# Postbuckling Ring Analysis

L. B. Sills[1] and B. Budiansky[2]

## Introduction

It is well known that the buckling behavior of circular rings depends on the type of pressure loading (e.g., hydrostatic, dead, central). The main purpose of this Note is to show how the initial postbuckling behavior of inextensional rings under various types of uniform pressure loading can be readily analyzed by the application of general postbuckling theory [1][3] in conjunction with suitable Lagrangian multipliers. Hydrostatic loading was considered in [1], with results that agreed with those obtained by Carrier [2] in an early analytic-numerical study. Dead loading has recently been considered by El-Naschie [3]. In the present Note we study the cases of constant-magnitude central loading ($C$) and central loading governed by an inverse-square law ($IS$). For comparison, we will also display the results for hydrostatic loading ($H$), found in [1].

## Ring Functionals

The potential energy of an inextensional ring subjected to a conservative pressure loading may be written as (see Fig. 1 for notation)

$$F = \frac{EI}{2R}\int_0^{2\pi}\left(\frac{d\theta}{d\alpha}\right)^2 d\alpha + qB(w, v) \tag{1}$$

where $q$ is the magnitude of the uniform pressure that *would* be imposed by the loading system if the ring did not deform. With $F \equiv (EI/R)\phi$, $\lambda \equiv qR^3/EI$, and $B(w, v) \equiv R^2A(w, v)$, a nondimensional potential energy functional may be defined as

$$\phi = \frac{1}{2}\int_0^{2\pi}\left(\frac{d\theta}{d\alpha}\right)^2 d\alpha + \lambda A(w, v) \tag{2}$$

The precise form of $A(w, v)$ depends on the type of pressure loading considered. For uniform central loading, we have

$$B = R^2\int_0^{2\pi}\{[(1 + w)^2 + v^2]^{1/2} - 1\}d\alpha$$

which gives

$$A = \int_0^{2\pi}\left[w + \frac{1}{2}v^2 - \frac{1}{2}wv^2 + \frac{1}{2}w^2v^2 - \frac{1}{2}v^4 + \dots\right]d\alpha \tag{3a}$$

An inverse-square central loading gives

$$B = R^2\int_0^{2\pi}\left\{1 - \frac{1}{[(1 + w)^2 + v^2]^{1/2}}\right\}d\alpha$$

so that

$$A = \int_0^{2\pi}\left[w - w^2 + \frac{1}{2}v^2 + w^3 - \frac{3}{2}wv^2 - \frac{3}{8}v^4 + 3v^2w^2 - w^4 + \dots\right]d\alpha \tag{3b}$$

Finally, for hydrostatic pressure, as in [1], we have

$$A = \int_0^{2\pi}\left[w - v\frac{dw}{d\alpha} + \frac{1}{2}w^2 + \frac{1}{2}v^2\right]d\alpha \tag{3c}$$

Equation (3$c$) is exact; the integrands of (3$a$) and (3$b$) are nonterminating series. The condition of inextensionality is

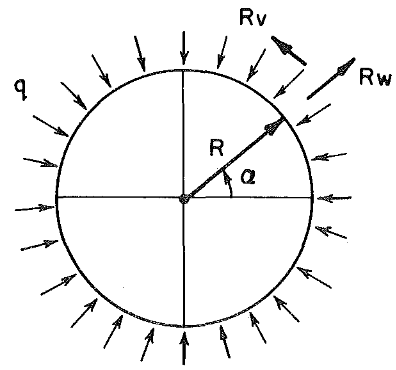


Fig. 1 Thin ring

$$C(w, v, \theta) = w + \frac{dv}{d\alpha} + \frac{1}{2}\left[\left(w + \frac{dv}{d\alpha}\right)^2 + \left(\frac{dw}{d\alpha} - v\right)^2\right] = 0 \tag{4}$$

and for an inextensional ring the rotation $\theta$ is related to $w$ and $v$ by

$$D(w, v, \theta) \equiv \sin\theta + \frac{dw}{d\alpha} - v = 0 \tag{5}$$

## General Theory

The variational equation of equilibrium, together with the constraining relations (4) and (5) are embodied in the assertion

$$\Omega'[U; \lambda]\delta U = 0 \tag{6}$$

where

$$\Omega \equiv \phi - \nu C - \omega D \tag{7}$$

$\nu(\alpha)$ and $\omega(\alpha)$ are Lagrangian multiplier functions, and $U$ is the vector of state variables $w$, $v$, $\theta$, $\nu$, and $\omega$. The prime in (6) denotes Frechet differentiation, and the $\delta U$ is an arbitrary admissible variation of the state variables. (Admissibility, here, is simply periodicity and an appropriate number of continuous derivatives.)

By the general theory [1], the solution of (6) is

$$U = U_0 + \xi U_1 + \xi^2 U_2 + \dots \tag{8}$$

where $U_0$ is the fundamental state, $U_1$ is the buckling mode, suitably normalized, and $\xi^2U_2 + \dots$ is chosen to be orthogonal, in some sense, to $U_1$. The postbuckling variation of the load parameter $\lambda$ is then

$$\lambda = \lambda_c + \xi\lambda_1 + \xi^2\lambda_2 + \dots \tag{9}$$

where $\lambda_c$ is the critical load. In the present example, the bifurcation is always symmetric, with $\lambda_1 = 0$. The variational equation for $U_1$ is

$$\Omega_c''U_1\delta U = 0 \tag{10}$$

where the subscript $c$ denotes evaluation at the fundamental state corresponding to $\lambda = \lambda_c$. For $\lambda_1 = 0$, the equation governing $U_2$ is

$$\Omega_c''U_2\delta U + \frac{1}{2}\Omega_c'''U_1^2\delta U = 0 \tag{11}$$

and the general formula for $\lambda_2$ is

$$\lambda_2 = \frac{-\frac{1}{6}\Omega_c^{iv}U_1^4 + 2\Omega_c''U_2^2}{\dot{\Omega}_c''U_1^2} \tag{12}$$

where $(\dot{\ }) \equiv (d/d\lambda)(\ )$.

## Solutions

In the three cases ($C$), ($IS$), ($H$), the fundamental state $U_0$ may be chosen as $w_0 = v_0 = \theta_0 = \omega_0 = 0$, $\nu_0 = \lambda$. With $D = d/d\alpha$, the differential equations that follow from the variational statement (10) are

$$LU_1 = 0 \tag{13}$$

---

[1] Division of Applied Sciences, Harvard University, Cambridge, Mass.; now, Division of Engineering, Brown University, Providence, R.I.

[2] Professor, Division of Applied Sciences, Harvard University, Cambridge, Mass.

[3] Numbers in brackets designate References at end of Note.

Manuscript received by ASME Applied Mechanics Division, August, 1977; final revision, October, 1977.

  

where

$$U_1 = \begin{bmatrix} w_1 \\ v_1 \\ \theta_1 \\ \nu_1 \\ \omega_1 \end{bmatrix} \tag{14}$$

and

$$L(D) = \left[\begin{array}{cc|ccc} \multicolumn{2}{c|}{} & 0 & -1 & D \\ \multicolumn{2}{c|}{M} & 0 & -D & -1 \\ \hline 0 & 0 & D^2 & 0 & 1 \\ -1 & -D & 0 & 0 & 0 \\ D & -1 & 1 & 0 & 0 \end{array}\right] \tag{15}$$

in which the $(2 \times 2)$ submatrix $M$ is given by

$$M = \lambda_c \begin{bmatrix} D^2 - 1 & -2D \\ -2D & -D^2 \end{bmatrix} \tag{16a}$$

for uniform central loading;

$$M = \lambda_c \begin{bmatrix} D^2 - 3 & -2D \\ -2D & -D^2 \end{bmatrix} \tag{16b}$$

for inverse-square central loading; and

$$M = \lambda_c \begin{bmatrix} D^2 & -D \\ -D & -D^2 \end{bmatrix} \tag{16c}$$

for hydrostatic pressure. In the determination of equations (16), use was made of the fact that $\nu_c = \lambda_c$.

In each of the loading cases, the eigenvector associated with the lowest nontrivial buckling load may be taken as

$$U_1 = \mathrm{Re}\,\{Ke^{2i\alpha}\} \tag{17}$$

where $K$ is a (complex) column vector. Substituting (17) into equation (13) leads to

$$L(2i)K = 0 \tag{18}$$

and so the eigenvalue $\lambda_c$ is the solution to the determinantal equation

$$|L(2i)| = 0 \tag{19}$$

For the three cases considered the results

$$\left.\begin{aligned} \lambda_c &= 9/2 \quad (C) \\ &= 9/4 \quad (IS) \\ &= 3 \quad\ \ (H) \end{aligned}\right\} \tag{20}$$

are recovered from (19). The solution of (18) may be conveniently normalized by choosing the first element of $K$ equal to unity. This gives

$$\begin{bmatrix} w_1 \\ v_1 \\ \theta_1 \\ \nu_1 \\ \omega_1 \end{bmatrix} = \mathrm{Re} \left\{ \begin{bmatrix} 1 \\ i/2 \\ -3i/2 \\ k_4 \\ -6i \end{bmatrix} e^{2i\alpha} \right\} \tag{21}$$

where the values of $k_4$ are

$$\left.\begin{aligned} k_4 &= -3/2 \quad (C) \\ &= 3/4 \quad\ \ (IS) \\ &= 3 \quad\ \ \ \ \ (H) \end{aligned}\right\} \tag{22}$$

for the three loading cases.

The variational statement (11) governing $U_2$ can now be evaluated and its Euler equations deduced. They are

$$LU_2 = F \tag{23}$$

where

$$F = \mathrm{Re} \left\{ \begin{bmatrix} 9/32 \\ 0 \\ 0 \\ 9/16 \\ 0 \end{bmatrix} + \begin{bmatrix} -153/32 \\ -9i/4 \\ 0 \\ -9/16 \\ 0 \end{bmatrix} e^{4i\alpha} \right\} \tag{24a}$$

for uniform central loading;

$$F = \mathrm{Re} \left\{ \begin{bmatrix} -189/64 \\ 0 \\ 0 \\ 9/16 \\ 0 \end{bmatrix} + \begin{bmatrix} -99/64 \\ -9i/8 \\ 0 \\ -9/16 \\ 0 \end{bmatrix} e^{4i\alpha} \right\} \tag{24b}$$

for inverse-square central loading, and

$$F = \mathrm{Re} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 9/16 \\ 0 \end{bmatrix} + \begin{bmatrix} 9 \\ 9i/4 \\ 0 \\ -9/16 \\ 0 \end{bmatrix} e^{4i\alpha} \right\} \tag{24c}$$

for hydrostatic pressure. The solution to equation (23) is given by

$$U_2 = \mathrm{Re} \left\{ \begin{bmatrix} -9/16 \\ 0 \\ 0 \\ 9/4 \\ 0 \end{bmatrix} + \begin{bmatrix} -1/16 \\ -5i/32 \\ 3i/32 \\ -33/16 \\ 3i/2 \end{bmatrix} e^{4i\alpha} \right\} \tag{25a}$$

for uniform central loading;

$$U_2 = \mathrm{Re} \left\{ \begin{bmatrix} -9/16 \\ 0 \\ 0 \\ 27/4 \\ 0 \end{bmatrix} + \begin{bmatrix} -5/112 \\ -17i/112 \\ 3i/112 \\ -111/112 \\ 3i/7 \end{bmatrix} e^{4i\alpha} \right\} \tag{25b}$$

for inverse-square central loading; and

$$U_2 = \mathrm{Re} \left\{ \begin{bmatrix} -9/16 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -9i/64 \\ -9i/64 \\ -27/16 \\ -9i/4 \end{bmatrix} e^{4i\alpha} \right\} \tag{25c}$$

for hydrostatic pressure. Equation (12) yields

$$\left.\begin{aligned} \lambda_2 &= 63/32 \quad\quad (C) \\ &= -999/112 \quad (IS) \\ &= 81/32 \quad\quad (H) \end{aligned}\right\} \tag{26}$$

## Conclusions

As expected, the critical buckling load for the ring whose central load is governed by an inverse square law is lower than that for the ring with uniform central load. Furthermore, the solution for the former predicts that the ring is imperfection sensitive.

This Note illustrates the way in which the general buckling theory [1] can easily be used to solve for the prebuckling and postbuckling behavior of the inextensional ring. The use of Lagrangian multipliers in conjunction with the general theory simplifies the calculations.

Finally, at the suggestion of an anonymous reviewer, we make reference, for the sake of completeness, to the study of postbuckling ring behavior by Rehfield [4], which was executed on the basis of a curvature-displacement relation that unjustifiably omits nonlinear terms. As a consequence, [4] predicts imperfection-sensitivity for rings under hydrostatic pressure, in contrast to the insensitivity shown here, and earlier in [1]. The same reviewer directed our attention to the suspect postbuckling results for central loading, different from ours, that have been given by El Naschie and El Nashai in [5]. At least one reason for this disagreement is the erroneous neglect in [5] of cubic and quartic terms in the energy of the loads. On the other hand, [5] does agree with our result for hydrostatic loading, because for this case the potential energy of loading is exactly quadratic.

## Acknowledgments

This work was supported in part by the National Science Foundation under Grant MPS76-15469, in part by the Air Force Office of Scientific Research under Grant AFOSR-77-3330, and by the Division of Applied Sciences, Harvard University.

## References

1 Budiansky, B., "Theory of Buckling and Postbuckling Behavior of Elastic Structures," *Advances in Applied Mechanics*, Vol. 14, 1974, pp. 1–65.
2 Carrier, G. F., "On the Buckling of Elastic Rings," *Journal of Mathematics and Physics*, Cambridge, Mass., Vol. 26, 1947, pp. 94–103.
3 El Naschie, M. S., "The Initial Postbuckling of an Extensional Ring Under External Pressure," *International Journal of the Mechanical Sciences*, Vol. 17, 1975, pp. 387–388.
4 Rehfield, L. W., "Initial Postbuckling of Circular Rings Under Pressure Loads," *AIAA Journal*, Vol. 10, 1975, pp. 1358–1359.
5 El Naschie, M. S., and El Nashai, A., "Influence of Loading Behavior on the Postbuckling of Circular Rings," *AIAA Journal*, Vol. 14, 1976, pp. 266–267.

# On Uniform Convergence of the Finite-Element Method

**R. W. McLay**[1]

## Introduction

In the author's original paper on the convergence of the finite-element method with singularity (See Johnson and McLay [1][2]) the uniform convergence of the displacement functions is shown with a bound (at least) of $c^{(1/2)-\lambda}$, where $\lambda$ is a measure of the singularity in the problem and $c$ is the mesh dimension on the model. It is the purpose of this Note to develop a sharper bound on the uniform convergence of the displacements with the concepts of completeness and the fundamental inequality associated with the minimum principle used.

## An Inequality

We make use of the fundamental inequality of equation (10) in [1], $\delta_R(U\text{-}u,\ V\text{-}v) \le \delta_R(\hat{U}\text{-}u,\ \hat{V}\text{-}v)$, where all notation is the same as in the original paper. Interest is centered on that portion of the finite-element model, $R_2$, shown in Fig. 1. Region $R_2$ is of finite size, has displacements $U,V$ chosen by the finite-element method and the remainder of the model has displacements $\hat{U},\hat{V}$ fitted to the exact solution by replacing the finite-element freedoms by the exact values at the nodal points in the same manner as is done in the proof of [1] (the completeness property of the finite-element functions proved by a Taylor's expansion). Note that each of the single strips of elements has $U,V$ on one side and $\hat{U},\hat{V}$ on the other. By using this model with the mixed functions $U,\hat{U}$, etc., it is possible to develop a further inequality from the one above which has some very interesting properties:

$$0 \le \delta_{R_2}(U\text{-}u,\ V\text{-}v) + \delta_{(\mathrm{Strips})}(\mathrm{Mixed}) + \delta_{\mathrm{Elsewhere}}(\hat{U}\text{-}u,\ \hat{V}\text{-}v) \le \delta_{R_2}(\hat{U}\text{-}u,\ \hat{V}\text{-}v) + \delta_{(\mathrm{Strips})}(\hat{U}\text{-}u,\ \hat{V}\text{-}v) + \delta_{\mathrm{Elsewhere}}(\hat{U}\text{-}u,\ \hat{V}\text{-}v) < Mc^{2(1-\lambda)} \quad (1)$$

Since all terms are positive and since $\delta_{R_2}(\hat{U}\text{-}u,\ \hat{V}\text{-}v) < Mc^2$ (the second derivatives of the exact solution are bounded in $R_2$) the inequality (1) reduces to

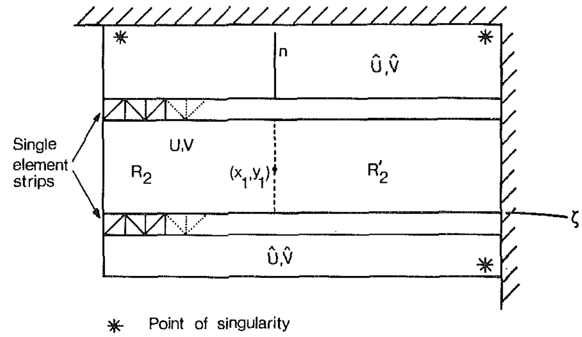


**Fig. 1 Subregions $R_2$ and $R'_2$**

$$0 \le \delta_{R_2}(U\text{-}u,\ V\text{-}v) \le \delta_{R_2}(\hat{U}\text{-}u,\ \hat{V}\text{-}v) + \delta_{(\mathrm{Strips})}(\hat{U}\text{-}u,\ \hat{V}\text{-}v), \le Mc^2 + Mc^{2(1-\lambda)}c \quad (2)$$

where the last term in (2) is generated by the fact that each strip has width $c$. But (2) proves the convergence of the energy in $R_2$ at a higher rate than that found [1], provided $R_2$ is a finite distance from each of the singularities. It is not difficult to show then that

$$0 \le \left|\left|\frac{\partial U}{\partial x} - \frac{\partial u}{\partial x}\right|\right|^2_{R_2} \le Mc^2 + Mc^{3-2\lambda} \quad (3)$$

with an identical statement for $||\partial V/\partial y - \partial v/\partial y||^2_{R_2}$. From [1], the displacements (in $R_2$) converge in the mean at a rate $0(C^1)$, independently of the singularities, and the displacements (in $R_2$) converge uniformly

$$|\hat{U}_1 - U_1| < \text{const}\ c^{1/2} \quad (4)$$

for $\lambda < 1/2$. More important, it is possible to use (4) to construct a sharper bound on the uniform convergence error in $R_2$.

Let us generalize equation (30) in [1] to the form

$$U_1 = \int_{R'_2}\int f(\eta)\frac{\partial U}{\partial \zeta}d\zeta d\eta \quad (5)$$

where $R'_2$ is the subregion of $R_2$ to the right of $(x_1,y_1)$, the point at which the nodal displacement $U_1$ is defined, and $f(\eta)$ is a function to be chosen. We integrate (5) to the form

$$U_1 = -\int f(\eta)U(x_1,\eta)d\eta \quad (6)$$

where the integral occurs on the $\eta$-axis.

We choose $f(\eta)$ as

$$f(\eta) = -\frac{U_1}{\int U(x_1,\sigma)d\sigma} \quad (7)$$

which, contrary to the development in [1], makes it a function of the problem itself. This does not replace the proof of [1] which requires a polynomial related to the trial functions. Instead, it builds on the proof of [1] with (3) and (4) to obtain

$$f(\eta) = -\frac{u_1}{\int u(x_1,\sigma)d\sigma} + 0(c^{1/2}) \quad (9)$$

which is problem dependent but is only weakly dependent on the finite-element mesh. With the assumption that $\int u(x_1,\sigma)d\sigma \ne 0$, true except for a special case of $(x_1,y_1)$ and of course repairable with a minor change in the extent of $R_2$, we know that $f(\eta) < M$. Note again that $M$ is problem dependent but that for each problem an $M$ can be chosen for any point in $R_2$. It follows then directly from [1] and equation (5)

$$|\hat{U}_1 - U_1| < \text{const}\ c^1 \quad (10)$$

with an identical development for $|\hat{V}_1 - V_1|$. Thus the uniform convergence in $R_2$ is of higher order than that shown in [1], a result der-

[1] Visiting Professor of Metallurgy and Materials Science, Cambridge University, England; Professor of Mechanical Engineering, University of Vermont, Burlington, Vt. Mem. ASME.

[2] Numbers in brackets designate References at end of Note.

Manuscript received by ASME Applied Mechanics Division, April, 1977; final revision, August, 1977.