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Global stability for a discrete SIS epidemic model with immigration of infectives

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Abstract. In this paper, we propose a discrete-time SIS epidemic model which is derived from continuous-time SIS epidemic models with immigration of infectives by the backward Euler method. For the discretized model, by applying new Lyapunov function techniques, we establish the global asymptotic stability of the disease-free equilibrium for $R_0 \leq 1$ and the endemic equilibrium for $R_0 > 1$, where R_0 is the basic reproduction number of the continuous-time model. This is just a discrete analogue of continuous SIS epidemic model with immigration of infectives.

Keywords: Backward Euler method, SIS epidemic model, global asymptotic stability, Lyapunov function.
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1 Introduction

In the literature of epidemiology, many authors have recently proposed mathematical models and studied the global behavior of the transmission of infectious disease for the models (see also [1–16] and references therein).

Brauer and van den Driessche [1] have formulated the following continuous SIS epidemic model with immigration of infectives:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = (1-p)A - \beta S(t)I(t) - dS(t) + \gamma I(t), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = pA + \beta S(t)I(t) - (d+\alpha+\gamma)I(t), \end{cases} \tag{1.1}$$

with the initial conditions $S(0) > 0,\ I(0) > 0$.

$S(t)$ and $I(t)$ denote the number of a population who are susceptible to a disease and infective members at time t, respectively. It is assumed that all newborns are susceptible. In addition, all recruitments are into the susceptible class at a constant rate $(1-p)A > 0$ and the infective class at a constant immigration rate $pA > 0$. The positive constant d represents the death rates of susceptible and infectious classes, and the positive constant α represents the rate at which the infective dies from the infection. The mass action coefficient is $\beta > 0$.

Let the basic reproduction number R_0 defined by

$$R_0 = \frac{\beta A}{d(d+\alpha+\gamma)}. \tag{1.2}$$

We here note that R_0 is the product of the population size at the disease-free steady state with no infectives (*i.e.* $p = 0$), the transmission coefficient and the mean infective period [1].

For the case $p = 0$, system (1.1) always has a disease-free equilibrium $E^0 = (A/d, 0)$. Furthermore, if $p = 0$ and $R_0 > 1$ or $0 < p \leq 1$, then system (1.1) has a unique endemic equilibrium $E^* = (S^*, I^*)$, where

$$S^* = \frac{A+\gamma I^*}{\beta I^* + d},\ I^* = \frac{\sigma + \sqrt{\sigma^2 + 4\beta dpA(d+\alpha)}}{2\beta(d+\alpha)},\ \sigma = (1-p)\beta A - d(d+\gamma+\alpha). \tag{1.3}$$

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By using Bendixson-Durac criterion [6, p. 373] and Poincaré-Bendixson theorem [6, p.366], stability results of the disease-free equilibrium E^0 and the endemic equilibrium E^* for system (1.1) has been established by Brauer and van den Driessche [1] as follows:

Theorem A *For system (1.1), if $p = 0$ and $R_0 < 1$, then the disease-free equilibrium E^0 is globally attractive, and if $p = 0$ and $R_0 > 1$ or $0 < p \leq 1$, then there exists a unique endemic equilibrium E^* which is globally asymptotically stable.*

Later, for a delayed SIS epidemic model with a wide class of nonlinear incidence rates, by using the special property $\lim_{t\to+\infty} N(t) = 1$ for the total population $N(t) = S(t) + I(t)$ and the model can be transformed into a form of the SIR epidemic model as in McCluskey [12], Huang and Takeuchi [7] have fully solved the global asymptotic stability of a disease-free equilibrium and a unique endemic equilibrium by the basic reproduction number of the model.

On the other hand, there occur situations such that constructing discrete epidemic models is more appropriate approach to understand disease transmission dynamics and to evaluate eradication policies because they permit arbitrary time-step units, preserving the basic features of corresponding continuous-time models. Furthermore, this allows better use of statistical data for numerical simulations due to the reason that the infection data are compiled at discrete given time intervals. For a discrete SIS epidemic model with immigration of infectives, by means of Micken's nonstandard discretization method (see Mickens [14]), Jang and Elaydi [10] showed the global asymptotic stability of a disease-free equilibrium, the local asymptotic stability of a unique endemic equilibrium and strong persistence of susceptible class of the model. A conjecture that one may construct a Lyapunov function to show the global stability of the endemic equilibrium for the model is also proposed. Using a discretization called "mixed type" formula in Izzo and Vecchio [8] and Izzo *et al.* [9], Sekiguchi [15] obtained the permanence of a class of SIR discrete epidemic models with one delay and SEIRS discrete epidemic models with two delays if an endemic equilibrium of each model exists. For the detailed property for a class of discrete epidemic models, we refer to [2, 3, 8–11, 15, 16].

However, in those cases, how to choose the discrete schemes which preserves the global asymptotic stability for the endemic equilibrium of corresponding continous-time models was still unsolved.

For a delayed SIR epidemic model, applying a variation of backward Euler discretization, Enatsu *et al.* [4] firstly solved this problem and established the complete global stability results by a discrete time analogue of a Lyapunov functional proposed by McCluskey [12]. Enatsu *et al.* [5] also have obtained the similar results for a discrete SIR epidemic model with a variation of backward Euler discretization which has a separable nonlinear incidence rate.

Motivated by the above results, in this paper, to preserve key properties of Lyapunov functional techniques in Enatsu *et al.* [4] for discretization, we apply the backward Euler method to the following iteration system;

$$S_{n+1} = \frac{(1-p)A + S_n + \gamma I_{n+1}}{1 + d + \beta I_{n+1}},\ I_{n+1} = \frac{-\tilde{B}_n + \sqrt{\tilde{B}_n^2 + 4\tilde{A}\tilde{C}_n}}{2\tilde{A}} = \frac{2\tilde{C}_n}{\tilde{B}_n + \sqrt{\tilde{B}_n^2 + 4\tilde{A}\tilde{C}_n}}, \quad n = 0, 1, 2, \cdots, \tag{1.4}$$

with the initial conditions

$$S_0 > 0, \text{ and } I_0 > 0, \tag{1.5}$$

where

$$\tilde{A} = \beta(1 + d + \alpha),\ \tilde{B}_n = (1 + d)(1 + d + \alpha + \gamma) - \beta(A + S_n + I_n),\ \tilde{C}_n = (1 + d)(pA + I_n). \tag{1.6}$$

For the initial conditions (1.5), let (S_n, I_n) $(n > 0)$ be the solutions of system (1.4). Then $S_n > 0$, $I_n > 0$ holds for any $n > 0$ (see Lemma 2.1). Moreover, (1.4) is equivalent to the following discrete SIS epidemic model:

$$\begin{cases} S_{n+1} - S_n = (1-p)A - \beta S_{n+1}I_{n+1} - dS_{n+1} + \gamma I_{n+1}, \\ I_{n+1} - I_n = pA + \beta S_{n+1}I_{n+1} - (d + \alpha + \gamma)I_{n+1}, \quad \text{and} \quad I_{n+1} > 0, \end{cases} \tag{1.7}$$

which is derived from system (1.1) by applying the backward Euler method.

Note that for any positive solution (S_n, I_n), there exist just two solutions (S_{n+1}, I_{n+1}) of (1.7) without the condition $I_{n+1} > 0$, one is $I_{n+1} < 0$ and the other is $I_{n+1} > 0$. Therefore, for any positive solution (S_n, I_n), we need the restriction $I_{n+1} > 0$ to consider only the positive solution (S_{n+1}, I_{n+1}) in (1.7).

Similar to the case of continuous system (1.1), for the case $p = 0$, system (1.7) always has a disease-free equilibrium $E^0 = (A/d, 0)$. Furthermore, if $p = 0$ and $R_0 > 1$, or $0 < p \leq 1$, then system (1.7) has a unique endemic equilibrium $E^* = (S^*, I^*)$ which is defined by (1.3).

Remark 1.1. To prove the positivity of S_n and I_n for $n > 0$ and apply key properties of Lyapunov functional techniques in Enatsu *et al.* [4], we need to use the backward Euler discretization instead of the forward Euler discretization which is a different discretization from that in Jang and Elaydi [10].

Using the same threshold $R_0 = \frac{\beta A}{d(d+\alpha+\gamma)}$ and $0 \leq p \leq 1$, we establish that the following global stability result:

Theorem 1.1. *For the case $p=0$ in system (1.7), there exists a unique disease-free equilibrium E^0 which is globally asymptotically stable, if and only if, $R_0 \leq 1$. For the case $p=0$ and $R_0>1$, or $0<p\leq 1$ in system (1.7), then there exists a unique endemic equilibrium E^* which is globally asymptotically stable.*

Remark 1.2. Theorem 1.1 for system (1.7) is just a discrete analogue of Theorem A for system (1.1).

The key ideas of our Lyapunov function techniques are as follows (see also Section 3).

(i) By letting $N_n := S_n + I_n$, we rewrite system (1.7) as follows.

$$\begin{cases} I_{n+1}-I_n = pA+\beta(N_{n+1}-I_{n+1})I_{n+1}-(d+\alpha+\gamma)I_{n+1},\ I_{n+1}>0, \\ N_{n+1}-N_n = A-dN_{n+1}-\alpha I_{n+1}. \end{cases}$$

(ii) Let $N=S+I,\ N^0=\frac{A}{d},\ N^*=S^*+I^*$, and define

$$U(n)=\frac{I}{\beta}g\left(\frac{I_n}{I}\right)+\frac{1}{\alpha N}\frac{(N_n-N)^2}{2}, \tag{1.8}$$

where $g(x)=x-1-\ln x\geq g(1)=0$ defined on $x>0$. By the relation

$$\lim_{x\to+0} xg\left(\frac{y}{x}\right)=y \text{ for each } y>0,$$

we offer a unified construction of discrete time analogue of Lyapunov functions $U^0(n)$ and $U^*(n)$ in the proof of global stability of the disease-free equilibrium and the endemic equilibrium as follows, respectively;

$$U^0(n)=\lim_{I\to+0,\ N\to N^0} U(n)=\frac{1}{\beta}I_n+\frac{1}{\alpha N^0}\frac{(N_n-N^0)^2}{2},$$

and

$$U^*(n)=\lim_{I\to I^*,\ N\to N^*} U(n)=\frac{I^*}{\beta}g\left(\frac{I_n}{I^*}\right)+\frac{1}{\alpha N^*}\frac{(N_n-N^*)^2}{2}.$$

(iii) Assume that $p=0$ and $R_0>1$ or $0<p\leq 1$. In order to prove the second part of Theorem 1.1, by using a key idea in Enatsu *et al.* [4], we use the following relation.

$$-\ln\frac{I_{n+1}}{I_n}=\ln\left\{1-\left(1-\frac{I_n}{I_{n+1}}\right)\right\}\leq-\left(1-\frac{I_n}{I_{n+1}}\right)=-\frac{I_{n+1}-I_n}{I_{n+1}}.$$

Adding

$$I^*\left(\frac{N_{n+1}}{N^*}-1\right)\left(\frac{I_{n+1}}{I^*}-1\right)$$

in $\frac{I^*}{\beta N^*}\left\{g\left(\frac{I_{n+1}}{I^*}\right)-g\left(\frac{I_n}{I^*}\right)\right\}$ to

$$-\frac{dN^*}{\alpha}\left(1-\frac{N_{n+1}}{N^*}\right)^2-I^*\left(\frac{N_{n+1}}{N^*}-1\right)\left(\frac{I_{n+1}}{I^*}-1\right)$$

in $\frac{1}{\alpha N^*}\left\{\frac{(N_{n+1}-N^*)^2}{2}-\frac{(N_n-N^*)^2}{2}\right\}$, we obtain $U^*(n+1)-U^*(n)\leq 0$ with equality if and only if $I_{n+1}=I^*$ and $N_{n+1}=N^*$.

(iv) Assume that $p=0$ and $R_0\leq 1$. In order to prove the second part of Theorem 1.1, along with the similar discussion in (i)-(iii), we have $U^0(n+1)-U^0(n)\leq 0$ with equality if and only if $I_{n+1}=0$ and $N_{n+1}=N^0$.

The organization of this paper is as follows. In Section 2, we offer some basic results for system (1.7). In particular, by Lemma 2.4, we offer a simplified proof for the permanence of system (1.7) for $p=0$ and $R_0>1$, or $0<p\leq 1$ (cf. Sekiguchi [15]). The first part of Theorem 1.1 concerning the global asymptotic stability of the disease-free equilibrium for $p=0$ and $R_0\leq 1$ and the second part of Theorem 1.1 concerning the permanence and the global stability of the endemic equilibrium for $p=0$ and $R_0>1$ or $0<p\leq 1$ are given in Section 3. Finally, we offer conclusion in Section 4.

2 Basic properties

In this section, we introduce basic lemmas as follows.

Lemma 2.1. *Let (S_n, I_n) be the solutions of system (1.7) with the initial conditions (1.5). Then $S_n > 0$, $I_n > 0$ for all $n > 0$.*

Proof. Assume that there exists a nonnegative integer $n_0 \geq 0$ such that $S_n,\ I_n > 0,\ n = 0, 1, \cdots, n_0$. Then, for $S_n > 0$ and $I_n > 0$, I_{n+1} is a unique positive solution of the following quadratic equation:

$$\begin{aligned} P(x) &= \{(1+d+\alpha+\gamma)x - (pA + I_n)\}(1+d+\beta x) - \beta\{(1-p)A + S_n + \gamma x\}x \\ &= \beta(1+d+\alpha)x^2 + \{(1+d)(1+d+\alpha+\gamma) - \beta(A + S_n + I_n)\}x - (1+d)(pA + I_n), \end{aligned} \tag{2.1}$$

Hence, by the first equation of (1.4), we have $S_{n_0+1} > 0$, and by the second equation of (1.4), we have $I_{n_0+1} > 0$. By induction, we prove this lemma. □

Lemma 2.2. *Any solution (S_n, I_n) of system (1.7) satisfies*

$$\limsup_{n\to+\infty}(S_n + I_n) \leq S^0 = A/d. \tag{2.2}$$

Proof. Let $N_n = S_n + I_n$. From system (1.7), we have that

$$N_{n+1} - N_n = A - d(S_{n+1} + I_{n+1}) - \alpha I_{n+1} \leq A - dN_{n+1} - \alpha I_{n+1},$$

from which we have

$$\limsup_{n\to+\infty} N_n \leq S^0 = \frac{A}{d}.$$

Hence, the proof is complete. □

Lemma 2.3. *Assume that $p = 0$ and $R_0 > 1$. If $I_{n+1} < I_n$, then $S_{n+1} < S^*$. Inversely, if $S_{n+1} \geq S^*$, then $I_{n+1} \geq I_n$.*

Proof. By the second equation of (1.7), we have

$$I_{n+1} = \frac{I_n - I_{n+1}}{d+\alpha+\gamma} + \frac{S_{n+1}}{S^*} I_{n+1}.$$

Therefore, if $I_{n+1} < I_n$, then we have

$$I_{n+1} > \frac{S_{n+1}}{S^*} I_{n+1},$$

from which we obtain $S_{n+1} < S^*$. The remained part of this lemma is evident. □

By Lemma 2.3, we obtain the following lemma which implies the permanence of system (1.7).

Lemma 2.4. *The following statements hold true.*

(i) Let $0 < p \leq 1$. Then, for any solution (S_n, I_n) of system (1.7), it holds that

$$\begin{aligned} &0 < \frac{(1-p)A}{1+d+\beta A/d} \leq \liminf_{n\to+\infty} S_n \leq \limsup_{n\to+\infty} S_n \leq \frac{(1-p)A + (1+\gamma)A/d}{1+d}, \\ &0 < \underline{\hat{I}} \leq \liminf_{n\to+\infty} I_n \leq \limsup_{n\to+\infty} I_n \leq \overline{\hat{I}}, \end{aligned}$$

where

$$\begin{cases} \underline{\hat{I}} = \dfrac{2\underline{C}}{\overline{B} + \sqrt{\overline{B}^2 + 4\tilde{A}\underline{C}}},\ \overline{\hat{I}} = \dfrac{2\overline{C}}{\underline{B} + \sqrt{\underline{B}^2 + 4\tilde{A}\overline{C}}}, \\ \tilde{A} = \beta(1+d+\alpha), \\ \underline{B} = (1+d)(1+d+\alpha+\gamma) - \beta(A + A/d),\ \overline{B} = (1+d)(1+d+\alpha+\gamma) - \beta A, \\ \underline{C} = (1+d)pA,\ \overline{C} = (1+d)(pA + A/d). \end{cases} \tag{2.3}$$

(ii) Let $p = 0$ and $R_0 > 1$. Then, for any solution (S_n, I_n) of system (1.7), it holds that

$$0 < \frac{A}{1+d+\beta\frac{A}{d}} \leq \liminf_{n\to+\infty} S_n \leq \limsup_{n\to+\infty} S_n \leq \frac{A + (1+\gamma)A/d}{1+d}, \tag{2.4}$$

$$0 < \left(\frac{1}{1+(d+\alpha+\gamma)}\right)^{l_0} qI^* \leq \liminf_{n\to+\infty} I_n \leq \limsup_{n\to+\infty} I_n \leq \overline{\hat{I}}, \tag{2.5}$$

where $0 < q < 1$ and $l_0 \geq 1$ satisfy $S^* < S^{\triangle} := \frac{A}{k}\{1 - (\frac{1}{1+k})^{l_0}\}$ for $k = d + \beta qI^*$.

Proof. Since for $p(x) = x + \sqrt{x^2+c}$ with $c > 0$, it holds that $p'(x) = 1 + \frac{x}{\sqrt{x^2+c}} > 0$, the function $p(x)$ is an increasing function of x on $(0 \le x < +\infty)$. By (1.4) and (2.2), we obtain the conclusion of (i) in this lemma. $\square$

From the proof of (i), it suffices to show that $\liminf_{n\to+\infty} I_n \ge (\frac{1}{1+(d+\alpha+\gamma)})^{l_0} qI^*$ holds. For any $0 < q < 1$, by (1.3), one can see that $S^* = \frac{A}{d+\beta I^*} < \frac{A}{d+\beta qI^*}$. We first prove the claim that any solution (S_n, I_n) of system (1.7) does not have the following property: there exists a nonnegative integer n_1 such that $I_n \le qI^*$ for all $n \ge n_1$. Suppose on the contrary that there exist a nonnegative integer n_1 such that $I_n \le qI^*$ for all $n \ge n_1$. From system (1.7), one can obtain that

$$S_{n+1} \ge \frac{S_n}{1+k} + \frac{A}{1+k} \quad \text{for all } n \ge n_1,$$

which yields that

$$S_{n+1} \ge \left(\frac{1}{1+k}\right)^{n+1-n_1} S_n + \frac{A}{1+k}\sum_{l=0}^{n-n_1}\left(\frac{1}{1+k}\right)^l \ge \frac{A}{1+k}\frac{1-(\frac{1}{1+k})^{n+1-n_1}}{1-\frac{1}{1+k}} \ge \frac{A}{k}\left\{1-\left(\frac{1}{1+k}\right)^{n+1-n_1}\right\}$$

for all $n \ge n_1$. Therefore, we have that

$$S_{n+1} \ge \frac{A}{k}\left\{1-\left(\frac{1}{1+k}\right)^{l_0}\right\} = S^{\triangle} > S^* \quad \text{for all } n \ge n_1 + l_0 - 1. \tag{2.6}$$

Then, by the second part of Lemma 2.3, we obtain that there exists a positive constant $\hat{i}$ such that $I_n \ge \hat{i}$ for any $n \ge n_1 + l_0 - 1$. Hence, one can see that

$$\begin{aligned} I_{n+1} - I_n &= \beta S_{n+1} I_{n+1} - (d+\alpha+\gamma) I_{n+1} \\ &> \{\beta S^{\triangle} - (d+\alpha+\gamma)\} I_{n+1} \\ &> \{\beta S^{\triangle} - (d+\alpha+\gamma)\}\hat{i} \text{ for all } n \ge n_1 + l_0 - 1, \end{aligned}$$

which implies by $\beta S^{\triangle} - (d+\alpha+\gamma) = \beta(S^{\triangle} - S^*) > 0$, that $\lim_{n\to+\infty} I_n = +\infty$. However, by Lemma 2.2, this leads a contradiction. Hence, the claim is proved.

By the claim, we are left to consider two possibilities. First, $I_n \ge qI^*$ holds for all n sufficiently large. Second, I_n oscillates about qI^* for all n sufficiently large. We now show that $I_n \ge (\frac{1}{1+(d+\alpha+\gamma)})^{l_0} qI^*$ for all n sufficiently large for the both cases. If the first case that $I_n \ge qI^*$ holds for all sufficiently large, then we immediately get the conclusion of the proof. For the second case that I_n oscillates about qI^* for all sufficiently large, let $n_2 < n_3$ be sufficiently large such that

$$I_{n_2}, I_{n_3} \ge qI^*, \quad \text{and} \quad I_n < qI^* \quad \text{for all } n_2 < n < n_3.$$

Then, by the second equation of system (1.7), we have that

$$I_{n+1} - I_n \ge -(d+\alpha+\gamma) I_{n+1}, \text{ that is, } I_{n+1} \ge \frac{1}{1+(d+\alpha+\gamma)} I_n$$

for all $n \ge n_2$, from which we have that

$$I_{n+1} \ge \left(\frac{1}{1+(d+\alpha+\gamma)}\right)^{n+1-n_2} I_{n_2} \ge \left(\frac{1}{1+(d+\alpha+\gamma)}\right)^{n+1-n_2} qI^*$$

for all $n \ge n_2$. Therefore, we obtain that

$$I_{n+1} \ge \left(\frac{1}{1+(d+\alpha+\gamma)}\right)^{l_0} qI^* \tag{2.7}$$

for all $n_2 \le n \le n_2 + l_0 - 1$. If $n_3 \ge n_2 + l_0$, then by applying the similar discussion to (2.6), we obtain $I_{n+1} \ge (\frac{1}{1+(d+\alpha+\gamma)})^{l_0} qI^*$ for $n_2 + l_0 \le n \le n_3$. Hence, we prove that $I_{n+1} \ge (\frac{1}{1+(d+\alpha+\gamma)})^{l_0} qI^*$ for $n_2 \le n \le n_3$. Since the interval $n_2 \le n \le n_3$ is arbitrarily chosen, we conclude that $I_{n+1} \ge (\frac{1}{1+(d+\alpha+\gamma)})^{l_0} qI^*$ for all n sufficiently large for the second case and obtain the conclusion of (ii) in this lemma. This completes the proof. $\square$

3 Global stability

In this section, by applying Lyapunov function techniques, we prove Theorem 1.1. By the relation $S_n = N_n - I_n$, system (1.7) is equivalent to the following system:

$$\begin{cases} I_{n+1} - I_n = pA + \beta(N_{n+1} - I_{n+1}) I_{n+1} - (d+\alpha+\gamma) I_{n+1}, \ I_{n+1} > 0, \\ N_{n+1} - N_n = A - dN_{n+1} - \alpha I_{n+1}, \end{cases} \tag{3.1}$$

with the initial conditions $I_0 > 0$ and $N_0 > 0$. If $p = 0$, then system (3.1) always has a disease-free equilibrium $\tilde{E}^0 = (0, N^0)$, $N^0 = \frac{A}{d}$ and if $p = 0$ and $R_0 > 1$, or $0 < p \leq 1$, then system (3.1) has a unique endemic equilibrium $\tilde{E}^* = (I^*, N^*)$. Therefore, in order to prove Theorem 1.1, it suffices to show the global stability of the disease-free equilibrium $\tilde{E}^0$ for $p = 0$ and $R_0 \leq 1$ (see Section 3.2) and the global stability of the endemic equilibrium $\tilde{E}^*$ for $p = 0$ and $R_0 > 1$, or $0 < p \leq 1$ (see Section 3.1).

3.1 The case $p = 0$ and $R_0 > 1$, or $0 < p \leq 1$

In this subsection, we prove the second part of Theorem 1.1.

Proof of the second part of Theorem 1.1. For the endemic equilibrium $\tilde{E}^*$ of system (3.1), we consider the following discrete time analogue of Lyapunov function:

$$U^*(n) = \frac{I^*}{\beta N^*} U_1^*(n) + \frac{1}{\alpha N^*} U_2^*(n), \tag{3.2}$$

with

$$U_1^*(n) = g\left(\frac{I_n}{I^*}\right), \quad \text{and} \quad U_2^*(n) = \frac{1}{2}(N_n - N^*)^2, \tag{3.3}$$

where $g(x) = x - 1 - \ln x \geq g(1) = 0$ defined on $x > 0$. From the equilibrium condition of (3.1), we have

$$d + \alpha + \gamma = \frac{pA}{I^*} + \beta(N^* - I^*). \tag{3.4}$$

By using a key idea in Enatsu *et al.* [4], we use the following relation.

$$-\ln\frac{I_{n+1}}{I_n} = \ln\left\{1 - \left(1 - \frac{I_n}{I_{n+1}}\right)\right\} \leq -\left(1 - \frac{I_n}{I_{n+1}}\right) = -\frac{I_{n+1} - I_n}{I_{n+1}}. \tag{3.5}$$

From (3.5), we obtain

$$\begin{aligned}
U_1^*(n+1) - U_1^*(n) &= \frac{I_{n+1} - I_n}{I^*} - \ln\frac{I_{n+1}}{I_n} \\
&\leq \frac{I_{n+1} - I_n}{I^*} - \frac{I_{n+1} - I_n}{I_{n+1}} \\
&= \frac{1}{I^*}\frac{I_{n+1} - I^*}{I_{n+1}}(I_{n+1} - I_n) \\
&= \frac{1}{I^*}\left(1 - \frac{I^*}{I_{n+1}}\right)\left\{pA + \beta S_{n+1} I_{n+1} - (d + \alpha + \gamma) I_{n+1}\right\}.
\end{aligned}$$

By using the relation (3.4), we have

$$\begin{aligned}
U_1^*(n+1) - U_1^*(n) &\leq \frac{1}{I^*}\left(1 - \frac{I^*}{I_{n+1}}\right)\left\{pA + \beta(N_{n+1} - I_{n+1})I_{n+1} - \left(\frac{pA}{I^*} + \beta(N^* - I^*)\right)I_{n+1}\right\} \\
&= \frac{1}{I^*}\left(1 - \frac{I^*}{I_{n+1}}\right)\left\{pA\left(1 - \frac{I_{n+1}}{I^*}\right) - \beta I_{n+1}(I_{n+1} - I^*) + \beta I_{n+1}(N_{n+1} - N^*)\right\} \\
&= \frac{pA}{I^*}\left(1 - \frac{I^*}{I_{n+1}}\right)\left(1 - \frac{I_{n+1}}{I^*}\right) - \beta I^*\left(\frac{I_{n+1}}{I^*} - 1\right)^2 + \beta N^*\left(\frac{I_{n+1}}{I^*} - 1\right)\left(\frac{N_{n+1}}{N^*} - 1\right).
\end{aligned}$$

Moreover, it holds that

$$\begin{aligned}
U_2^*(n+1) - U_2^*(n) &= \frac{1}{2}\left(N_{n+1} + N_n - 2N^*\right)\left(N_{n+1} - N_n\right) \\
&= \left(N_{n+1} - N^*\right)\left(N_{n+1} - N_n\right) - \frac{1}{2}\left(N_{n+1} - N_n\right)^2 \\
&\leq \left(N_{n+1} - N^*\right)\left(N_{n+1} - N_n\right) \\
&= \left(N_{n+1} - N^*\right)\left\{A - dN_{n+1} - \alpha I_{n+1}\right\} \\
&= \left(N_{n+1} - N^*\right)\left\{-d(N_{n+1} - N^*) - \alpha(I_{n+1} - I^*)\right\} \\
&= -d\left(N_{n+1} - N^*\right)^2 - \alpha\left(N_{n+1} - N^*\right)\left(I_{n+1} - I^*\right) \\
&= -d(N^*)^2\left(1 - \frac{N_{n+1}}{N^*}\right)^2 - \alpha N^* I^*\left(\frac{N_{n+1}}{N^*} - 1\right)\left(\frac{I_{n+1}}{I^*} - 1\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned} U^*(n+1) - U^*(n) &\le \frac{pA}{\beta N^*}\left(1-\frac{I^*}{I_{n+1}}\right)\left(1-\frac{I_{n+1}}{I^*}\right) - \frac{(I^*)^2}{N^*}\left(\frac{I_{n+1}}{I^*}-1\right)^2 + I^*\left(\frac{N_{n+1}}{N^*}-1\right)\left(\frac{I_{n+1}}{I^*}-1\right) \\ &\quad -\frac{dN^*}{\alpha}\left(1-\frac{N_{n+1}}{N^*}\right)^2 - I^*\left(\frac{N_{n+1}}{N^*}-1\right)\left(\frac{I_{n+1}}{I^*}-1\right) \\ &= \frac{pA}{\beta N^*}\left(1-\frac{I^*}{I_{n+1}}\right)\left(1-\frac{I_{n+1}}{I^*}\right) - \frac{(I^*)^2}{N^*}\left(\frac{I_{n+1}}{I^*}-1\right)^2 - \frac{dN^*}{\alpha}\left(1-\frac{N_{n+1}}{N^*}\right)^2 \le 0 \end{aligned}$$

for all $n \ge 0$. Since $U^*(n) \ge 0$ is monotone decreasing sequence, there is a limit $\lim_{n\to+\infty} U^*(n) \ge 0$. Then, we have $\lim_{n\to+\infty}(U^*(n+1) - U^*(n)) = 0$, which implies that $\lim_{n\to+\infty} I_{n+1} = I^*$ and $\lim_{n\to+\infty} N_{n+1} = N^*$. Since $U^*(n) \le U^*(0)$ for all $n \ge 0$, $\tilde{E}^*$ is uniformly stable. Hence, $\tilde{E}^*$ is globally asymptotically stable. □

3.2 The case $p = 0$ and $R_0 \le 1$

In this section, we prove the first part of Theorem 1.1.

Proof of the first part of Theorem 1.1. For the disease-free equilibrium $\tilde{E}^0$ of system (3.1), we consider the following discrete time analogue of Lyapunov function:

$$U^0(n) = \frac{1}{\beta N^0} I_n + \frac{1}{\alpha N^0} U_1^0(n), \tag{3.6}$$

with

$$U_1^0(n) = \frac{1}{2}(N_n - N^0)^2. \tag{3.7}$$

Then, from $I_{n+1} - I_n = \beta(N_{n+1} - I_{n+1})I_{n+1} - (d+\alpha+\gamma)I_{n+1}$ and

$$\begin{aligned} U_1^0(n+1) - U_1^0(n) &= \frac{1}{2}\left(N_{n+1} + N_n - 2N^0\right)\left(N_{n+1} - N_n\right) \\ &= \left(N_{n+1} - N^0\right)\left(N_{n+1} - N_n\right) - \frac{1}{2}\left(N_{n+1} - N_n\right)^2 \\ &\le \left(N_{n+1} - N^0\right)\left(N_{n+1} - N_n\right) \\ &= \left(N_{n+1} - N^0\right)\left\{-d(N_{n+1} - N^0) - \alpha I_{n+1}\right\} \\ &= -d\left(N_{n+1} - N^0\right)^2 - \alpha\left(N_{n+1} - N^0\right) I_{n+1} \\ &= -d(N^0)^2\left(1 - \frac{N_{n+1}}{N^0}\right)^2 - \alpha I_{n+1}\left(N_{n+1} - N^0\right), \end{aligned}$$

we have

$$\begin{aligned} U^0(n+1) - U^0(n) &\le \frac{1}{N^0}(N_{n+1} - I_{n+1})I_{n+1} - \frac{d+\alpha+\gamma}{\beta N^0} I_{n+1} - \frac{dN^0}{\alpha}\left(1 - \frac{N_{n+1}}{N^0}\right)^2 - I_{n+1}\left(\frac{N_{n+1}}{N^0} - 1\right) \\ &= -\frac{I_{n+1}^2}{N^0} + \left(1 - \frac{d+\alpha+\gamma}{\beta N^0}\right) I_{n+1} - \frac{dN^0}{\alpha}\left(1 - \frac{N_{n+1}}{N^0}\right)^2 \\ &= -\frac{I_{n+1}^2}{N^0} + \left(1 - \frac{1}{R_0}\right) I_{n+1} - \frac{dN^0}{\alpha}\left(1 - \frac{N_{n+1}}{N^0}\right)^2 \le 0 \end{aligned}$$

for all $n \ge 0$. Since $U^0(n) \ge 0$ is monotone decreasing sequence, there is a limit $\lim_{n\to+\infty} U^0(n) \ge 0$. Then, $\lim_{n\to+\infty}(U^0(n+1) - U^0(n)) = 0$, from which we obtain that $\lim_{n\to+\infty} I_{n+1} = 0$ and $\lim_{n\to+\infty} N_{n+1} = N^0$. Since $U^0(n) \le U^0(0)$ for all $n \ge 0$, $\tilde{E}^0$ is uniformly stable. Hence, $\tilde{E}^0$ is globally asymptotically stable. □

4 Conclusion

Recently, it was still unsolved how to choose the discrete schemes which preserves the global asymptotic stability for the endemic equilibrium of corresponding continuous models. By applying a discrete time analogue of a Lyapunov functional proposed by McCluskey [12], Enatsu *et al.* [4] established the complete analysis of global stability of equilibria for a discrete SIR epidemic model with a variation of the backward Euler discretization. On the other hand, for a continuous delayed SIS epidemic model which has the special property $\lim_{t\to+\infty} N(t) = 1$ for the total population $N(t) = S(t)+I(t)$ and can be transformed into a form of a delayed SIR epidemic model as in McCluskey [12], Huang and Takeuchi [7] have fully solved the global asymptotic stability of a disease-free equilibrium and a unique endemic equilibrium by the basic reproduction number of the model.

In this paper, in order to preserve key properties of Lyapunov functional techniques in Enatsu *et al.* [4] for discretization, we use the backward Euler method on a continuous SIS epidemic model with immigration of infectives in Brauer and van den Driessche [1]. Moreover, by means of a unified construction of discretized Lyapunov functions $U^0(n)$ and $U^*(n)$, we establish the global stability of the disease-free equilibrium E^0 when $R_0 \leq 1$ and the endemic equilibrium E^* when $R_0 > 1$ for the discrete SIS epidemic model (1.7), respectively. This is just a discrete analogue of continuous SIS epidemic model with immigration of infectives.

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