NON SMOOTH LAGRANGIAN SETS AND ESTIMATIONS OF MICRO-SUPPORT

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Abstract. A constant sheaf along a smooth submanifold is characterized by the property that its micro-support is contained in the conormal bundle to this submanifold. More generally, a characterization of those sheaves whose micro-support is locally contained in a smooth Lagrangian manifold is given in [K-S, Chapter 6].

The aim of this paper is to extend this result to the non smooth case of Lagrangian sets associated with convex subsets. Geometric objects of this kind appears naturally in the study of boundary value problems.

1. Notations and review. Let X be a real C^1 manifold and let $Y \subset X$ be a closed submanifold. One denotes by $\pi : T^*X \to X$ the cotangent bundle to X and by T^*_YX the conormal bundle to Y in X.

One denotes by $\mathrm{D}^b(X)$ the derived category of the category of bounded complexes of sheaves of $\mathbf{C}$-vector spaces on X. For F an object of $\mathrm{D}^b(X)$, one denotes by $\mathrm{SS}(F)$ its micro-support, a closed, conic, involutive subset of T^*X.

Let $A \subset X$ be a closed C^1-convex subset at $x_\circ \in A$ (i.e. A is convex for a choice of local C^1 coordinates at $x_\circ$). One denotes by $\mathbf{C}_A$ the sheaf which is zero on $X \setminus A$ and the constant sheaf with fiber $\mathbf{C}$ on A. In order to describe $\mathrm{SS}(\mathbf{C}_A)$ fix a local system of coordinates $(x) = (x', x'')$ at $x_\circ$ so that A is convex and $Y = \{x \in X;\ x'' = 0\}$ is its linear hull. Denote by $j : Y \to X$ the embedding and by ${}^tj' : Y \times_X T^*X \to T^*Y$ the associated projection. One has

$$\mathrm{SS}(\mathbf{C}_A) = {}^tj'(N^*_Y(A)),$$

where $N^*_Y(A)$ denotes the conormal cone to A in Y. In other words, $(x;\xi) \in \mathrm{SS}(\mathbf{C}_A)$ if and only if $x \in A$ and the half space $\{y \in X;\ \langle y - x, \xi\rangle \geq 0\}$ contains A. By analogy with the smooth case, we set $T^*_AX = \mathrm{SS}(\mathbf{C}_A)$.

For $p \in T^*X$, $\mathrm{D}^b(X;p)$ denotes the localization of $\mathrm{D}^b(X)$ with respect to the null system $\{F \in \mathrm{D}^b(X);\ p \notin \mathrm{SS}(F)\}$. One also considers the microlocalization bifunctor $\mu hom(\cdot,\cdot)$ which is defined in [K-S].

Remark 1.1. In [K-S] the bifunctor μhom is considered only for C^2 manifolds but it is clear that its definition is possible for a C^1 manifold as well. Roughly speaking, this functor is the composition of the specialization functor (which is defined as long as the normal deformation is defined, i.e. for C^1 manifolds) and the Fourier-Sato transform which is defined for vector bundles over any locally compact space.

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If X is of class C^2 one has the following estimate

$$\mathrm{SS}(\mu hom(F,G)) \subset C(\mathrm{SS}(F), \mathrm{SS}(G)), \tag{1.1}$$

where $C(\cdot,\cdot) \subset TT^*X \cong T^*T^*X$ denotes the strict normal cone.

Assume X of class C^2 and let $\chi : T^*X \to T^*X$ be a germ of homogeneous contact transformation at $p \in T^*X$, i.e. a diffeomorphism at p preserving the canonical one-form. Set $\Lambda^a_\chi = \{(x,y;\xi,\eta);\ \chi(x;\xi) = (y;-\eta)\}$, the antipodal of the graph of χ. It is possible to consider "quantizations" of χ in order to make contact transformations operate on sheaves.

Theorem 1.2. *(cf* [K-S, Chapter 7]*) There exists $K \in \mathrm{D}^b(X \times X)$ with $\mathrm{SS}(K) \subset \Lambda^a_\chi$ in a neighborhood of $(p, \chi(p)^a)$. This complex induces an equivalence of categories $\Phi_K : \mathrm{D}^b(X;p) \to \mathrm{D}^b(X;\chi(p))$ defined by $\Phi_K(F) = \mathrm{R}q_{2*}(K \otimes q_1^{-1}F)$ where q_i is the i-th projection from $X \times X$ to X. Moreover one has the relations*

$$\mathrm{SS}(\Phi_K(F)) = \chi(\mathrm{SS}(F)), \tag{1.2}$$

$$\chi_*\mu hom(F,G) \cong \mu hom(\Phi_K(F), \Phi_K(G)) \quad \textit{near } \chi(p). \tag{1.3}$$

2. The main theorem. The characterization of those sheaves whose micro-support is contained in a smooth Lagrangian is given by the following theorem.

Theorem 2.1. *(cf* [K-S, Theorem 6.6.1]*) Let X be a real C^2 manifold, let $Y \subset X$ be a closed submanifold and take $p \in T^*_YX$. Let F be an object of $\mathrm{D}^b(X)$ such that*

$$\mathrm{SS}(F) \subset T^*_YX \qquad \textit{in a neighborhood of } p.$$

Then one has $F \cong M_Y$ in $\mathrm{D}^b(X;p)$ for a complex M of $\mathbf{C}$-vector spaces.

Remark 2.2. The extension from the C^2 to the C^1 frame has already been given in the paper [D'A-Z]. Concerning this extension, we point out the following fact. Let $Y \subset X$ be a hypersurface of regularity $C^1 \setminus C^2$ and let Y^+ be the closed half space with boundary Y such that $p \in \mathrm{SS}(A_{Y^+})$. The crucial point here is that, even though $T^*_YX \mathbin{\widehat{+}} T^*_YX \supset \pi^{-1}\pi(p)$, nevertheless $N^*(Y^+) \mathbin{\widehat{+}} N^*(Y^+) \subset N^*(Y^+)$.

Here we give the following extension of this result.

Theorem 2.3. *Let X be a real C^1 manifold, let $A \subset X$ be a closed C^1-convex subset at $x_\circ$ and take $p \in (T^*_AX)_{x_\circ}$. Let F and G be objects of $\mathrm{D}^b(X)$ such that*

$$\mathrm{SS}(F),\ \mathrm{SS}(G) \subset T^*_AX \qquad \textit{in a neighborhood of } p.$$

Then

(i) *$\mu hom(F,G) \cong N_{T^*_AX}$ for a complex N of $\mathbf{C}$-vector spaces;*
(ii) *$F \cong M_A$ in $\mathrm{D}^b(X;p)$ for a complex M of $\mathbf{C}$-vector spaces;*
(iii) *for M as in (ii), one has $M \cong \mu hom(\mathbf{C}_A, F)_p$.*

Remark 2.4. Let X be a real C^2 manifold and $Y \subset X$ a closed submanifold. In this context, the assertion (ii) already appears in [U-Z] for any closed subset $A \subset Y$ satisfying $N^*_Y(A)_{x_\circ} \neq T^*_{x_\circ}Y$ (which holds true, in particular, for C^1-convex subsets at $x_\circ$), but only for $p \in T^*_Y X \cap T^*_A X$.

proof of Theorem 2.3. The problem being local, fix a system of local coordinates at $x_\circ$ so that A is convex in $X \subset \mathbf{R}^n$ with coordinates $(x) = (x_1, \ldots, x_n)$. Let $(x;\xi)$ be the associated symplectic coordinates of T^*X and consider the contact transformation

$$\begin{aligned}\chi :&T^*X \to T^*X\\ &(x;\xi) \mapsto (x - \varepsilon\frac{\xi}{|\xi|};\xi).\end{aligned}$$

The set $A_\varepsilon = \{x \in X; \mathrm{dist}(x, A) \leq \varepsilon\}$ has a C^1 boundary for $0 < \varepsilon << 1$ and one has $\chi(T^*_A X) = T^*_{A_\varepsilon}X$ near $\chi(p)$. It is also easy to verify that, setting

$$S = \{(x, y) \in X \times X;\ \mathrm{dist}(x, y) \leq \varepsilon\},$$

the complex $K = \mathbf{C}_S$ verifies the hypothesis of Theorem 1.2 for such a χ.

Let $\phi : X \to X$ be a C^1 diffeomorphism so that $\phi(A_\varepsilon) = \{x \in X;\ x_1 \leq 0\}$ and set $Z = \{x \in X;\ x_1 = 0\}$. By (1.2) one has that $\mathrm{SS}(\phi_*(\Phi_K(\cdot))) \subset T^*_Z X$ near ${}^t\phi'(\chi(p))$ for $(\cdot = F,\ G)$ and hence, by (1.1),

$$\begin{aligned}\mathrm{SS}(\mu hom(\phi_*(\Phi_K(F)), \phi_*(\Phi_K(G)))) &\subset C(T^*_Z X, T^*_Z X)\\ &\cong T_{T^*_Z X}T^*X.\end{aligned}$$

By Theorem 2.1 one then has

$$\mu hom(\phi_*(\Phi_K(F)), \phi_*(\Phi_K(G))) \cong N_{T^*_Z X}$$

for a complex N of $\mathbf{C}$-vector spaces. It follows by (1.3) that

$$\mu hom(F, G) \cong {}^t\phi'^{-1}(\chi^{-1}(N_{T^*_Z X})) \cong N_{T^*_A X}$$

which proves (i).

For any complex M of $\mathbf{C}$-vector spaces let us now compute $\Phi_K(M_A)$. There is an isomorphism $(\mathrm{R}q_{2!}M_{S\cap(A\times X)})_x \cong \mathrm{R}\Gamma_c(q_2^{-1}(x); M_{S\cap(A\times X)})$. Since $q_2^{-1}(x) \cap S \cap (A \times X)$ is either empty if $x \notin A_\varepsilon$ or compact convex if $x \in A_\varepsilon$, one has:

$$\Phi_K(M_A) \cong M_{A_\varepsilon}. \tag{2.1}$$

Moreover notice that

$$\begin{aligned}\phi_*(\Phi_K(F)) &\cong M_Z\\ &\cong \phi_*(M_{A_\varepsilon})\\ &\cong \phi_*(\Phi_K(M_A)),\end{aligned}$$

where the first isomorphism follows from Theorem 2.1 and the third from (2.1). Since $\phi_* \circ \Phi_K$ is an equivalence of categories, assertion (ii) follows.

As for (iii), one has the chain of isomorphisms:

$$\begin{aligned}\mu hom(\mathbf{C}_A, F)_p &\cong \mu hom(\mathbf{C}_Z, M_Z)_{{}^t\phi'(\chi(p))} \\ &\cong \mu hom(\mathbf{C}_{x_1\leq 0}, M_{x_1\leq 0})_{{}^t\phi'(\chi(p))} \\ &\cong \mathrm{R}\Gamma_{x_1\leq 0}(M_{x_1\leq 0})_{\pi({}^t\phi'(\chi(p)))} \\ &\cong M.\end{aligned}$$

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