Comparison theory of Lorentzian distance with applications to spacelike hypersurfaces

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Abstract. In this note we summarize some comparison results for the Lorentzian distance function in spacetimes, with applications to the study of the geometric analysis of the Lorentzian distance function on spacelike hypersurfaces. In particular, we will consider spacelike hypersufaces whose image under the immersion is bounded in the ambient spacetime and derive sharp estimates for the mean curvature of such hypersurfaces under appropriate hypotheses on the curvature of the ambient spacetime. The results in this note are part of our recent paper [1], where complete details and further related results may be found.

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THE LORENTZIAN DISTANCE FUNCTION

Consider M^{n+1} an $(n+1)$-dimensional spacetime, and let p,q be points in M. Using the standard terminology and notation from Lorentzian geometry, one says that q is in the chronological future of p, written $p \ll q$, if there exists a future-directed timelike curve from p to q. Similarly, q is in the causal future of p, written $p < q$, if there exists a future-directed causal (i.e., nonspacelike) curve from p to q. Obviously, $p \ll q$ implies $p < q$. As usual, $p \leq q$ means that either $p < q$ or $p = q$.

For a subset $S \subset M$, one defines the chronological future of S as $I^+(S) = \{q \in M : p \ll q \text{ for some } p \in S\}$, and the causal future of S as $J^+(S) = \{q \in M : p \leq q \text{ for some } p \in S\}$. Thus $S \cup I^+(S) \subset J^+(S)$. In particular, the chronological future $I^+(p)$ and the causal future $J^+(p)$ of a point $p \in M$ are

$$I^+(p) = \{q \in M : p \ll q\}, \quad \text{and} \quad J^+(p) = \{q \in M : p \leq q\}.$$

As is well-known, $I^+(p)$ is always open, while $J^+(p)$ is neither open nor closed in general.

If $q \in J^+(p)$, then the Lorentzian distance $d(p,q)$ is the supremum of the Lorentzian lengths of all the future-directed causal curves from p to q (possibly, $d(p,q) = +\infty$). If $q \notin J^+(p)$, then the Lorentzian distance $d(p,q) = 0$ by definition. In particular, $d(p,q) > 0$ if and only is $q \in I^+(p)$. Let us recall that the Lorentzian distance function $d : M \times M \to [0,+\infty]$ for an arbitrary spacetime may fail to be continuous in general, and may also fail to be finite valued; globally hyperbolic spacetimes turn out to be the natural class of spacetimes for which the Lorentzian distance function is finite-valued and continuous (see, for instance, [3] and [2]).

Given a point $p \in M$, one can define the Lorentzian distance function from p by $d_p(q) = d(p,q)$. In order to guarantee the smoothness of d_p as a function on M, one needs to restrict this function on certain special subsets of M. Consider

$$T_{-1}M|_p = \{v \in T_pM : v \text{ is a future-directed timelike unit vector}\}$$

the fiber of the unit future observer bundle of M at p, and set $s_p : T_{-1}M|_p \to [0,+\infty]$ the function given by

$$s_p(v) = \sup\{t \geq 0 : d_p(\gamma_v(t)) = t\},$$

where $\gamma_v : [0,a) \to M$ is the future inextendible geodesic starting at p with initial velocity v. Then, one can define the subset $\tilde{\mathscr{I}}^+(p) \subset T_pM$ as

$$\tilde{\mathscr{I}}^+(p) = \{tv : \text{ for all } v \in T_{-1}M|_p \text{ and } 0 < t < s_p(v)\}$$

and consider the subset $\mathscr{I}^+(p) = \exp_p(\mathrm{int}(\tilde{\mathscr{I}}^+(p))) \subset I^+(p)$. Observe that $\exp_p : \mathrm{int}(\tilde{\mathscr{I}}^+(p)) \to \mathscr{I}^+(p)$ is a diffeomorphism and $\mathscr{I}^+(p) \subset M$ is an open subset (possible empty). In the following result we summarize the main properties about the Lorentzian distance function (see [4, Section 3.1]).

Lemma 1 *Let M be a spacetime and $p \in M$.*

1. *If M is strongly causal at p, then $s_p(v) > 0$ for all $v \in T_{-1}M|_p$ and $\mathscr{I}^+(p) \neq \emptyset$.*
2. *If $\mathscr{I}^+(p) \neq \emptyset$, then the Lorentzian distance function d_p is smooth on $\mathscr{I}^+(p)$ and its gradient $\overline{\nabla} d_p$ is a past-directed timelike (geodesic) unit vector field on $\mathscr{I}^+(p)$.*

COMPARISON RESULTS FOR THE LORENTZIAN DISTANCE FROM A POINT

For every $c \in \mathbb{R}$, let us define

$$f_c(s) = \begin{cases} \sqrt{c}\coth(\sqrt{c}\,s) & \text{if } c > 0 \text{ and } s > 0 \\ 1/s & \text{if } c = 0 \text{ and } s > 0 \\ \sqrt{-c}\cot(\sqrt{-c}\,s) & \text{if } c < 0 \text{ and } 0 < s < \pi/\sqrt{-c}. \end{cases}$$

It is worth pointing out that the index of a Jacobi field J_c along a timelike geodesic γ_c in a Lorentzian space form of constant curvature c is given by $I_{\gamma_c}(J_c,J_c) = -f_c(s)\langle x,x\rangle$, where $J_c(0) = 0$ and $J_c(s) = x \perp \gamma_c'(s)$. On the other hand, when $\mathscr{I}^+(p) \neq \emptyset$, $f_c(s)$ is the future mean curvature of the level set $\Sigma_c(s) = \{q \in \mathscr{I}^+(p) : d_p(q) = s\} \subset M_c^{n+1}$.

Our first result assumes that the sectional curvatures of the timelike planes of M are bounded from above by a constant c and reads as follows.

Lemma 2 *[1, Lemma 3.1] Let M be a spacetime such that $K_M(\Pi) \leq c$, $c \in \mathbb{R}$, for all timelike planes in M. Assume that there exists a point $p \in M$ such that $\mathscr{I}^+(p) \neq \emptyset$,* *and let $q \in \mathscr{I}^+(p)$ (with $d_p(q) < \pi/\sqrt{-c}$ when $c < 0$). Then for every spacelike vector $x \in T_qM$ orthogonal to $\overline{\nabla} d_p(q)$ it holds that*

$$\overline{\nabla}^2 d_p(x,x) \geq -f_c(d_p(q))\langle x,x\rangle,$$

where $\overline{\nabla}^2$ stands for the Hessian operator on M.

Observe that if $K_M(\Pi) \leq c$ for all timelike planes in an $(n+1)$-dimensional spacetime M, then for every unit timelike vector $Z \in TM$ one gets that $\mathrm{Ric}_M(Z,Z) \geq -nc$. Our next result holds under this weaker hypothesis on the Ricci curvature of M. When $c = 0$ this is nothing but the so called timelike convergence condition.

Lemma 3 *[1, Lemma 3.3] Let M be an $(n+1)$-dimensional spacetime such that $\mathrm{Ric}_M(Z,Z) \geq -nc$, $c \in \mathbb{R}$, for every unit timelike vector Z. Assume that there exists a point $p \in M$ such that $\mathscr{I}^+(p) \neq \emptyset$, and let $q \in \mathscr{I}^+(p)$ (with $d_p(q) < \pi/\sqrt{-c}$ when $c < 0$). Then*

$$\overline{\Delta} d_p(q) \geq -n f_c(d_p(q)),$$

where $\overline{\Delta}$ stands for the (Lorentzian) Laplacian operator on M.

On the other hand, under the assumption that the sectional curvatures of the timelike planes of M are bounded from below by a constant c, we get the following result.

Lemma 4 *[1, Lemma 3.2] Let M be a spacetime such that $K_M(\Pi) \geq c$ $c \in \mathbb{R}$, for all timelike planes in M. Assume that there exists a point $p \in M$ such that $\mathscr{I}^+(p) \neq \emptyset$, and let $q \in \mathscr{I}^+(p)$ (with $d_p(q) < \pi/\sqrt{-c}$ when $c < 0$). Then for every spacelike vector $x \in T_qM$ orthogonal to $\overline{\nabla} d_p(q)$ it holds that*

$$\overline{\nabla}^2 d_p(x,x) \leq -f_c(d_p(q))\langle x,x\rangle,$$

where $\overline{\nabla}^2$ stands for the Hessian operator on M.

The proofs of Lemma 2, Lemma 3 and Lemma 4 follow from the fact that

$$\overline{\nabla}^2 d_p(x,x) = -\int_0^s (\langle J'(t),J'(t)\rangle - \langle R(J(t),\gamma'(t))\gamma'(t),J(t)\rangle)dt = I_\gamma(J,J)$$

where γ is the radial future directed unit timelike geodesic from p to q and J is the Jacobi field along γ with $J(0) = 0$ and $J(s) = x$, and it is strongly based on the maximality of the index of Jacobi fields. For the details, see [1, Section 3].

SPACELIKE HYPERSURFACES CONTAINED IN $\mathscr{I}^+(p)$

Consider $\psi : \Sigma^n \to M^{n+1}$ a spacelike hypersurface immersed into a spacetime M. Since M is time-oriented, there exists a unique future-directed timelike unit normal field N globally defined on Σ. Let A stand for the shape operator of Σ with respect to N. We will assume that there exists a point $p \in M$ such that $\mathscr{I}^+(p) \neq \emptyset$ and that

$\psi(\Sigma) \subset \mathscr{I}^+(p)$. Let $r = d_p$ denote the Lorentzian distance function with respect to p, and let $u = r \circ \psi : \Sigma \to (0, \infty)$ be the function r along the hypersurface, which is a smooth function on Σ. Our first objective is to compute the Laplacian of u. To do that, observe that

$$\overline{\nabla} r = \nabla u - \langle \overline{\nabla} r, N \rangle N$$

along Σ, where ∇u stands for the gradient of u on Σ. In particular,

$$\langle \overline{\nabla} r, N \rangle = \sqrt{1 + |\nabla u|^2} \geq 1.$$

Moreover,

$$\overline{\nabla}^2 r(X, X) = \nabla^2 u(X, X) + \sqrt{1 + |\nabla u|^2} \langle AX, X \rangle \tag{1}$$

for every tangent vector field $X \in T\Sigma$, where $\overline{\nabla}^2 r$ and $\nabla^2 u$ stand for the Hessian of r and u in M and Σ, respectively. Assume now that $K_M(\Pi) \leq c$ (resp. $K_M(\Pi) \geq c$) for all timelike planes in M, and that $u < \pi/\sqrt{-c}$ on Σ when $c < 0$. Then by the Hessian comparison results for r given in Lemma 2 (resp. Lemma 4), one gets that

$$\overline{\nabla}^2 r(X, X) \geq (\leq) - f_c(u)(1 + \langle X, \nabla u \rangle^2)$$

for every unit tangent vector field $X \in T\Sigma$, and therefore by (1)

$$\nabla^2 u(X, X) \geq (\leq) - f_c(u)(1 + \langle X, \nabla u \rangle^2) - \sqrt{1 + |\nabla u|^2} \langle AX, X \rangle.$$

Tracing this inequality, one gets the following inequality for the Laplacian of u

$$\Delta u \geq (\leq) - f_c(u)(n + |\nabla u|^2) + nH\sqrt{1 + |\nabla u|^2},$$

where $H = -(1/n)\mathrm{tr}(A)$ is the mean curvature of Σ. Similarly, under the assumption $\mathrm{Ric}_M(Z, Z) \geq -nc$, $c \in \mathbb{R}$, for every unit timelike vector Z, we know from the Laplacian comparison result given in Lemma 3 that $\overline{\Delta} r \geq -n f_c(r)$ along the hypersurface. Therefore, we conclude that

$$\Delta u = \overline{\Delta} r + \overline{\nabla}^2 r(N, N) + nH\sqrt{1 + |\nabla u|^2} \geq -n f_c(u) + \overline{\nabla}^2 r(N, N) + nH\sqrt{1 + |\nabla u|^2}.$$

Summarizing, if $\psi(\Sigma) \subset \mathscr{I}^+(p)$ (and $u < \pi/\sqrt{-c}$ on Σ when $c < 0$)

(a) $K_M(\Pi) \leq c$ implies that $\Delta u \geq -f_c(u)(n + |\nabla u|^2) + nH\sqrt{1 + |\nabla u|^2}$;

(b) $K_M(\Pi) \geq c$ implies that $\Delta u \leq -f_c(u)(n + |\nabla u|^2) + nH\sqrt{1 + |\nabla u|^2}$; and

(c) $\mathrm{Ric}_M(Z, Z) \geq -nc$ implies $\Delta u \geq -n f_c(u) + \overline{\nabla}^2 r(N, N) + nH\sqrt{1 + |\nabla u|^2}$.

For further details, see [1, Section 3].

SPACELIKE HYPERSURFACES BOUNDED BY A LEVEL SET OF THE LORENTZIAN DISTANCE

For the applications of our comparison results to the estimate of the mean curvature of spacelike hypersurfaces, we will make use of a generalized version of the well-known Omori-Yau maximum principle. Following the terminology introduced by Pigola, Rigoli and Setti [5, Definition 1.10], the *Omori-Yau maximum principle* is said to hold on an n-dimensional Riemannian manifold Σ^n if, for any smooth function $u \in \mathscr{C}^\infty(\Sigma)$ with $u^* = \sup_\Sigma u < +\infty$ there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in Σ with the properties

$$\text{(i) } u(p_k) > u^* - \frac{1}{k}, \quad \text{(ii) } |\nabla u(p_k)| < \frac{1}{k}, \quad \text{and} \quad \text{(iii) } \Delta u(p_k) < \frac{1}{k}.$$

Equivalently, for any $u \in \mathscr{C}^\infty(\Sigma)$ with $u_* = \inf_\Sigma u > -\infty$ there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in Σ satisfying

$$\text{(i) } u(p_k) < u_* + \frac{1}{k}, \quad \text{(ii) } |\nabla u(p_k)| < \frac{1}{k}, \quad \text{and} \quad \text{(iii) } \Delta u(p_k) > -\frac{1}{k}.$$

In this sense, the classical maximum principle given by Omori [6] and Yau [7] states that the Omori-Yau maximum principle holds on every complete Riemannian manifold with Ricci curvature bounded from below. More generally, as shown by Pigola, Rigoli and Setti [5, Example 1.13], a sufficiently controlled decay of the radial Ricci curvature suffices to imply the validity of the Omori-Yau maximum principle. Now we are ready to give our main results.

Theorem 5 *[1, Theorem 4.1] Let M be an $(n+1)$-dimensional spacetime such that $\mathrm{Ric}_M(Z,Z) \geq -nc$, $c \in \mathbb{R}$, for every unit timelike vector Z. Let $p \in M$ be such that $\mathscr{I}^+(p) \neq \emptyset$, and let $\psi : \Sigma \to M^{n+1}$ be a spacelike hypersurface such that $\psi(\Sigma) \subset \mathscr{I}^+(p) \cap B^+(p,\delta)$ for some $\delta > 0$ (with $\delta \leq \pi/\sqrt{-c}$ when $c < 0$), where $B^+(p,\delta)$ denotes the future inner ball of radius δ,*

$$B^+(p,\delta) = \{q \in I^+(p) : d_p(q) < \delta\}.$$

If the Omori-Yau maximum principle holds on Σ, then its future mean curvature H satisfies

$$\inf_\Sigma H \leq f_c(\sup_\Sigma u),$$

where u denotes the Lorentzian distance d_p along the hypersurface.

For a sketch of the proof, observe that since $\mathrm{Ric}_M(Z,Z) \geq -nc$, we have that

$$\Delta u \geq -nf_c(u) + \overline{\nabla}^2 r(N,N) + nH\sqrt{1+|\nabla u|^2}.$$

Applying the Omori-Yau maximum principle to the function u, we get that

$$\frac{1}{k} > \Delta u(p_k) \geq -nf_c(u(p_k)) + \overline{\nabla}^2 r(N(p_k),N(p_k)) + nH(p_k)\sqrt{1+|\nabla u(p_k)|^2}.$$

That is,

$$\inf_\Sigma H \leq H(p_k) \leq \frac{1/k + n f_c(u(p_k)) - \overline{\nabla}^2 r(N(p_k), N(p_k))}{n\sqrt{1+|\nabla u(p_k)|^2}}.$$

On the other hand,

$$\begin{aligned} N(p_k) &= N^*(p_k) - \langle N(p_k), \overline{\nabla} r(p_k)\rangle \overline{\nabla} r(p_k), \\ \overline{\nabla} r(p_k) &= \nabla u(p_k) - \langle \overline{\nabla} r(p_k), N(p_k)\rangle N(p_k), \end{aligned}$$

with $N^*(p_k)$ orthogonal to $\overline{\nabla} r(p_k)$. Then, $|N^*(p_k)|^2 = |\nabla u(p_k)|^2$ and $\lim_{k\to\infty} N^*(p_k) = 0$. Finally, taking into account that $\overline{\nabla}^2 r(N(p_k), N(p_k)) = \overline{\nabla}^2 r(N^*(p_k), N^*(p_k))$ and making $k \to \infty$ we get the result. For further details, see [1, Section 4].

Theorem 6 *[1, Theorem 4.2] Let M be an $(n+1)$-dimensional spacetime such that $K_M(\Pi) \geq c$, $c \in \mathbb{R}$, for all timelike planes in M. Let $p \in M$ be such that $\mathscr{I}^+(p) \neq \emptyset$, and let $\psi : \Sigma \to M^{n+1}$ be a spacelike hypersurface such that $\psi(\Sigma) \subset \mathscr{I}^+(p)$. If the Omori-Yau maximum principle holds on Σ (and $\inf_\Sigma u < \pi/\sqrt{-c}$ when $c < 0$), then its future mean curvature H satisfies*

$$\sup_\Sigma H \geq f_c(\inf_\Sigma u),$$

where u denotes the Lorentzian distance d_p along the hypersurface. In particular, if $\inf_\Sigma u = 0$ then $\sup_\Sigma H = +\infty$.

As a direct application of Theorem 6 we get the following.

Corollary 7 *[1, Corollary 4.3] Under the assumptions of Theorem 6, if H is bounded from above on Σ, then there exists some $\delta > 0$ such that $\psi(\Sigma) \subset O^+(p,\delta)$, where $O^+(p,\delta)$ denotes the future outer ball of radius δ,*

$$O^+(p,\delta) = \{q \in I^+(p) : d_p(q) > \delta\}.$$

For a sketch of the proof of Theorem 6, observe that since $K_M(\Pi) \geq c$, we know that

$$\Delta u \leq -f_c(u)(n + |\nabla u|^2) + nH\sqrt{1+|\nabla u|^2}.$$

Applying the Omori-Yau maximum principle to the positive function u, we get that

$$-\frac{1}{k} < \Delta u(p_k) \leq -f_c(u(p_k))(n + |\nabla u(p_k)|^2) + nH(p_k)\sqrt{1+|\nabla u(p_k)|^2}.$$

It follows from here that

$$\sup_\Sigma H \geq H(p_k) \geq \frac{-1/k + f_c(u(p_k))(n + |\nabla u(p_k)|^2)}{n\sqrt{1+|\nabla u(p_k)|^2}}.$$

Therefore, making $k \to \infty$ here we get the result. The last assertion follows from the fact that $\lim_{s\to 0} f_c(s) = +\infty$. On the other hand, for a proof of Corollary 7, simply observe that $\sup_\Sigma H < +\infty$ implies that $\inf_\Sigma u > 0$. For further details, see [1, Section 4].

In particular, when the ambient spacetime is a Lorentzian space form, Theorems 5 and 6 yield the following consequences.

Theorem 8 *[1, Theorem 4.5] Let M_c^{n+1} be a Lorentzian space form of constant sectional curvature c and let $p \in M_c^{n+1}$. Let us consider $\psi : \Sigma \to M_c^{n+1}$ a spacelike hypersurface such that $\psi(\Sigma) \subset \mathscr{I}^+(p) \cap B^+(p,\delta)$ for some $\delta > 0$ (with $\delta \leq \pi/\sqrt{-c}$ if $c < 0$). If the Omori-Yau maximum principle holds on Σ, then*

$$\inf_\Sigma H \leq f_c(\sup_\Sigma u) \leq f_c(\inf_\Sigma u) \leq \sup_\Sigma H,$$

where u denotes the Lorentzian distance d_p along the hypersurface.

Corollary 9 *[1, Corollary 4.6] Let M_c^{n+1} be a Lorentzian space form of constant sectional curvature c and let $p \in M_c^{n+1}$. If Σ is a complete spacelike hypersurface in M_c^{n+1} with constant mean curvature H which is contained in $\mathscr{I}^+(p)$ and bounded from above by a level set of the Lorentzian distance function d_p (with $d_p < \pi/\sqrt{-c}$ if $c < 0$), then Σ is necessarily a level set of d_p.*

For a proof simply observe that the Ricci curvature of a spacelike hypersurface Σ in an arbitrary spacetime M is given by

$$\begin{aligned}
\mathrm{Ric}_\Sigma(X,X) &= \mathrm{Ric}_M(X,X) - \left(K_M(X \wedge N) + \frac{n^2H^2}{4}\right)|X|^2 + |AX + \frac{n}{2}X|^2 \\
&\geq \mathrm{Ric}_M(X,X) - \left(K_M(X \wedge N) + \frac{n^2H^2}{4}\right)|X|^2.
\end{aligned}$$

In particular, if M_c^{n+1} is a Lorentzian space form of constant sectional curvature c then $\mathrm{Ric}_\Sigma(X,X) \geq \left((n-1)c - n^2H^2/4\right)|X|^2$. Thus, every spacelike hypersurface Σ with bounded mean curvature in M_c^{n+1} has Ricci curvature bounded from below. Hence, if complete, it satisfies the Omori-Yau maximum principle.

As observed in [1, Remark 1], our last results have specially simple and illustrative consequences when the ambient is the Lorentz-Minkowski spacetime. For instance, we can state the following improvement of Theorem 2 in [8].

Corollary 10 *[1, Corllary 4.7] The only complete spacelike hypersurfaces with constant mean curvature in the Minkowski space $\mathbb{L}^{n+1}$ which are contained in $\mathscr{I}^+(p)$ (for some fixed $p \in \mathbb{L}^{n+1}$) and bounded from above by a hyperbolic space centered at p are precisely the hyperbolic spaces centered at p.*

THE LORENTZIAN DISTANCE FUNCTION FROM AN ACHRONAL HYPERSURFACE

Given $S \subset M^{n+1}$ an achronal spacelike hypersurface, one can define the Lorentzian distance function from S by $d_S(q) = \sup\{d(p,q) : p \in S\}$. As in the previous case of the Lorentzian distance from a point, to guarantee the smoothness of d_S, one needs to restrict this function on certain special subsets of M. Let η be the future-directed Gauss map of S, and let $s : S \to [0,+\infty]$ the function given by

$$s(p) = \sup\{t \geq 0 : d_S(\gamma_p(t)) = t\},$$

where $\gamma_p : [0,a) \to M$ is the future inextendible geodesic starting at p with initial velocity η_p. Then, one can define

$$\tilde{\mathscr{I}}^+(S) = \{t\eta_p : \text{ for all } p \in S \text{ and } 0 < t < s(p)\}$$

and consider the subset $\mathscr{I}^+(S) = \exp_S(\mathrm{int}(\tilde{\mathscr{I}}^+(S))) \subset I^+(S)$, where $\exp_S$ denotes the exponential map with respect to the hypersurface S. Below we collect some interesting properties about d_S (see [4, Section 3.2]).

Lemma 11 *Let S be an achronal spacelike hypersurface in a spacetime M.*

1. *If S is compact and M is globally hyperbolic, then $s(p) > 0$ for all $p \in S$ and $\mathscr{I}^+(S) \neq \emptyset$.*
2. *If $\mathscr{I}^+(S) \neq \emptyset$, then d_S is smooth on $\mathscr{I}^+(S)$ and its gradient $\overline{\nabla} d_S$ is a past-directed timelike (geodesic) unit vector field on $\mathscr{I}^+(S)$.*

Doing now a similar analysis of the Lorentzian distance function to an achronal hypersurface, one can derive also sharp estimates for the mean curvature of spacelike hypersurfaces which contained in its chronological future. Further details about this may be found in [1].

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REFERENCES

1. L.J. Alías, A. Hurtado, and V. Palmer, *Trans. Amer. Math. Soc.* in press. Available at http://arxiv.org/pdf/0802.4376.
2. B. O'Neill, *Semi-Riemannian Geometry; With Applications to Relativity*, Academic Press, New York, 1983.
3. J.K. Beem, P.E. Ehrlich, and K.L. Easey, *Global Lorentzian Geometry*, Marcel Dekker Inc., New York, 1996.
4. F. Erkekoglu, E. García-Río, and D. N. Kupeli, *Gen. Relativ. Gravit.* **35**, 1597–1615 (2003).
5. S. Pigola, M. Rigoli, and A.G. Setti, *Memoirs Amer. Math. Soc.* **822** (2005).
6. H. Omori, *J. Math. Soc. Japan* **19**, 205–214 (1967).
7. S.T. Yau, *Commun. Pure Appl. Math.* **28**, 201–228 (1975).
8. J. A. Aledo, and L. J. Alías, *Manuscripta Math.* **101**, 401–413 (2000).