}$ which is irreducible.

Definition 1.38 *We define a seminorm on $\mathcal{R}$*

$$\|R\| = \sup_{\pi} \|\pi(R)\|, \quad R \in \mathcal{R}, \tag{78}$$

where the supremum is over all cyclic representations of $\mathcal{R}$. (A cyclic representation is a one containing a cyclic vector. In particular, irreducible representations are cyclic). The resolvent C^*-algebra $\widetilde{\mathcal{R}}$ is defined as the completion of $\mathcal{R}/\ker \|\cdot\|$.

Remark 1.39 *The supremum is finite because for any representation π we have*

$$\|\pi(R(\lambda, z))\| \leq \frac{1}{\lambda}, \quad \textit{(Homework)}. \tag{79}$$

and thus $\|\pi(R)\|$ for any $R \in \mathcal{R}$ is finite. It is not known if $\ker \|\cdot\|$ is trivial. To show that it would suffice to exhibit one representation of $\mathcal{R}$ which is faithful *(i.e. injective: $\pi(R) = 0$ implies $R = 0$). A natural candidate is the Schrödinger representation. In this case one would have to check that if*

$$\sum_{finite} c_{i_1,\ldots i_n} \pi_1\big(R(\lambda_{i_1}, z_{i_1}) \cdots R(\lambda_{i_n}, z_{i_n})\big) = 0 \tag{80}$$

Then all $c_{i_1,\ldots i_n} = 0$.

Definition 1.40 *A representation $(\pi, \mathcal{H})$ of $\tilde{\mathcal{R}}$ is regular if there exist self-adjoint operators P_i, Q_j on $\mathcal{H}$ s.t. for $\lambda \in \mathbb{R}\backslash\{0\}$*

$$\pi(R(\lambda, z)) = (i\lambda - uP - vQ)^{-1}. \tag{81}$$

For example, the Schrödinger representation π_1 (of $\widetilde{\mathcal{R}}$) is regular.

Fact: Any regular irreducible representation π of $\mathcal{R}$ is faithful [4]. Hence, the Schrödinger representation of $\widetilde{\mathcal{R}}$ is faithful. This does *not* imply however that the Schrödinger representation of $\mathcal{R}$ is faithful since we divided by ker $\|\cdot\|$!

Proposition 1.41 *There is a one-to-one correspondence between regular representations of $\widetilde{\mathcal{R}}$ and representations of $\widetilde{\mathcal{W}}$ satisfying the Criterion. (The latter are also called "regular"). Hence, by the Stone-von Neumann uniqueness theorem, any irreducible regular representation of $\widetilde{\mathcal{R}}$ is unitarily equivalent to the Schrödinger representation.*

Proof. (Idea). Use the Laplace transformation

$$\pi(R(\lambda, z)) = -i \int_0^{\sigma\lambda} e^{-\lambda t} \pi(W(-tz))dt, \quad \sigma = \text{sgn}\lambda \tag{82}$$

to construct a regular representation of $\tilde{\mathcal{R}}$ out of a regular representation of $\tilde{\mathcal{W}}$. □

Remark 1.42 *The Laplace transform can also be useful in checking if* ker $\|\cdot\|$ *is trivial.*

Up to now, we found no essential difference between the Weyl algebra and the resolvent algebra. An important difference is that the Weyl C^*-algebra $\widetilde{\mathcal{W}}$ is simple, i.e. it has no non-trivial two sided ideals. The resolvent C^*-algebra has many ideals. They help to accommodate interesting dynamics.

Theorem 1.43 *There is a closed two-sided ideal $\mathcal{J} \subset \widetilde{\mathcal{R}}$ s.t. in any irreducible regular representation $(\pi, \mathcal{H})$ one has $\pi(\mathcal{J}) = \mathcal{K}(\mathcal{H})$ where $\mathcal{K}(\mathcal{H})$ is the algebra of compact operators on $\mathcal{H}$.*

Remark 1.44 *We recall:*

- *A is a compact operator if it maps bounded operators into pre-compact operators. (On a separable Hilbert space if it is a norm limit of a sequence of finite rank operators).*

- *A is Hilbert-Schmidt ($A \in \mathcal{K}_2(\mathcal{H})$) if $\|A\|_2 := \mathrm{Tr}(A^*A)^{1/2} < \infty$. Hilbert-Schmidt operators are compact.*

- *A convenient way to show that an operator on $L^2(\mathbb{R}^n)$ is Hilbert-Schmidt is to study its integral kernel K, defined by the relation:*

$$(Af)(p) = \int dp'\, K(p,p')f(p'). \tag{83}$$

  If K is in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ then $A \in \mathcal{K}_2(L^2(\mathbb{R}^n))$ and $\|A\|_2 = \|K\|_2$.

- *For example, consider $A = f(Q)g(P)$. Its integral kernel in momentum space is determined as follows:*

$$\begin{aligned}(f(Q)g(P)\Psi)(p) &= \frac{1}{\sqrt{2\pi}}\int dp'\, e^{iQp'}(\mathcal{F}f)(p')(g(P)\Psi)(p)\\ &= \frac{1}{\sqrt{2\pi}}\int dp'\,(\mathcal{F}f)(p')(g(P)\Psi)(p-p')\\ &= \frac{1}{\sqrt{2\pi}}\int dp'\,(\mathcal{F}f)(p')g(p-p')\Psi(p-p')\\ &= \frac{1}{\sqrt{2\pi}}\int dp'\,(\mathcal{F}f)(p-p')g(p')\Psi(p'). \end{aligned} \tag{84}$$

  Hence the integral kernel of $f(Q)g(P)$ if $K(p,p') = (\mathcal{F}f)(p-p')g(p')$. If f, g are square-integrable, so is K.

Proof. (Idea). By the von Neumann uniqueness theorem we can assume that π is the Schrödinger representation π_1. Then it is easy to show that $\pi(\widetilde{\mathcal{R}})$ contains some compact operators: For example, set $u_i = (\underbrace{0,\ldots,1}_{i},\ldots,0)$ and $v_i = (\underbrace{0,\ldots,1}_{i},\ldots,0)$. Then the operator

$$\begin{aligned} A &:= \pi_1(R(\lambda_1, iv_1)R(\mu_1,u_1)\ldots R(\lambda_n, iv_n)R(\mu_n,u_n))\\ &= \prod_{j=1}^{n}(i\lambda_j - Q_j)^{-1}\prod_{k=1}^{n}(i\mu_j - P_j)^{-1} \end{aligned} \tag{85}$$

is Hilbert-Schmidt for all $\lambda_i, \mu_i \in \mathbb{R}\backslash\{0\}$. (This can be shown by checking that it has a square-integrabe kernel). In particular it is compact. Now it is a general fact in the theory of C^*-algebras that if the image of an irreducibe representation contains one non-zero compact operator then it contains all of them (Howework or Corollary 4.1.10 of [6]). Thus, since π_1 is faithful, we can set $\mathcal{J} = \pi_1^{-1}(\mathcal{K}(\mathcal{H}))$. This is a closed two-sided ideal in $\widetilde{\mathcal{R}}$ since $\mathcal{K}(\mathcal{H})$ is a closed two-sided ideal in $B(\mathcal{H})$. □

Theorem 1.45 *Let $n = 1$, $H = P^2 + V(Q)$, where $V \in C_0(\mathbb{R})_{\mathbb{R}}$ real, continuous vanishing at infinity and $U(t) = e^{itH}$. Then*

$$U(t)\pi_1(R)U(t)^{-1} \in \pi_1(\widetilde{\mathcal{R}}), \qquad \textit{for all } R \in \widetilde{\mathcal{R}}, t \in \mathbb{R}. \tag{86}$$

Remark 1.46 *Since π_1 is faithful, we can define the group of automorphisms of $\mathcal{R}$*

$$\alpha_t(R) := \pi_1^{-1}\big(U(t)\pi_1(R)U(t)^{-1}\big), \tag{87}$$

which is the dynamics governed by the Hamiltonian H.

Remark 1.47 *For simplicity, we assume that $V \in S(\mathbb{R})_{\mathbb{R}}$ and $\int dx\, V(x) = 0$. General case follows from the fact that such functions are dense in $C_0(\mathbb{R})_{\mathbb{R}}$ in supremum norm.*

Proof. Let $U_0(t) = e^{itH_0}$, where $H_0 = P^2$. Since this is a quadratic Hamiltonian, we have

$$U_0(t)\pi_1(\widetilde{\mathcal{R}})U_0(t)^{-1} \subset \pi_1(\widetilde{\mathcal{R}}). \tag{88}$$

Now we consider $\Gamma_V(t) := U(t)U_0(t)^{-1}$. It suffices to show that $\Gamma_V(t) - 1$ are compact for all $V \in C_0(\mathbb{R})_{\mathbb{R}}$ since then $\Gamma_V(t) \in \pi_1(\widetilde{\mathcal{R}})$ by Theorem 1.43 and hence

$$U(t)\pi_1(\widetilde{\mathcal{R}})U(t)^{-1} = \Gamma_V(t)U_0(t)\pi_1(\widetilde{\mathcal{R}})U_0(t)^{-1}\Gamma_V(t)^{-1} \in \pi_1(\widetilde{\mathcal{R}}), \tag{89}$$

using $\Gamma_V(t)^{-1} = \Gamma_V(t)^* \in \pi_1(\widetilde{\mathcal{R}})$.

We use the Dyson perturbation series of $\Gamma_V(t)$:

$$\Gamma_V(t) = \sum_{n=0}^{\infty} i^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \dots \int_0^{t_2} dt_1\, V_{t_1}V_{t_2}\dots V_{t_n}, \tag{90}$$

where $V_t := U_0(t)V(Q)U_0(t)^{-1}$ and the integrals are defined in the strong-operator topology, that is exist on any fixed vector. (Cf. Proposition 1.50 below).

The key observation is that $\int_0^t ds\, V_s$ are Hilbert-Schmidt. To this end compute the integral kernel K_s of V_s:

$$(K_s)(p_1, p_2) = \frac{1}{\sqrt{2\pi}} e^{ip_1^2 s}(\mathcal{F}V)(p_1 - p_2)e^{-ip_2^2 s}. \tag{91}$$

This is clearly not Hilbert-Schmidt. Now let us compute the integral kernel $\hat{K}_s$ of $\int_0^t ds\, V_s$:

$$(\hat{K}_s)(p_1, p_2) = \int_0^t ds\, (K_s)(p_1, p_2) = \frac{1}{\sqrt{2\pi}} \frac{e^{i(p_1^2 - p_2^2)t} - 1}{i(p_1^2 - p_2^2)}(\mathcal{F}V)(p_1 - p_2). \tag{92}$$

This is Hilbert-Schmidt. In fact:

$$\begin{aligned}\int dp_1 dp_2\, |(\hat{K}_s)(p_1,p_2)|^2 &= c\int dq_1\, |(\mathcal{F}V)(q_1)|^2 \int dq_2 \frac{\sin^2(tq_1q_2)}{(q_1q_2)^2} \\ &= c\int dq_1\, |(\mathcal{F}V)(q_1)|^2 \frac{|t|}{|q_1|}\int dr \frac{\sin^2(r)}{r^2} \\ &= c'|t|\int dq_1\, \frac{|(\mathcal{F}V)(q_1)|^2}{|q_1|} \end{aligned} \tag{93}$$

Since $(\mathcal{F}V)(0) = 0$ we have $(\mathcal{F}V)(q_1) \leq c|q_1|$ near zero so the integral exists.

Consequently, the strong-operator continuous functions

$$\mathbb{R}^{n-1} \ni (t_2,\ldots,t_n) \mapsto \int_0^{t_2} dt_1\, V_{t_1}V_{t_2}\ldots V_{t_n} \tag{94}$$

have values in the Hilbert-Schmidt class and their Hilbert-Schmidt (HS) norms are bounded by

$$\left(c'|t_2|\int dq_1\, \frac{|(\mathcal{F}V)(q_1)|^2}{|q_1|}\right)^{1/2} \|V\|^{n-1} \tag{95}$$

(since $\|AB\|_2 \leq \|A\|_2\|B\|$). The integral of any strong-operator continuous HS-valued function with uniformly bounded (on compact sets) HS norm is again HS. (See Lemma 1.49 below). So each term in the Dyson expansion (apart from $n = 0$) is in $\pi_1(\widetilde{\mathcal{R}})$ and the expansion converges uniformly in norm. So $\Gamma_V(t) - 1$ is a compact operator. □

Remark 1.48 *The resolvent algebra admits dynamics corresponding to $H = P^2 + V(Q)$. But there are other interesting Hamiltonians which are not covered e.g. $H = \sqrt{P^2 + M^2}$. So there remain open questions...*

In the above proof we used two facts, which we will now verify:

Lemma 1.49 *Let $\mathbb{R}^n \ni t \mapsto F(t) \in \mathcal{K}_2(\mathcal{H})$ be continuous in the strong operator topology and suppose that for some compact set $K \subset \mathbb{R}^n$ we have*

$$\sup_{t\in K} \|F(t)\|_2 < \infty, \tag{96}$$

where $\|F(t)\|_2 = \mathrm{Tr}(F(t)^*F(t))^{1/2}$. Then

$$\hat{F} := \int_K dt\, F(t) \tag{97}$$

is again Hilbert-Schmidt.

Proof. We have

$$\begin{aligned}\|\hat{F}\|_2^2 &= \operatorname{Tr}\hat{F}^*\hat{F} = |\sum_i \int_{K\times K} dt_1 dt_2 \langle e_i, F(t_1)^* F(t_2) e_i\rangle| \\ &\leq \sum_i \int_{K\times K} dt_1 dt_2 |\langle e_i, F(t_1)^* F(t_2) e_i\rangle| \\ &\leq \sum_i \int_{K\times K} dt_1 dt_2 \|F(t_1)e_i\|\, \|F(t_2)e_i\|. \end{aligned} \tag{98}$$

Since the summands/integrals are positive, I can exchange the order of integration/summation. By Cauchy-Schwarz inequality:

$$\begin{aligned}\|\hat{F}\|_2^2 &\leq \int_{K\times K} dt_1 dt_2 \big(\sum_i \|F(t_1)e_i\|^2\big)^{1/2} \big(\sum_i \|F(t_2)e_i\|^2\big)^{1/2} \\ &= \int_{K\times K} dt_1 dt_2\, \|F(t_1)\|_2 \|F(t_2)\|_2 \\ &\leq |K|^2 \sup_{t\in K} \|F(t)\|_2^2 < \infty. \end{aligned} \tag{99}$$

Where in the last step we use the assumption (96). □

Lemma 1.50 *(Special case of Theorem 3.1.33 of [1]) Let $\mathbb{R} \ni t \mapsto U_0(t)$ be a strongly continuous group of unitaries on $\mathcal{H}$ with generator H_0 (i.e. $U_0(t) = e^{itH_0}$, above we had $H_0 = P^2$) and let V be a bounded s.a. operator on $\mathcal{H}$. Then $H_0 + V$ generates a strongly continuous group of unitaries U s.t.*

$$\begin{aligned}&U(t)\Psi = U_0(t)\Psi \\ &+\sum_{n\geq 1} i^n \int_{0\leq t_1\leq\cdots\leq t_n\leq t} dt_1 \ldots dt_n\, U_0(t_1) V U_0(t_2 - t_1) V \cdots U_0(t_n - t_{n-1}) V U_0(t - t_n)\Psi \end{aligned} \tag{100}$$

For any $\Psi \in \mathcal{H}$. (To get the expression for $\Gamma_V(t)$ it suffices to set $\Psi = U_0(t)^{-1}\Psi'$).

Proof. Strategy: we will treat (100) as a definition of a $t \geq 0$ dependent family of operators $t \mapsto U(t)$. We will use this definition to show that it can be naturally extended to a group of unitaries parametrized by $t \in \mathbb{R}$. Then, by differentiation, we will check that its generator is $H_0 + V$. Hence, by Stone's theorem we will have $U(t) = e^{it(H_0+V)}$.

Let $U^{(n)}(t)$ be the n-th term of the series of U. We have, by a change of variables,

$$U^{(0)}(t) = U_0(t), \quad U^{(n)}(t) = \int_0^t dt_1\, U_0(t_1) i V U^{(n-1)}(t - t_1). \tag{101}$$

Iteratively, one can show that all $U^{(n)}(t)$ are well defined and strongly continuous. It is easy to check that this is a series of bounded operators which converges in norm: In fact

$$\|U^{(n)}(t)\Psi\| \leq \frac{t^n}{n!}\|V\|^n\|\Psi\|, \text{ hence } \sum_n \|U^{(n)}(t)\Psi\| < \infty. \tag{102}$$

By taking the sum of both sides of the recursion relation (101), we get

$$U(t) = U_0(t) + \int_0^t ds\, U_0(s)iVU(t-s). \tag{103}$$

Now we want to show the (semi-)group property:

$$\begin{aligned} U(t_1)U(t_2) &= U_0(t_1)U(t_2) + \int_0^{t_1} ds\, U_0(s)iVU(t_1-s)U(t_2) \\ &= U_0(t_1+t_2) + \int_0^{t_2} ds\, U_0(t_1+s)iVU(t_2-s) \\ &\quad + \int_0^{t_1} ds\, U_0(s)iVU(t_1-s)U(t_2) \\ &= U(t_1+t_2) + \int_0^{t_2} ds\, U_0(t_1+s)iVU(t_2-s) \\ &\quad + \int_0^{t_1} ds\, U_0(s)iVU(t_1-s)U(t_2) \\ &\quad - \int_0^{t_1+t_2} ds\, U_0(s)iVU(t_1+t_2-s) \end{aligned} \tag{104}$$

Now $\int_{t_1}^{t_1+t_2}$ part of the last integral cancels the $\int_0^{t_2}$ integral (change of variables). We are left with

$$U(t_1)U(t_2) - U(t_1+t_2) = \int_0^{t_1} ds\, U_0(s)iV\big(U(t_1-s)U(t_2) - U(t_1+t_2-s)\big). \tag{105}$$

Now let $U^\lambda(t)$ be defined by replacing V with λV in (100), $\lambda \in \mathbb{R}$. It is clear from (100) that the function

$$F_{t_1}(\lambda) = U^\lambda(t_1)U^\lambda(t_2) - U^\lambda(t_1+t_2) \tag{106}$$

is real-analytic. By (105) we get

$$F_{t_1}(\lambda) = \lambda \int_0^{t_1} ds\, U_0(s)iVF_{t_1-s}(\lambda). \tag{107}$$

Clearly, $F_{t_1}(0) = 0$. Using this, and differentiating the above equation w.r.t. λ at 0, we get $\partial_\lambda F_{t_1}(0) = 0$. By iterating we get that all the Taylor series coefficients of F_{t_1} at zero are zero and thus $F_{t_1}(\lambda) = 0$ by analyticity. We conclude that the semigroup property holds i.e.

$$U(t_1+t_2) = U(t_1)U(t_2). \tag{108}$$

Now we want to show that $U(t)$ are unitaries. A candidate for an inverse of $U(t)$ is $U'(t)$ defined by replacing H_0 with $H_0' := -H_0$ and V by $V' = -V$. (JUMP DOWN). We also set $U_0'(t) = e^{i(-H_0)t}$. Let $t_2 \geq t_1$. Then

$$\begin{aligned}
U(t_1)U'(t_2) &= U_0(t_1)U'(t_2) + \int_0^{t_1} ds\, U_0(s)iVU(t_1-s)U'(t_2) \\
&= U_0(t_1-t_2) + \int_0^{t_2} ds\, U_0'(-t_1+s)iV'U'(t_2-s) \\
&\quad + \int_0^{t_1} ds\, U_0(s)iVU(t_1-s)U'(t_2) \\
&= U'(t_2-t_1) + \int_0^{t_2} ds\, U_0'(-t_1+s)iV'U'(t_2-s) \\
&\quad + \int_0^{t_1} ds\, U_0(s)iVU(t_1-s)U'(t_2) \\
&\quad - \int_0^{t_2-t_1} ds\, U_0'(s)iV'U'(t_2-t_1-s) \qquad (109)
\end{aligned}$$

In the last integral the part $-\int_0^{-t_1}$ combines with the second line and $-\int_{-t_1}^{-t_1+t_2}$ cancels the first line. Thus we get

$$U(t_1)U'(t_2) - U'(t_2-t_1) = \int_0^{t_1} ds\, U_0(s)iV\big(U(t_1-s)U'(t_2) - U'(t_2-(t_1-s))\big) \qquad (110)$$

(JUMP TO HERE). By an analogous argument as above we obtain

$$U(t_1)U'(t_2) = U'(t_2-t_1), \qquad (111)$$

In particular, $U(t)U'(t) = 1$ and we can consistently set $U(-t) := U'(t)$ for $t \geq 0$. Moreover, it is easily seeen from (100), by a change of variables, that $U'(t) = U(t)^*$. Thus we have a group of unitaries. By Stone's theorem it has a generator which can be obtained by differentiation: Clearly we have for Ψ in the domain of H_0:

$$\partial_t|_{t=0} U_0(t)\Psi = iH_0\Psi \qquad (112)$$

Now we write

$$I_t := \sum_{n\geq 1} i^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \dots \int_0^{t_2} dt_1\, V_{t_1}V_{t_2}\dots V_{t_n}U_0(t)\Psi \qquad (113)$$

We have

$$\begin{aligned}
\partial_t I_t &= i\sum_{n\geq 1} i^{n-1} \int_0^t dt_{n-1} \dots \int_0^{t_2} dt_1\, V_{t_1}V_{t_2}\dots V_{t_{n-1}}V_tU_0(t)\Psi \\
&+ \sum_{n\geq 1} i^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \dots \int_0^{t_2} dt_1\, V_{t_1}V_{t_2}\dots V_{t_n}U_0(t)iH_0\Psi. \qquad (114)
\end{aligned}$$

Taking the limit $t \to 0$ the second term tends to zero and the first term tends to zero apart from $n = 1$ (since then there are no integrals). The $n = 1$ term gives $iV\Psi$, thus, together with (112) we get that the generator of U is $H_0 + V$. □

1.2 Algebra of bounded operators on a Hilbert space

Motivation: Most algebras of interest in physics (e.g. C^*-algebras, W^*-algebras) can be realized as certain subalgebras of the algebra $B(\mathcal{H})$ of all bounded operators on some suitable Hilbert space. Important advantage: on a Hilbert space it is easy to introduce various concepts of convergence (strong-operator, weak-operator topology).

Definition 1.51 *$B(\mathcal{H})$ is the space of linear maps $A : \mathcal{H} \to \mathcal{H}$ s.t.*

$$\|A\| := \sup_{\|\Psi\|=1} \|A\Psi\| = \sup_{\|\Psi\|=1, \|\Phi\|=1} |\langle \Phi, A\Psi \rangle| < \infty. \tag{115}$$

Lemma 1.52 *(Basic properties):*

1. *$B(\mathcal{H})$ is a normed complex vector space which is complete. (Banach space).*
2. *$B(\mathcal{H})$ is equipped with operator product $B(\mathcal{H}) \cdot B(\mathcal{H}) \subset B(\mathcal{H})$. We have*

   $$\|AB\| \leq \|A\| \, \|B\|. \tag{116}$$

   i.e. $B(\mathcal{H})$ is a Banach algebra (B-algebra).
3. *$B(\mathcal{H})$ is equipped with $*$-operation $B(\mathcal{H})^* \subset B(\mathcal{H})$. We have*

   $$\|A^*\| = \|A\| \tag{117}$$

   i.e. $B(\mathcal{H})$ is a Banach* algebra (B^*-algebra).
4. *C^*-property:*

   $$\|A^*A\| = \|A\|^2. \tag{118}$$

   i.e. $B(\mathcal{H})$ is a C^*-algebra.

Proof. (Of the C^*-property). On the one hand

$$\|A^*A\| = \sup_{\Phi,\Psi \in \mathcal{H}_1} |\langle \Phi, A^*A\Psi \rangle| \leq \sup_{\Phi,\Psi \in \mathcal{H}_1} \|A\Psi\| \, \|A\Phi\| = \|A\|^2. \tag{119}$$

On the other hand

$$\|A^*A\| \geq \sup_{\Phi \in \mathcal{H}_1} |\langle \Phi, A^*A\Phi \rangle| = \|A\|^2. \tag{120}$$

□

Basic terminology in the theory of bounded operators:

- self-adjoint: $A = A^*$.
- positive: $(A \geq 0)$ if $\langle \Phi, A\Phi \rangle \geq 0$, $\Phi \in \mathcal{H}$. (Positive eigenvalues).

- projection: $A^* = A = A^2$.
- isometry: $\|A\Phi\| = \|\Phi\|$, $\Phi \in \mathcal{H}$. (Equivalently, $A^*A = 1$).
- partial isometry $A^*A = E$, E-projection. (Then also $AA^* = F$, F projection).
- unitary: $A^*A = AA^* = 1$.
- finite rank: $\dim(A\mathcal{H}) = n < \infty$.
- compact operators $\mathcal{K}(\mathcal{H})$: A maps bounded sets into pre-compact. Equivalently, on a separable Hilbert space, $\|A - A_n\| < \varepsilon$ for operators A_n of finite rank n and sufficiently large n (dep. on ε).
- Hilbert-Schmidt $\mathcal{K}_2(\mathcal{H})$: $\|A\|_2 := \big(\mathrm{Tr}\, A^*A\big)^{1/2} < \infty$.
- Trace-class $\mathcal{K}_1(\mathcal{H})$: $A = B^*C$, B, C are Hilbert-Schmidt. If A positive, $\mathrm{Tr}\, A < \infty$.

Useful facts:

- $A \geq 0$ iff there is a (non-unique) B s.t. $A = B^*B$. If we require that $B \geq 0$ then it is unique and we write $B = \sqrt{A}$.
- polar decomposition: $A = U|A|$, where $|A| := \sqrt{A^*A}$ and U partial isometry. U^*U projection onto $\overline{A^*\mathcal{H}}$, UU^* projection onto $\overline{A\mathcal{H}}$. Decomposition is unique.

Let us look at $B(\mathcal{H})$ as an abstract algebra (defined by its relations) and consider its representations:

- The defining representation of $B(\mathcal{H})$ is denoted $(\iota, \mathcal{H})$, i.e.

$$\iota(A)\Phi := A\Phi, \quad A \in B(\mathcal{H}), \quad \Phi \in \mathcal{H}. \tag{121}$$

- Note that $\mathcal{K}_2(\mathcal{H})$, equipped with the scalar product

$$\langle H_1 | H_2 \rangle = \mathrm{Tr}\, H_1^* H_2, \tag{122}$$

  is a Hilbert space. Also $\mathcal{K}_2(\mathcal{H})$ is a left and right $*$-ideal in $B(\mathcal{H})$, that is

$$B(\mathcal{H}) \cdot \mathcal{K}_2(\mathcal{H}) = \mathcal{K}_2(\mathcal{H}) \cdot B(\mathcal{H}) = \mathcal{K}_2(\mathcal{H}), \quad \mathcal{K}_2(\mathcal{H})^* = \mathcal{K}_2(\mathcal{H}). \tag{123}$$

  Thus one can define a $*$-representation of $B(\mathcal{H})$ in $\mathcal{K}_2(\mathcal{H})$ as follows:

$$\pi_{\mathrm{HS}}(A)|H\rangle := |AH\rangle. \tag{124}$$

  Note that

$$\langle H | \pi_{\mathrm{HS}}(A) H \rangle = \langle H | AH \rangle = \mathrm{Tr}\, H^* A H = \mathrm{Tr}\, H H^* A \tag{125}$$

  Note that HH^* is positive and $\mathrm{Tr} HH^* < \infty$. If it is normalized, i.e. $\mathrm{Tr}\, HH^* = 1$, then $\rho := HH^*$ is a density matrix. Hence all mixed states in QM can be described in the Hilbert space formalism using this representation.

Remark 1.53 *An abstract state ω (positive, normalized, linear functional) on a C^*-algebra is called pure if the equation*

$$\omega = p\omega' + (1-p)\omega'', \quad \text{where} \quad 0 < p < 1, \quad \omega', \omega'' \quad \text{states}, \tag{126}$$

has only one solution: $\omega = \omega' = \omega''$. General fact: ω is pure iff its GNS representation π_ω is irreducible. In an irreducible representation the physicists' definition of pure states as $\rho_{\text{pure}} = |\Psi\rangle\langle\Psi|$ and mixed states as $\rho_{\text{mixed}} = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$ works.

Remark 1.54 *In terms of Theorem 1.20 (GNS construction) the situation is the following: Consider a state $\omega(A) = \operatorname{Tr} \rho A$, $A \in B(\mathcal{H})$, where ρ is a density matrix (mixed in the physicists' sense). This state induces a cyclic representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ s.t.*

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle. \tag{127}$$

This representation is unitarily equivalent to a subrepresentation of π_{HS}. The isometry $V : \mathcal{H}_\omega \to \mathcal{K}_2(\mathcal{H})$ given by

$$V\pi_\omega(A)\Omega_\omega = |A\sqrt{\rho}\rangle \tag{128}$$

satisfies $V\pi_\omega(A) = \pi_{\text{HS}}(A)V$. Hence π_{HS} is reducible.

- Pathological representations/states: By the Hahn-Banach theorem there exist positive, linear and normalized functionals σ on $B(\mathcal{H})$ s.t. $\sigma(C) = 0$ for any $C \in \mathcal{K}(\mathcal{H})$ but $\sigma(1) = 1$. The GNS construction gives a representation π_σ which maps all compact operators to zero.

  Also $\sigma(\,\cdot\,)$ is 'less continuous' than $\omega(\,\cdot\,) = \operatorname{Tr} \rho(\,\cdot\,)$. Any state on a C^*-algebra is continuous w.r.t. the norm topology, but not necessarily in terms of the weak topology (i.e. convergence of matrix elements).

$$1 = \sigma(1) = \sigma(\lim_{N\to\infty} \sum_{n=0}^{N} |e_n\rangle\langle e_n|) = \lim_{N\to\infty} \sigma(|e_n\rangle\langle e_n|) = 0. \tag{129}$$

  On the other hand

$$\begin{aligned} 1 &= \omega(1) = \operatorname{Tr} \rho(\lim_N \sum_{n=0}^{N} |e_n\rangle\langle e_n|) = \sum_\ell \lim_N \sum_{n=0}^{N} \langle e'_\ell, \sqrt{\rho} e_n\rangle\langle e_n, \sqrt{\rho} e'_\ell\rangle \\ &= \lim_N \sum_\ell \sum_{n=0}^{N} \langle e'_\ell, \sqrt{\rho} e_n\rangle\langle e_n, \sqrt{\rho} e'_\ell\rangle = \lim_N \operatorname{Tr} \rho(\sum_{n=0}^{N} |e_n\rangle\langle e_n|) \end{aligned} \tag{130}$$

  To exchange $\lim_N$ with $\sum_\ell$ we used the dominated convergence theorem and the bound

$$\langle e'_\ell, \sqrt{\rho} \sum_{n=0}^{N} e_n\rangle\langle e_n, \sqrt{\rho} e'_\ell\rangle \leq \langle e'_\ell, \rho e'_\ell\rangle. \tag{131}$$

Let us consider more systematically various notions of convergence in $B(\mathcal{H})$. A sequence $\{A_n \in B(\mathcal{H})\}_{n\in\mathbb{N}}$ is said to be convergent to $A \in B(\mathcal{H})$ in:

(a) weak operator topology ("weakly") if $\langle \Psi, (A - A_n)\Phi \rangle \to 0$ for any $\Psi, \Phi \in \mathcal{H}$.

(b) strong operator topology ("strongly") if $\|(A - A_n)\Psi\| \to 0$ for any $\Psi \in \mathcal{H}$.

(c) norm if $\|A - A_n\| \to 0$.

For example, $\lim_{N\to\infty} \sum_{n=1}^{N} |e_n\rangle\langle e_n| = 1$ exists in weak and strong operator topology, but not in norm (if $\dim \mathcal{H} = \infty$). In general $(c) \Rightarrow (b) \Rightarrow (a)$ but the converse implications do not hold.

Definition 1.55 *A positive linear and normalized functional $\omega : B(\mathcal{H}) \to \mathbb{C}$ (state) is called normal if for every sequence of projections $Q_n, n \in \mathbb{N}$, which converges strongly to some projection Q one has*

$$\omega(\lim_n Q_n) = \omega(Q) = \lim_n \omega(Q_n). \tag{132}$$

Note: σ is not normal in this sense. ($\mathcal{H}$ is assumed to be separable here. For non-separable $\mathcal{H}$ one has to use generalized sequences ('nets') $\{Q_i\}_{i\in\mathbb{I}}$. Here $\mathbb{I}$ is an index set together with a partial ordering (reflexive, transitive amd antisymmetric) which satisfies: For any $i, i' \in \mathbb{I}$ there is j s.t. $j > i$, $j > i'$).

Proposition 1.56 *[1] Let ω be a normal state on $B(\mathcal{H})$. Then there exists a density matrix ρ_ω s.t.*

$$\omega(A) = \mathrm{Tr}\, \rho_\omega A, \quad A \in B(\mathcal{H}). \tag{133}$$

It turns out that topological and algebraic concepts are closely tied for $*$-subalgebras of $B(\mathcal{H})$:

Theorem 1.57 *[1] (von Neumann bicommutant theorem) Let $\mathcal{A}$ be a unital $*$-algebra of operators on a Hilbert space. Then $\mathcal{A}$ is dense in $\mathcal{A}''$ in the weak and strong topology.*

Remark 1.58 *We note/recall the following:*

1. *The commutant of $\mathcal{A}$ in $B(\mathcal{H})$ is defined as follows:*

$$\mathcal{A}' = \{\, B \in B(\mathcal{H}) \,|\, [B, A] = 0 \;\textit{for all}\; A \in \mathcal{A} \,\}. \tag{134}$$

2. *A unital $*$-algebra of operators on a Hilbert space s.t. $\mathcal{A}'' = \mathcal{A}$ is called a von Neumann algebra. In particular, it is a C^*-algebra.*

3. *For separable $\mathcal{H}$ it suffices to add limits of strongly convergent sequences to obtain the strong closure of a $*$-algebra. (Nets not needed).*

1.3 Weyl algebra for systems with infinitely many degrees of freedom

Algebraic approach is adventageous in order to perform the transition from finite to infinite systems.

- Finite systems: $\mathbb{C}^n$, $\langle\,\cdot\,,\,\cdot\,\rangle$, $\sigma(z,z')=\mathrm{Im}\langle z,z'\rangle$. Pre-Weyl algebra $\mathcal{W}$ is the free $*$-algebra generated by $W(z), z\in\mathbb{C}^n$, subject to relations

$$W(z)W(z')=e^{\frac{i}{2}\sigma(z,z')}W(z+z'),\quad W(z)^*=W(-z),\quad z\in\mathbb{C}^n. \qquad (135)$$

  Remark 1.59 *This form of Weyl relations corresponds to $W(z)=e^{i(uP+vQ)}$, $z=u+iv$ via BCH. If we wanted $W_{\mathrm{new}}(z)=e^{i(vP+uQ)}$, $z=u+iv$, that would lead to a minus sign in front of σ:*

$$W_{\mathrm{new}}(z)W_{\mathrm{new}}(z')=e^{-\frac{i}{2}\sigma(z,z')}W_{\mathrm{new}}(z+z') \qquad (136)$$

  This convention will be more convenient in the case of systems with infinitely many degrees of freedom.

- Infinite systems: infinite dimensional complex-linear space $\mathcal{D}$ with scalar product $\langle\,\cdot\,,\,\cdot\,\rangle$ (pre-Hilbert space). Define the symplectic form $\sigma(f,g)=\mathrm{Im}\,\langle f,g\rangle$, $f,g\in\mathcal{D}$. Pre-Weyl algebra $\mathcal{W}$ is the free $*$-algebra generated by $W(f), f\in\mathcal{D}$, subject to relations

$$W(f)W(g)=e^{-\frac{i}{2}\sigma(f,g)}W(f+g),\quad W(f)^*=W(-f),\quad f,g\in\mathcal{D}. \qquad (137)$$

Example 1.60 *: $\mathcal{D}=S(\mathbb{R}^d)$,*

$$\langle f,g\rangle=\int d^dx\,\overline{f(x)}g(x). \qquad (138)$$

Heuristics: $W(f)=$ "$e^{i\left(\varphi(\mathrm{Re}\,f)+\pi(\mathrm{Im}\,f)\right)}$", where

$$\varphi(g):=\int d^dx\,g(x)\varphi(x),\quad \pi(h):=\int d^dx\,h(x)\pi(x) \qquad (139)$$

are spatial means of the quantum "field operator" $\varphi(x)$ and its "canonical conjugate momentum" $\pi(x)$. The fields φ, π satisfy formally

$$[\varphi(x),\pi(y)] = i\delta(x-y)1, \qquad (140)$$
$$[\varphi(x),\varphi(y)] = [\pi(x),\pi(y)]=0. \qquad (141)$$

$\varphi(x),\pi(y)$ are not expected to be operators, but only operator valued distributions. But $\varphi(g),\pi(h)$ are expected to be operators and we have

$$[\varphi(g),\pi(h)]=i\int d^dx\,g(x)h(x)1=i\langle\overline{g},h\rangle 1. \qquad (142)$$

Example 1.61 *: $\mathcal{D} = S(\mathbb{R}^d)$,*

$$\langle f, g \rangle = \int d^d p\, \overline{f(p)} g(p). \tag{143}$$

Here $\mathbb{R}^d$ is interpreted as momentum space.
Heuristic interpretation: $W(f) = e^{\frac{i}{\sqrt{2}}(a^*(f)+a(f))}$ where

$$a^*(f) = \int d^d p\, f(p) a^*(p), \quad a(f) = \int d^d p\, \overline{f(p)} a(p). \tag{144}$$

are creation and annihilation operators of particles with momentum in the support of f. The commutation relations are

$$[a(p), a^*(q)] = \delta(p-q)1, \tag{145}$$
$$[a(p), a(q)] = [a(p), a^*(q)] = 0. \tag{146}$$

Similarly as before a priori these are only operator valued distributions. For smeared versions we have:

$$[a(g), a^*(h)] = \int d^d p\, \overline{g(p)} h(p) 1 = \langle g, h \rangle 1. \tag{147}$$

1.3.1 Fock space

We recall the definition and basic properties of a Fock space over $\mathfrak{h} := L^2(\mathbb{R}^d, d^d x)$. We have for $n \in \mathbb{N}$

$$\otimes^n \mathfrak{h} = \mathfrak{h} \otimes \cdots \otimes \mathfrak{h} = L^2(\mathbb{R}^{nd}, d^{nd} x), \tag{148}$$
$$\otimes_s^n \mathfrak{h} = S_n(\mathfrak{h} \otimes \cdots \otimes \mathfrak{h}) = L_s^2(\mathbb{R}^{nd}, d^{nd} x), \tag{149}$$
$$\otimes_s^0 \mathfrak{h} := \mathbb{C}\Omega, \text{ where } \Omega \text{ is called the vacuum vector.} \tag{150}$$

Here S_n is the symmetrization operator defined by

$$S_n = \frac{1}{n!} \sum_{\sigma \in P_n} \sigma, \text{ where } \sigma(f_1 \otimes \cdots \otimes f_n) = f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}, \tag{151}$$

P_n is the set of all permutations and $L_s^2(\mathbb{R}^{nd}, d^{nd} x)$ is the subspace of symmetric (w.r.t. permutations of variables) square integrable functions. The (symmetric) Fock space is given by

$$\Gamma(\mathfrak{h}) := \oplus_{n \geq 0} \otimes_s^n \mathfrak{h} = \oplus_{n \geq 0} L_s^2(\mathbb{R}^{nd}, d^{nd} x). \tag{152}$$

We can write $\Psi \in \Gamma(\mathfrak{h})$ in terms of its Fock space components $\Psi = \{\Psi^{(n)}\}_{n \geq 0}$. We define a dense subspace $\Gamma_{\mathrm{fin}}(\mathfrak{h}) \subset \Gamma(\mathfrak{h})$ consisting of such Ψ that $\Psi^{(n)} = 0$ except for finitely many n. Next, we define a domain

$$D := \{\, \Psi \in \Gamma_{\mathrm{fin}}(\mathfrak{h}) \,|\, \Psi^{(n)} \in S(\mathbb{R}^{nd}) \text{ for all } n \,\}. \tag{153}$$

Now, for each $p \in \mathbb{R}^d$ we define an operator $a(p) : D \to \Gamma(\mathfrak{h})$ by

$$(a(p)\Psi)^{(n)}(k_1, \dots, k_n) = \sqrt{n+1}\,\Psi^{(n+1)}(p, k_1, \dots, k_n),$$
$$\text{In particular } a(p)\Omega = 0. \tag{154}$$

Note that the adjoint of $a(p)$ is not densely defined, since formally

$$(a^*(p)\Psi)^{(n)}(k_1, \dots, k_n) = \frac{1}{\sqrt{n}} \sum_{\ell=1}^{n} \delta(p - k_\ell)\Psi^{(n-1)}(k_1, \dots, k_{\ell-1}, k_{\ell+1}, \dots, k_n) \tag{155}$$

However, $a^*(p)$ is well defined as a quadratic form on $D \times D$. Expressions

$$a(g) = \int d^d p\, a(p)\overline{g(p)}, \quad a^*(g) = \int d^d p\, a^*(p) g(p), \quad g \in S(\mathbb{R}^d), \tag{156}$$

give well-defined operators on D which can be extended to $\Gamma_{\mathrm{fin}}(\mathfrak{h})$. On this domain they act as follows

$$(a(g)\Psi)^{(n)}(k_1, \dots, k_n) = \sqrt{n+1} \int d^d p\, \overline{g}(p)\Psi^{(n+1)}(p, k_1, \dots, k_n), \tag{157}$$

$$(a^*(g)\Psi)^{(n)}(k_1, \dots, k_n) = \frac{1}{\sqrt{n}} \sum_{\ell=1}^{n} g(k_\ell)\Psi^{(n-1)}(k_1, \dots, k_{\ell-1}, k_{\ell+1}, \dots, k_n). \tag{158}$$

These expressions can be used to define $a(g), a^*(g)$ for $g \in L^2(\mathbb{R}^d)$. Since these operators leave $\Gamma_{\mathrm{fin}}(\mathfrak{h})$ invariant, one can compute on this domain:

$$[a(f), a^*(g)] = \langle f, g \rangle 1 \tag{159}$$

for $f, g \in L^2(\mathbb{R}^d)$. (Formally, this follows from $[a(p), a^*(q)] = \delta(p - q)$).

Now we are ready to define canonical fields and momenta: Let $\mu : \mathbb{R}^d \mapsto \mathbb{R}_+$ be positive, measurable function of momentum s.t. if $f \in S(\mathbb{R}^d)$ then $\mu^{1/2} f, \mu^{-1/2} f \in L^2(\mathbb{R}^d)$. (Examples: $\mu(p) = 1, \mu_m(p) = \sqrt{p^2 + m^2}$, $m \geq 0$). We set for $f, g \in S(\mathbb{R}^d)$

$$\varphi_\mu(f) := \frac{1}{\sqrt{2}}\big(a^*(\mu^{-1/2}\hat{f}) + a(\mu^{-1/2}\hat{\bar{f}})\big), \tag{160}$$

$$\pi_\mu(g) := \frac{1}{\sqrt{2}}\big(a^*(i\mu^{1/2}\hat{g}) + a(i\mu^{1/2}\hat{\bar{g}})\big), \tag{161}$$

where $\hat{f}(p) := (\mathcal{F}f)(p)$. For $\mu := \mu_m$ this is the canonical field and momentum of the free scalar relativistic quantum field theory of mass $m \geq 0$. From (159) we have

$$[\varphi_\mu(f), \pi_\mu(g)] = \frac{1}{2}\big(- \langle i\hat{\bar{g}}, \hat{f} \rangle + \langle \hat{\bar{f}}, i\hat{g} \rangle\big) = \frac{i}{2}(\langle \hat{\bar{g}}, \hat{f} \rangle + \langle \hat{\bar{f}}, \hat{g} \rangle) = i\langle \bar{f}, g \rangle, \tag{162}$$

where in the last step we made use of Plancherel theorem and

$$\langle \bar{g}, f \rangle = \int d^d x\, g(x) f(x) = \langle \bar{f}, g \rangle. \tag{163}$$

Remark 1.62 *Note that (160), (161) arise by smearing the operator-valued distributions:*

$$\varphi_{\mu_m}(x) = \frac{1}{(2\pi)^{d/2}} \int \frac{d^d k}{\sqrt{2\mu_m(k)}} \big(e^{-ikx} a^*(k) + e^{ikx} a(k)\big), \tag{164}$$

$$\pi_{\mu_m}(x) = \frac{i}{(2\pi)^{d/2}} \int d^d k \sqrt{\frac{\mu_m(k)}{2}} \big(e^{-ikx} a^*(k) - e^{ikx} a(k)\big). \tag{165}$$

Consider a unitary operator u on $\mathfrak{h}$. Then, its 'second quantization' is the following operator on the Fock space:

$$\Gamma(u)|_{\Gamma^{(n)}(\mathfrak{h})} = u \otimes \cdots \otimes u, \tag{166}$$

$$\Gamma(u)\Omega = \Omega. \tag{167}$$

where $\Gamma^{(n)}(\mathfrak{h})$ is the n-particle subspace. We have the useful relations:

$$\Gamma(u) a^*(h) \Gamma(u)^* = a^*(uh), \quad \Gamma(u) a(h) \Gamma(u)^* = a(uh). \tag{168}$$

(Note that $a^*(h)^* = a(h)$).

Consider a self-adjoint operator b on $\mathfrak{h}$. Then, its 'second quantization' is the following operator on the Fock space:

$$\mathrm{d}\Gamma(b)|_{\Gamma^{(n)}(\mathfrak{h})} = \sum_{i=1}^{n} 1 \otimes \cdots b \cdots \otimes 1, \tag{169}$$

$$\mathrm{d}\Gamma(b)\Omega = 0. \tag{170}$$

Suppose that $b = b(k)$ is a multiplication operator in momentum space on $\mathfrak{h} = L^2(\mathbb{R}^d)$. Then as an equality of quadratic forms on $D \times D$ we have

$$\mathrm{d}\Gamma(b) = \int d^d k\, b(k) a^*(k) a(k). \tag{171}$$

Moreover, suppose that $U(t) = e^{itb}$. Then

$$\Gamma(U(t)) = e^{it\mathrm{d}\Gamma(b)}. \tag{172}$$

1.3.2 Representations of the Weyl algebra

Now we are ready to define several representatons of $\mathcal{W}$ on $\Gamma(\mathfrak{h})$. We set $\mathcal{D} = S(\mathbb{R}^d)$ and $\sigma(f, g) := \mathrm{Im}\,\langle f, g\rangle$ with standard scalar product in $L^2(\mathbb{R}^d)$:

Definition 1.63 *Let μ be as above. The corresponding Fock space representation of $\mathcal{W}$ is given by*

$$\rho_\mu(W(f)) = e^{i(\varphi_\mu(\mathrm{Re}\, f) + \pi_\mu(\mathrm{Im}\, f))}. \tag{173}$$

In terms of creation and annihilation operators, we have

$$\rho_\mu(W(f)) = e^{\frac{i}{\sqrt{2}}(a^*(\hat{f}_\mu) + a(\hat{f}_\mu))}, \tag{174}$$

where $\hat{f}_\mu(p) := (\mu^{-\frac{1}{2}} \widehat{\mathrm{Re} f} + i\mu^{\frac{1}{2}} \widehat{\mathrm{Im} f})(p)$. Note that for $\mu = 1$ we have $\hat{f}_\mu(p) = \hat{f}(p)$ and thus we reproduce Examples 1.60,1.61.

Theorem 1.64 *Representations ρ_{μ_m} are faithful, irreducible and $\rho_{\mu_{m_1}}$ is not unitarily equivalent to $\rho_{\mu_{m_2}}$ for $m_1 \neq m_2$. (So Stone-von Neumann uniqueness theorem does not hold for systems with infinitely many degrees of freedom).*

Proof. See Theorem X.46 of [7].

1.3.3 Symmetries

Symmetries are represented by their automorphic action on the algebra.

Definition 1.65 *Let $(\mathcal{D}, \sigma)$ be a symplectic space. A symplectic transformation S is a linear bijection $S : \mathcal{D} \to \mathcal{D}$ s.t.*

$$\sigma(Sf, Sg) = \sigma(f, g), \quad f, g \in \mathcal{D}. \tag{175}$$

Note that S^{-1} is also a symplectic transformation.

Fact: Every symplectic transformation induces an automorphism of $\mathcal{W}$ according to the relation:

$$\alpha_S(W(f)) = W(Sf), \quad f \in \mathcal{D}. \tag{176}$$

Proposition 1.66 *Let S be a symplectic transformation s.t. also $\|\widehat{(Sf)}_\mu\| = \|\hat{f}_\mu\|$. (For $\mu = 1$ this is just unitarity of S w.r.t. the scalar product in $L^2(\mathbb{R}^d)$). Then there exists a unitary operator $U_{\mu,S}$ on $\Gamma(\mathfrak{h})$ s.t.*

$$U_{\mu,S}\rho_\mu(W)U_{\mu,S}^* = \rho_\mu(\alpha_S(W)), \quad W \in \mathcal{W}, \tag{177}$$

and $U_{\mu,S}\Omega = \Omega$. (Converse also true).

Proof. We skip the index μ. Since we know that $\mathcal{W}$ acts irreducibly on $\Gamma(\mathfrak{h})$, we have that

$$D := \{\, \rho(W)\Omega \,|\, W \in \mathcal{W} \,\} \tag{178}$$

is dense in $\Gamma(\mathfrak{h})$. On this domain we set

$$U_S\rho(W(f))\Omega = \rho(W(Sf))\Omega, \tag{179}$$

and extend by linearity to $\mathcal{W}$. By invertibility of S this has a dense range. We check that it is an isometry on this domain. For this it suffices to verify

$$\langle U_S\rho(W(f))\Omega, U_S\rho(W(g))\Omega\rangle = \langle \rho(W(f))\Omega, \rho(W(g))\Omega\rangle. \tag{180}$$

We have

$$\begin{aligned} \text{l.h.s.} &= \langle \rho(W(Sf))\Omega, \rho(W(Sg))\Omega\rangle = \langle \Omega, \rho(W(-Sf)W(Sg))\Omega\rangle \\ &= e^{\frac{i}{2}\mathrm{Im}\langle f,g\rangle}\langle \Omega, \rho(W(S(g-f)))\Omega\rangle, \end{aligned} \tag{181}$$

where we made use of the fact that S is symplectic. Let us set $h := S(g-f)$. We have

$$\begin{aligned}\langle\Omega, \rho(W(h))\Omega\rangle &= \langle\Omega, e^{\frac{i}{\sqrt{2}}(a^*(\hat{h}_\mu)+a(\hat{h}_\mu))}\Omega\rangle \\ &= e^{-\frac{1}{2}\|\hat{h}_\mu\|^2}\langle\Omega, e^{\frac{i}{\sqrt{2}}a^*(\hat{h}_\mu)}e^{\frac{i}{\sqrt{2}}a(\hat{h}_\mu)}\Omega\rangle \\ &= e^{-\frac{1}{2}\|\hat{h}_\mu\|^2} = e^{-\frac{1}{2}\|(\widehat{S(g-f)})_\mu\|^2} = e^{-\frac{1}{2}\|(\widehat{g-f})_\mu\|^2} \\ &= \text{r.h.s. of (180)},\end{aligned} \tag{182}$$

where we used Baker-Campbell-Hausdorff (which can be justified by expanding exponentials into convergent series) and the additional assumption on S.

Now the converse: suppose that α_S is unitarily implemented in ρ_μ by a unitary $U_{\mu,S}$ s.t. $U_{\mu,S}\Omega = \Omega$. Then, in particular,

$$\begin{aligned}\langle\Omega, \rho_\mu(W(Sf))\Omega\rangle &= \langle\Omega, \rho_\mu(\alpha_S(W(f)))\Omega\rangle \\ &= \langle\Omega, U_{\mu,S}\rho_\mu(W(f))U_{\mu,S}^*\Omega\rangle = \langle\Omega, \rho_\mu(W(f))\Omega\rangle.\end{aligned} \tag{183}$$

Hence,

$$e^{-\frac{1}{2}\|(\widehat{Sf})_\mu\|^2} = e^{-\frac{1}{2}\|\widehat{f}_\mu\|^2} \tag{184}$$

which concludes the proof. □

1.3.4 Symmetries in the case $\mu = 1$ (”non-local” quantum field)

We set $\mathcal{D} = S(\mathbb{R}^d)$, $\langle f, g\rangle = \int d^dx\overline{f(x)}g(x)$, $\sigma(f,g) = \mathrm{Im}\,\langle f, g\rangle$, $m > 0$.

- Note that any unitary u on $\mathfrak{h} = L^2(\mathbb{R}^d)$, which preserves $\mathcal{D}$, gives rise to a symplectic transformation $S = u|_{\mathcal{D}}$.
- By Proposition 1.66, the automorphism induced by S is unitarily implemented on $\Gamma(\mathfrak{h})$.
- A natural candidate for the implementing unitary is $\Gamma(u)$.

1. Space translations: $(S_af)(x) = f(x-a)$ (or $\widehat{(S_af)}(p) = e^{-iak}\hat{f}(p)$). Obviously

$$\langle(S_af), (S_ag)\rangle = \int d^dx\,\overline{f(x-a)}g(x-a) = \langle f, g\rangle. \tag{185}$$

(This implies that S_a is symplectic). The implementing unitary is $U(a) = \Gamma(e^{-ipa}) = e^{-ia\mathrm{d}\Gamma(p)}$, where ’p’ means the corresponding multiplication operator on $L^2(\mathbb{R}^d, d^dp)$. $P := \mathrm{d}\Gamma(p) = \int d^3k\,ka^*(k)a(k)$ can be called the ’total momentum operator’. Indeed by (168):

$$\begin{aligned}\alpha_a(W(f)) &= W(S_af) = e^{\frac{i}{\sqrt{2}}\left(a^*(e^{-iap}\hat{f})+a(e^{-iap}\hat{f})\right)} \\ &\quad = \Gamma(e^{-ipa})e^{\frac{i}{\sqrt{2}}\left(a^*(\hat{f})+a(\hat{f})\right)}\Gamma(e^{-ipa})^*.\end{aligned} \tag{186}$$

2. Rotations: $(S_R f)(x) = f(R^{-1}x)$, $R \in SO(d)$.

$$\langle (S_R f), (S_R g) \rangle = \int d^d x \, \overline{f(R^{-1}x)} g(R^{-1}x) = \langle f, g \rangle \tag{187}$$

The implementing unitary is $U(R) = \Gamma(u_R)$, where $(u_R g)(x) = g(R^{-1}x)$ is a unitary representation of rotations on $L^2(\mathbb{R}^d)$.

3. Time translations: $\widehat{(S_t f)}(p) = e^{it\omega(p)} \hat{f}(p)$ where $\omega(p)$ is a reasonable dispersion relation of a particle. Since we want to build a relativistic theory, we set $\omega(p) = \sqrt{p^2 + m^2}$, $m > 0$. Clearly:

$$\langle (S_t f), (S_t g) \rangle = \langle f, g \rangle. \tag{188}$$

The implementing unitary is $U(t) = \Gamma(e^{it\omega(p)}) = e^{it \mathrm{d}\Gamma(\omega(p))}$, where

$$H := \mathrm{d}\Gamma(\omega(p)) = \int d^3 k \, \omega(k) a^*(k) a(k), \tag{189}$$

can be called the 'total energy operator' or the Hamiltonian.

Remark 1.67 *Note that $f_t := S_{-t} f$ satisfies the Schrödinger equation:*

$$i \partial_t f_t(x) = \omega(-i\nabla) f_t. \tag{190}$$

4. Lorentz transformations

   - Minkowski spacetime: $(\mathbb{R}^{d+1}, g)$, $g = (1, -1, -1, -1)$.
   - Lorentz group: $\mathcal{L} = O(1, d) = \{ \Lambda \in GL(1+d) \, | \, \Lambda g \Lambda^T = g \}$
   - Proper Lorentz group: $\mathcal{L}_+ = SO(1, d) = \{ \Lambda \in O(1, d) \, | \, \det \Lambda = 1 \}$ (preserves orientation).
   - Ortochronous Lorentz group: $\mathcal{L}^\uparrow = \{ \Lambda \in O(1, d) \, | \, e^T \Lambda e \geq 0 \}$, where $e = (1, 0, 0, 0)$. (Preserves the direction of time)
   - Proper ortochronous Lorentz group: $\mathcal{L}_+^\uparrow = \mathcal{L}^\uparrow \cap \mathcal{L}_+$ is a symmetry group of the SM of particle physics.
   - The full Lorentz group consists of four disjoint components:

   $$\mathcal{L} = \mathcal{L}_+^\uparrow \cup \mathcal{L}_+^\downarrow \cup \mathcal{L}_-^\uparrow \cup \mathcal{L}_-^\downarrow \tag{191}$$

   For $d = 3$ they can be defined using time reversal $T(t, x) = (-t, x)$ and parity $P(t, x) = (t, -x)$ transformations:

   $$\mathcal{L}_+^\downarrow = TP\mathcal{L}_+^\uparrow, \quad \mathcal{L}_-^\uparrow = P\mathcal{L}_+^\uparrow, \quad \mathcal{L}_-^\downarrow = T\mathcal{L}_+^\uparrow. \tag{192}$$

Now we set

$$(S_\Lambda f)(p) = \sqrt{\frac{\omega(\Lambda^{-1}p)}{\omega(p)}} f(\Lambda^{-1}p), \quad f \in \mathcal{D}, \tag{193}$$

where $\Lambda^{-1}p$ is defined by $\Lambda^{-1}(\omega(p), p) = (\omega(\Lambda^{-1}p), \Lambda^{-1}p)$. We have

$$\langle (S_\Lambda f), (S_\Lambda f) \rangle = \langle f, g \rangle. \tag{194}$$

This can be shown (Homework) using that $\frac{d^d p}{\omega(p)}$ is a Lorentz invariant measure (unique for a fixed m and normalization, see Theorem IX.37 of [7]). Formally

$$\int d^{d+1}\tilde{p}\, \delta(\tilde{p}^2 - m^2)\theta(\tilde{p}^0) F(\tilde{p}) = \int \frac{d^d p}{2\omega(p)} F(\omega(p), p), \tag{195}$$

where $\tilde{p} = (p^0, p)$, $\tilde{p}^2 = (p^0)^2 - p^2$.

S_Λ arises by restriction to $\mathcal{D}$ of a unitary representation u_Λ of $\mathcal{L}_+^\uparrow$ acting on $\mathfrak{h} = L^2(\mathbb{R}^d)$ by formula (193). The implementing unitary is $U(\Lambda) := \Gamma(u_\Lambda)$.

5. Poincaré transformations: The (proper ortochronous) Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^{d+1} \rtimes \mathcal{L}_+^\uparrow$ is a set of pairs $(\tilde{x}, \Lambda)$ with the multiplication:

$$(\tilde{x}_1, \Lambda_1)(\tilde{x}_2, \Lambda_2) = (\tilde{x}_1 + \Lambda_1 \tilde{x}_2, \Lambda_1 \Lambda_2). \tag{196}$$

It acts naturally on $\mathbb{R}^{d+1}$ by $(\tilde{x}, \Lambda)\tilde{y} = \Lambda \tilde{y} + \tilde{x}$. (Here we set $\tilde{x} = (t, x)$).

Note that $(\tilde{x}, \Lambda) = (\tilde{x}, I)(0, \Lambda)$. Accordingly, we define

$$S_{(\tilde{x},\Lambda)} := S_{\tilde{x}} \circ S_\Lambda = S_t \circ S_x \circ S_\Lambda \tag{197}$$

as a symplectic transformation on $\mathcal{D}$ corresponding to $(\tilde{x}, \Lambda)$. We still have to check if $(\tilde{x}, \Lambda) \mapsto \alpha_{S_{(\tilde{x},\Lambda)}}$ is a representation of a group, that is whether

$$\alpha_{S_{(\tilde{x}_1,\Lambda_1)}} \circ \alpha_{S_{(\tilde{x}_2,\Lambda_2)}} = \alpha_{S_{(\tilde{x}_1,\Lambda_1)(\tilde{x}_2,\Lambda_2)}}. \tag{198}$$

We use the fact that all these automorphisms can be implemented in the (faithful) representation $\rho_{\mu=1}$. We have

$$\begin{aligned} \rho_1(\alpha_{(\tilde{x},\Lambda)}(W(f))) &= \rho_1\big(W(S_{(\tilde{x},\Lambda)} f)\big) = \rho_1\big(W(S_t \circ S_x \circ S_\Lambda f)\big) \\ &= U(t)U(x)U(\Lambda)\rho_1\big(W(f)\big)\big(U(t)U(x)U(\Lambda)\big)^* \end{aligned} \tag{199}$$

To verify (198) it suffices to check that

$$\begin{aligned} U(\tilde{x}, \Lambda) &:= U(t)U(x)U(\Lambda) = \Gamma(e^{i\omega(p)t})\Gamma(e^{-ipx})\Gamma(u_\Lambda) \\ &= \Gamma(e^{i\omega(p)t} e^{-ipx} u_\Lambda) \end{aligned} \tag{200}$$

is a unitary representation of $\mathcal{P}_+^\uparrow$ on $\Gamma(\mathfrak{h})$. For this it suffices that

$$u_{(\tilde{x},\Lambda)} = e^{i\omega(p)t} e^{-ipx} u_\Lambda \tag{201}$$

is a unitary representation of $\mathcal{P}_+^\uparrow$ on $\mathfrak{h} = L^2(\mathbb{R}^d)$. (Homework).

Summing up, for any $m > 0$ we have a representation $P_+^\uparrow \ni (\tilde{x}, \Lambda) \mapsto \alpha^{(m,\mu=1)}_{(\tilde{x},\Lambda)}$ of the Poincaré group in $\mathrm{Aut}\,\mathcal{W}$. In the representation $\rho_{\mu=1}$ automorphisms $\alpha^{(m,\mu=1)}$ are unitarily implemented by a representation $P_+^\uparrow \ni (\tilde{x}, \Lambda) \mapsto U(\tilde{x}, \Lambda)$.

Nevertheless, $(\mathcal{W}, \alpha^{(m,\mu=1)}, \rho_{\mu=1})$ does not give rise to a decent (local) relativistic QFT. Problem with causality:

- $W(f)$, $\mathrm{supp} f \subset O$ should be an observable localized in an open bounded region $O \subset \mathbb{R}^d$ at $t = 0$.
- $\alpha_t(W(f))$ should be localized in $\{O + |t|\vec{n}\,,\, |\vec{n}| = 1\}$ in a causal theory.
- However, $\alpha_t(W(f)) = W(S_t f)$, $\widehat{(S_t f)}(p) = e^{i\omega(p)t}\hat{f}(p)$ thus $S_t f$ is not compactly supported. (Infinite propagation speed of the Schrödinger equation). In fact, since $e^{i\omega(p)t}$ is not entire analytic (cut at $p = im$), its inverse Fourier transform cannot be a compactly supported distribution (see Theorem IX.12 of [7]).

1.3.5 Symmetries in the case $\mu(p) = \sqrt{p^2 + m^2}$ ("local" quantum field)

We set $\mathcal{D} = S(\mathbb{R}^d)$, $\langle f, g\rangle = \int d^d x \overline{f}(x) g(x)$, $\sigma(f, g) = \mathrm{Im}\,\langle f, g\rangle$.

- Recall that we need symplectic transformations S s.t. $\|(Sf)_\mu\| = \|f_\mu\|$, where $\hat{f}_\mu(p) := (\mu^{-\frac{1}{2}}\widehat{\mathrm{Re} f} + i\mu^{\frac{1}{2}}\widehat{\mathrm{Im} f})(p)$.
- Note that $\|(Sf)_\mu\| = \|f_\mu\|$ does not imply in this case that S is symplectic.
- Strategy: Take the unitary u on $\mathfrak{h}$ corresponding to a given symmetry (which we know from $\mu = 1$ case) and find S s.t. $u f_\mu = (Sf)_\mu$. Then check that S is symplectic.

1. Space translations: We have $\widehat{\mathrm{Re}(S_a f)}(p) = \widehat{S_a \mathrm{Re} f}(p) = e^{-iap}\widehat{\mathrm{Re} f}(p)$ and analogously for Im. Thus $\widehat{(S_a f)}_\mu(p) = e^{-iap}\hat{f}_\mu(p)$ and therefore $\|\widehat{(S_a f)}_\mu\| = \|\hat{f}_\mu\|$ so the symmetry is unitarily implemented. The implementing unitary is the same as in the $\mu = 1$ case.
2. Rotations: Again $\widehat{\mathrm{Re}(S_R f)}(p) = \widehat{S_R \mathrm{Re} f}(p) = u_R \widehat{\mathrm{Re} f}(p)$ and analogously for Im. Since μ is rotation invariant (depends only on p^2), we have $u_R \mu(p) u_R^* = \mu(p)$ and therefore $\widehat{S_R f}_\mu(p) = (u_R \hat{f}_\mu)(p)$. Thus $\|\widehat{(S_R f)}_\mu\| = \|\hat{f}_\mu\|$ so the symmetry is unitarily implemented. The implementing unitary is the same as in the $\mu = 1$ case.
3. Time translations: First note that $\widehat{(S'_t f)}(p) = e^{it\omega(p)}\hat{f}(p)$ does NOT satisfy the additional condition. For example, for f real we have

$$\widehat{(S'_t f)}_\mu(p) = \big(\mu^{-\frac{1}{2}}(p)\cos(\omega(p)t) + i\mu^{\frac{1}{2}}(p)\sin(\omega(p)t)\big)\hat{f}(p). \tag{202}$$

The L^2 norm of this $(S'_t f)_\mu$ does depend on t. (Thus $\alpha_{S'_t}$ is not implemented in this representation by unitaries preserving the vacuum).

Instead, we consider the following group of transformations:

$$\begin{aligned}(S_t f)(x) &= (\cos(t\mu) + i\mu^{-1}\sin(t\mu))\mathrm{Re}\, f(x) \\ &\quad + i(\cos(t\mu) + i\mu\sin(t\mu))\mathrm{Im}\, f(x).\end{aligned} \tag{203}$$

Think of μ as a function of $p^2 = -\nabla_x^2$. Thus we can compute real and imaginary parts as for functions:

$$\mathrm{Re}\,(S_t f) = \cos(t\mu)\mathrm{Re}\, f - \mu\sin(t\mu)\mathrm{Im}\, f, \tag{204}$$

$$\mathrm{Im}\,(S_t f) = \mu^{-1}\sin(t\mu))\mathrm{Re}\, f + \cos(t\mu)\mathrm{Im}\, f \tag{205}$$

This is a symplectic transformation

$$\sigma(S_t f, S_t g) = \langle \mathrm{Re}\,(S_t f), \mathrm{Im}(S_t g)\rangle - (f \leftrightarrow g) \tag{206}$$

We note that terms involving $\mathrm{Re}\, f \mathrm{Re}\, g$ and $\mathrm{Im}\, f \mathrm{Im}\, g$ cancel because are invariant under $(f \leftrightarrow g)$. The remaining two terms give

$$\begin{aligned}\sigma(S_t f, S_t g) &= \langle \cos^2(t\mu)\mathrm{Re}\, f, \mathrm{Im}\, g\rangle - \langle \sin^2(t\mu)\mathrm{Im}\, f, \mathrm{Re}\, g\rangle - (f \leftrightarrow g) \\ &= \langle \mathrm{Re}\, f, \mathrm{Im}\, g\rangle - \langle \mathrm{Im}\, f, \mathrm{Re}\, g\rangle = \sigma(f, g).\end{aligned} \tag{207}$$

Next, we check $\|(S_t f)_\mu\| = \|f_\mu\|$:

$$\begin{aligned}(S_t f)_\mu &= \mu^{-\frac{1}{2}}\mathrm{Re}\,(S_t f) + i\mu^{\frac{1}{2}}\mathrm{Im}\,(S_t f) \\ &= \big(\cos(t\mu) + i\sin(t\mu)\big)\big(\mu^{-\frac{1}{2}}\mathrm{Re}\, f + i\mu^{\frac{1}{2}}\mathrm{Im}\, f\big) \\ &= e^{i\mu t} f_\mu.\end{aligned} \tag{208}$$

Hence clearly $\|(S_t f)_\mu\| = \|f_\mu\|$ and this group of automorphisms is unitarily implemented on Fock space by unitaries preserving the vacuum. They are given by $U(t) = \Gamma(e^{i\mu t}) = e^{i\mathrm{d}\Gamma(\mu(p))}$. Thus the Hamiltonian is $\mathrm{d}\Gamma(\mu(p)) = \int d^d k\, \mu(k) a^*(k) a(k)$.

Remark 1.68 *Note that $f_t := (S_t f)$ in (203) is the unique solution of the Klein-Gordon equation:*

$$(\partial_t^2 - \nabla_x^2 + m^2) f_t(x) = 0 \tag{209}$$

with the initial conditions $f_{t=0}(x) = f(x)$ and $(\partial_t f)_{t=0}(x) = (\nabla_x^2 - m^2)\mathrm{Im}\, f(x) + i\mathrm{Re}\, f(x)$. In contrast to the Schrödinger equation, KG equation has finite propagation speed: If $supp f_{t=0}, supp\, \partial_t f_{t=0} \subset O$ then $supp\, f_t \subset \{O + |t|\vec{n}\,, |\vec{n}| = 1\}$. This theory has good chances to be local.

4. Lorentz transformations: There exist symplectic transformations S_Λ which satisfy $\|(S_\Lambda f)_\mu\| = \|(S_\Lambda)_\mu\|$ and preserve localization (for $f \in C_0^\infty(\mathbb{R}^d)$ we have $(S_\Lambda f) \in C_0^\infty(\mathbb{R}^d)$) (Homework).

5. Poincaré transformations: For $(\tilde{x}, \Lambda) \in \mathcal{P}_+^\uparrow$ we define

$$S_{(\tilde{x},\Lambda)} := S_{\tilde{x}} \circ S_\Lambda = S_t \circ S_x \circ S_\Lambda \tag{210}$$

as a symplectic transformation on $\mathcal{D}$ corresponding to $(\tilde{x}, \Lambda)$. Obviously, $\|(S_{(\tilde{x},\Lambda)} f)_\mu\| = \|f_\mu\|$, since the individual factors satisfy this. (We note that S_x is as in the $\mu = 1$ case but S_t, S_Λ are different). The proof that $(\tilde{x}, \Lambda) \mapsto \alpha_{S_{(\tilde{x},\Lambda)}}$ is a representation of a group goes as in $\mu = 1$ case, exploiting that these automorphisms are implemented on Fock space by the same group of unitaries as in the $\mu = 1$ case.

Summing up, for any $m \geq 0$ we have a representation $P_+^\uparrow \ni (\tilde{x}, \Lambda) \mapsto \alpha^{(m)}_{(\tilde{x},\Lambda)}$ of the Poincaré group in $\mathrm{Aut}\,\mathcal{W}$. In the representation ρ_{μ_m} automorphisms $\alpha^{(m)}$ are unitarily implemented by the representation $P_+^\uparrow \ni (\tilde{x}, \Lambda) \mapsto U(\tilde{x}, \Lambda)$, the same as in the $\mu = 1$ case. Time evolution is governed by the KG equation which has finite propagation speed and Lorentz transformations act locally: we expect that $(\mathcal{W}, \alpha^{(m)}, \rho_{\mu_m})$ gives rise to a local (causal) relativistic QFT.

1.3.6 Spectrum condition (positivity of energy)

In this subsection we study the spectrum of the group of unitaries on $\Gamma(\mathfrak{h})$ implementing translations in ρ_μ, $\mu = \sqrt{p^2 + m^2}$. (The discussion below is equally valid for $\rho_{\mu=1}$ since $\mu_m(p) = \omega(p)$, hence unitaries implementing translations are the same in both representations).

$$U(t, x) = e^{iHt - iPx} = e^{i\mathrm{d}\Gamma(\mu(p))t - i\mathrm{d}\Gamma(p)x} \tag{211}$$

$H, P^1, \ldots, P^d$ is a family of commuting s.a. operators on $\Gamma(\mathfrak{h})$. Such a family has a joint spectral measure E: Let $\Delta \in \mathbb{R}^{d+1}$ be a Borel set and χ_Δ its characteristic function. Then $E(\Delta) := \chi_\Delta(H, P^1, \ldots, P^d)$. The joint spectrum of $H, P^1, \ldots, P^d$, denoted $\mathrm{Sp}\,(H, P)$ is defined as the support of E. Physically, these are the measurable values of total energy and momentum of our system.

Theorem 1.69 *$Sp\,(H, P) \subset \overline{V}_+$, where $\overline{V}_+ = \{\, (p^0, p) \in \mathbb{R}^{d+1} \,|\, p_0 \geq |p| \,\}$ is the closed future lightcone.*

Proof. We have to show that for $\Delta \cap \overline{V}_+ = \emptyset$, Δ bounded Borel set, we have $E(\Delta) = 0$. Let $\chi_\Delta^\varepsilon \in C_0^\infty(\mathbb{R}^{d+1})$ approximate χ_Δ pointwise as $\varepsilon \to 0$. (This regularization is needed because the Fourier transform of a sharp characteristic function may not be L^1). Note that $\chi_\Delta(H, P)$ leaves $\Gamma^{(n)}(\mathfrak{h})$ invariant, thus it suffices to show that its matrix elements vanish on these subspaces. We have for

$\Psi, \Phi \in \Gamma^{(n)}(\mathfrak{h})$:

$$
\begin{aligned}
&\langle \Psi, \chi_\Delta(H,P)\Phi\rangle \\
&= \lim_{\varepsilon\to 0}\langle \Psi, \chi^\varepsilon_\Delta(H,P)\Phi\rangle \\
&= \lim_{\varepsilon\to 0}(2\pi)^{-\frac{(d+1)}{2}}\int dt\, dx \langle \Psi, U(t,x)\Phi\rangle \check{\chi}^\varepsilon{}_\Delta(t,x) \\
&= \lim_{\varepsilon\to 0}(2\pi)^{-\frac{(d+1)}{2}}\int dt\, dx \int d^{nd}p\, (\overline{\Psi}\cdot\Phi)(p_1,\dots,p_n)e^{i(\tilde{p}_1+\dots+\tilde{p}_n)\cdot\tilde{x}}\check{\chi}^\varepsilon{}_\Delta(t,x) \\
&= \int d^{nd}p\, (\overline{\Psi}\cdot\Phi)(p_1,\dots,p_n)\chi_\Delta(\tilde{p}_1+\dots+\tilde{p}_n), \qquad (212)
\end{aligned}
$$

where we made use of Fubini and dominated convergence. Note that $\tilde{p} = (\mu(p), p) \in \overline{V}_+$ for all $p \in \mathbb{R}^d$. Since $\overline{V}_+$ is a cone, also $\tilde{p}_1 + \dots + \tilde{p}_n \in \overline{V}_+$. Thus the last expression is zero if $\Delta \cap \overline{V}_+ = \emptyset$. □

Remark 1.70 *In the proof above we used the following conventions for the Fourier transform on $\mathbb{R}^{d+1}$:*

$$\hat{f}(p^0,p) := (2\pi)^{-\frac{(d+1)}{2}}\int d^dx dt\, e^{ip^0t-ipx}f(t,x), \qquad (213)$$

$$\check{f}(t,x) := (2\pi)^{-\frac{(d+1)}{2}}\int d^dp dp^0\, e^{-ip^0t+ipx}f(p^0,p). \qquad (214)$$

A more detailed analysis of the spectrum exhibits that

- for $m > 0$

$$
\begin{aligned}
\mathrm{Sp}\,(H,P) &= \{0\}\cup\{H_m\}\cup G_{2m}, \text{ where} &(215)\\
H_m &:= \{\,(p^0,p)\in\mathbb{R}^{d+1}\,|\,p^0 = \sqrt{p^2+m^2}\}, &(216)\\
G_{2m} &:= \{\,(p^0,p)\in\mathbb{R}^{d+1}\,|\,p^0 \geq \sqrt{p^2+(2m)^2}\}. &(217)
\end{aligned}
$$

  $\{0\}$ is a simple eigenvalue corresponding to the vacuum vector Ω. H_m is called the mass hyperboloid. The corresponding spectral subspace $E(H_m)\Gamma(\mathfrak{h})$ satisfies

$$E(H_m)\Gamma(\mathfrak{h}) = \Gamma^{(1)}(\mathfrak{h}) = \mathfrak{h}. \qquad (218)$$

  Thus it is invariant under $(\tilde{x},\Lambda)\mapsto U(\tilde{x},\Lambda)$. In fact it carries the familiar irreducible representation of $u_{(x,\Lambda)}$ given by (201). According to Wigner's definition of a particle, $E(H_m)\Gamma(\mathfrak{h})$ describes single-particle states of a particle of mass m and spin 0. G_{2m} can be called the multiparticle spectrum. (PICTURE).

- For $m = 0$ we have

$$\mathrm{Sp}\,(H,P) = \overline{V_+}. \qquad (219)$$

  Again, there is a simple eigenvalue at $\{0\}$ (embedded in the multiparticle spectrum) which corresponds to the vacuum vector Ω. $H_{m=0}$ is the boundary of $\overline{V_+}$. The subspace $E(H_{m=0})\Gamma(\mathfrak{h}) = \mathfrak{h}$ carries states of a single massless particle of mass zero.

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