Micromechanical Definition of the Strain Tensor for Granular Materials

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In order to develop constitutive relations for granular materials from the micromechanical viewpoint, general expressions relating macroscopic stress and strain to contact forces and particle displacements are required. Such an expression for the stress tensor under quasi-static conditions is well established in the literature, but a corresponding expression for the strain tensor has been lacking so far. This paper presents such an expression for two-dimensional assemblies. This expression is verified by computer simulations of biaxial and shear tests. As a demonstration of the use of the developed expression, a study is made of the elastic moduli of two-dimensional, isotropic assemblies of bonded, nonrotating disks. Theoretical expressions are given for the elastic moduli in terms of micromechanical parameters, such as coordination number and contact stiffnesses. Comparison with the results from computer simulations show that the agreement is fairly good over a wide range of coordination numbers and contact stiffness ratios.

Introduction

Constitutive relations describing the behavior of granular materials are of great importance to various geotechnical and industrial applications. Usually these constitutive relations are developed from the continuum-mechanical viewpoint and do not recognize the discrete nature of granular materials. The resulting relations are frequently phenomenological in nature.

As an alternative to the continuum-mechanical approach, the micromechanical approach to constitutive modeling of granular materials under quasi-static conditions is being developed (for example, Cundall et al., 1982; Bathurst and Rothenburg, 1988a, b; Rothenburg and Bathurst, 1989; Rothenburg et al., 1989; Mehrabadi et al., 1993). Herein a granular material is modeled as an assembly of semi-rigid particles interacting by means of contact forces. Development of constitutive relations is performed using suitable averaging techniques.

In order to link the behavior on the micro (particle) level to the macro (continuum) level, general micromechanical expressions for the stress and strain tensors are required. The expression for the stress tensor is well established (Drescher and De Josselin de Jong, 1972; Strack and Cundall, 1978; Rothenburg and Selvadurai, 1981), but a similar expression for the strain tensor has been lacking so far in the literature. The principal aim of this paper is to develop such a general expression for the two-dimensional case. This expression will be verified by computer simulation of biaxial and direct shear tests.

As a demonstration of the use of the resulting micromechanical expressions for the stress and strain tensor, a study is made of the elastic moduli of two-dimensional, isotropic assemblies of bonded, nonrotating disks. Theoretical expressions will be derived for the elastic moduli in terms of micromechanical parameters, such as contact stiffnesses and coordination number, i.e., the average number of contacts per particle. These theoretical expressions will be compared with the results of computer simulations.

The usual sign convention from continuum mechanics for stress and strain is adopted: tensile stresses and strains are counted positive. The summation convention is adopted: summation is implied over repeated indices.

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Micromechanical Stress Tensor

Many authors (Drescher and De Josselin de Jong, 1972; Strack and Cundall, 1978; Rothenburg and Selvadurai, 1981) have proposed an expression for the average stress tensor in terms of the forces acting at the contacts between particles and the geometry of the assembly of particles. A "derivation" of this expression is repeated here, since it suggest an analogous way to "derive" the expression for the average strain tensor.

The derivation of the expression for the average stress tensor proceeds in two steps. In the first step the average stress tensor is related to quantities involving forces exerted on the particles by the boundary that encloses the assembly of particles. The second step equates these quantities involving external forces to quantities involving internal forces. The result is the micromechanical expression for the average stress tensor.

Average Stress Tensor in Terms Involving External Forces. The expression for the average stress tensor is derived under conditions of quasi-static equilibrium and in the absence of body forces. Then the (continuum) equilibrium conditions are

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0. \tag{1}$$

The two-dimensional average stress tensor in area S with boundary B is defined by

$$\overline{\sigma}_{ij} = \frac{1}{S} \int_S \sigma_{ij} dS. \tag{2}$$

From (1) and Gauss' theorem it follows

$$\overline{\sigma}_{ij} = \frac{1}{S} \int_B n_k \sigma_{ik} x_j dS \tag{3}$$

where n_i is the vector normal to the boundary; the vector tangential to the boundary is denoted by t_i. Considering the loads to be point loads on the boundary B, it follows:

$$\bar{\sigma}_{ij} = \frac{1}{S} \sum_{\beta \in B} f_i^\beta x_j^\beta \quad (4)$$

where f_i^β is the boundary force exerted on boundary particle β. This is the expression for the average stress tensor in terms involving external forces.

Average Stress Tensor in Terms Involving Internal Forces. The equilibrium conditions for particle p in the absence of body forces read

$$\sum_q f_i^{pq} = 0 \quad (5)$$

where the summation is over the particles q that are in contact with particle p and f_i^{pq} is the force exerted by particle q on particle p.

Multiplication of (5) by the position vector X_j^p of the center of mass of particle p and addition of all equations gives

$$\sum_p \sum_q f_i^{pq} X_j^p = 0. \quad (6)$$

This double sum contains one term for each boundary contact β with particle p $f_i^\beta X_j^p$, which can be rewritten as $f_i^\beta (x_j^\beta - l_j^{p\beta})$, where $l_j^{p\beta}$ is the so-called *contact vector* connecting the center of mass of particle p to the boundary contact point β.

Each internal contact between particles p and q contributes a term $(f_i^{pq} X_j^p + f_i^{qp} X_j^q)$. Since $f_i^{pq} = -f_i^{qp}$, terms corresponding to internal contacts can be written as $-f_i^{pq}(X_j^q - X_j^p)$ or $-f_i^{pq} l_j^{pq}$, where l_j^{pq} is the so-called *contact vector* connecting the centres of the particles p and q. Combinations $f_i^{pq} l_j^{pq}$ can be written as $f_i^c l_j^c$, since $f_i^{pq} = -f_i^{qp}$ and $l_i^{pq} = -l_i^{qp}$. As a result it follows that

$$\sum_{\beta \in B} f_i^\beta x_j^\beta = \sum_{c \in S} f_i^c l_j^c. \quad (7)$$

Hence it follows from (4),

$$\bar{\sigma}_{ij} = \frac{1}{S} \sum_{c \in S} f_i^c l_j^c. \quad (8)$$

This is the expression for the average stress tensor in terms involving internal forces, i.e., the micromechanical expression for the average stress tensor.

Average Stress Tensor in Terms of Group Averages. Bathurst and Rothenburg (1988a, b) have suggested grouping the contacts within a finite number of orientation classes. Then group averages $\overline{f_i^c l_j^c}(\varphi_g)$ can be calculated and relation (8) can be rewritten in terms of group averages as

$$\bar{\sigma}_{ij} = \frac{M_S}{S} \sum_g \overline{f_i^c l_j^c}(\varphi_g) E(\varphi_g) \Delta\varphi \quad (9)$$

where M_S is the total number of contacts in area S and $E(\varphi)$ is the contact orientation distribution function, as proposed by Horne (1965). $E(\varphi)\Delta\varphi$ is the fraction of contacts with orientations within $(\varphi, \varphi + \Delta\varphi)$.

The continuous form of (9), valid for an infinite assembly, becomes

$$\bar{\sigma}_{ij} = m_S \int_0^{2\pi} \overline{f_i^c l_j^c}(\varphi) E(\varphi) d\varphi \quad (10)$$

where $m_S = M_S/S$ is the contact density with respect to assembly area.

Micromechanical Strain Tensor

In analogy to the expression for the average stress tensor, an expression for the average displacement gradient is derived here. The strain tensor is obtained by taking the symmetric part of the displacement gradient tensor.

The derivation of the expression for the average displacement gradient tensor proceeds in two steps. In the first step the average displacement gradient tensor is related to quantities involving relative displacements of the boundary particles. The second step equates these quantities involving external relative displacements to quantities involving internal relative displacements. The result is the micromechanical expression for the average displacement gradient tensor, and hence for the average strain tensor.

Average Displacement Gradient Tensor in Terms Involving External Relative Displacements. The average displacement gradient tensor is defined by

$$\bar{\theta}_{ij} = \frac{1}{S} \int_S \frac{\partial u_i}{\partial x_j} dS \quad (11)$$

where u_i is the displacement vector. Using Gauss' theorem it follows that

$$\bar{\theta}_{ij} = \frac{1}{S} \int_B u_i n_j ds. \quad (12)$$

A relation based on (12) has been proposed by Strack and Cundall (1978). Constitutive relations at the contact will involve *relative* displacements between particles. Therefore it is desirable to transform (12) to a form containing derivatives of the displacements. This is done using the following identity:

$$\int_B u_i t_k ds = \int_B u_i \frac{dx_k}{ds} ds = -\int_B \frac{du_i}{ds} x_k ds. \quad (13)$$

Combining (12) and (13) gives

$$\bar{\theta}_{ij} = -\frac{1}{S} e_{jk} \int_B \frac{du_i}{ds} x_k ds \quad (14)$$

where e_{ij} is the two-dimensional permutation tensor. The discrete formulation of (14) in terms of relative displacements at the boundary is

$$\bar{\theta}_{ij} = -\frac{1}{S} e_{jk} \sum_{\alpha \in B} \Delta l_i^\alpha x_k^\alpha. \quad (15)$$

This expression for the average displacement gradient tensor is analogous to Eq. (4) for the average stress tensor. It gives the average displacement gradient tensor in terms involving external relative displacements.

Average Displacement Gradient Tensor in Terms Involving Internal Relative Displacements. The derivation of the stress tensor employed the equilibrium conditions for the particles. The equivalents for the displacement gradient tensor are the compatibility conditions for polygons. These polygons arise as a way of dividing the plane network of particle centers of mass and contacts into polygons, as depicted in Fig. 1. Various properties associated with such a subdivision of the assembly into polygons were studied by Satake (1992).

Since the polygons form closed loops, the compatibility conditions for polygon r are

$$\sum_s \Delta l_i^{rs} = 0 \quad (16)$$

where the summation is over the sides of polygon r and Δl_i^{rs} is the relative displacement between particles comprising side s of polygon r. Multiplication of (16) by the position vector

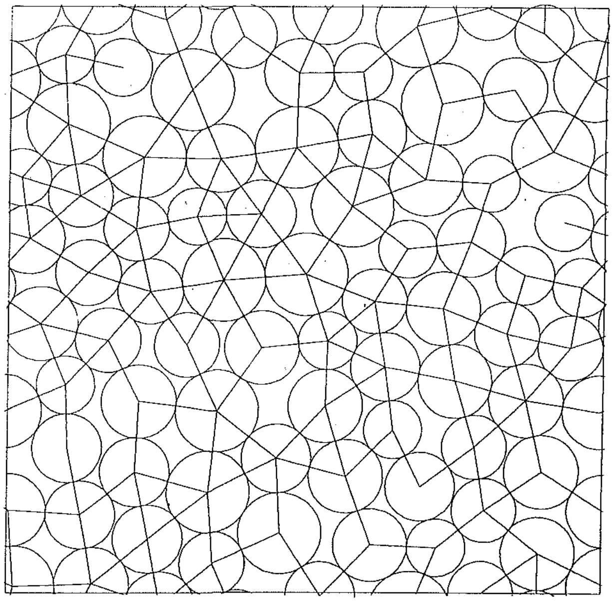

Fig. 1 Tessellation of the area into polygons, based on the contacts of the particles

V_i^r of the center of gravity of polygon r and addition of all equations gives

$$\sum_r \sum_s \Delta l_i^{rs} V_j^r = 0. \tag{17}$$

This double sum contains one term for each external side α of polygon r $\Delta l_i^{r\alpha} V_j^r$, which can be rewritten as $\Delta l_i^\alpha (x_j^\alpha - g_j^{r\alpha})$, where $g_j^{r\alpha}$ is the vector connecting the center of gravity of polygon r to boundary point α.

Each internal side contributes a term $(\Delta l_i^{rs} V_j^r + \Delta l_i^{sr} V_j^s)$. Since $\Delta l_i^{rs} = -\Delta l_i^{sr}$, terms corresponding to internal contacts can be written as $-\Delta l_i^{rs}(V_j^s - V_j^r)$ or $-\Delta l_i^{rs} g_j^{rs}$, where g_j^{rs} is the vector connecting the centers of gravity of polygons r and s. Combinations $\Delta l_i^{rs} g_j^{rs}$ can be written as $\Delta l_i^c g_j^c$, since $\Delta l_i^{rs} = -\Delta l_i^{sr}$, and $g_j^{rs} = -g_j^{sr}$. The resulting expression for (17) becomes

$$\sum_{\alpha \in B} \Delta l_i^\alpha x_j^\alpha = \sum_{c \in S} \Delta l_i^c g_j^c. \tag{18}$$

Hence it follows from (15)

$$\bar{\theta}_{ij} = \frac{1}{S} \sum_{c \in S} \Delta l_i^c h_j^c \tag{19}$$

where the so-called *polygon vector* h_j^c is defined by

$$h_j^c = -e_{jk} g_k^c. \tag{20}$$

Equation (19) is the expression for the average displacement gradient tensor in terms involving internal relative displacements. This micromechanical expression for the average displacement gradient tensor is analogous to the micromechanical expression for the average stress tensor (8). Equation (19) was first reported by Rothenburg (1980).

The expression for the average strain tensor then becomes

$$\bar{\epsilon}_{ij} = \frac{1}{S} \sum_{c \in S} \frac{1}{2} (\Delta l_i^c h_j^c + \Delta l_j^c h_i^c). \tag{21}$$

Average Strain Tensor in Terms of Group Averages. In analogy to Eq. (10) for the average stress tensor, the average strain tensor can be expressed in terms of group averages. The continuous form for the average strain tensor, valid for an infinite assembly, is

$$\bar{\epsilon}_{ij} = m_S \int_0^{2\pi} \frac{1}{2} \overline{\Delta l_i^c h_j^c + \Delta l_j^c h_i^c}(\varphi) E(\varphi) d\varphi. \tag{22}$$

Geometrical Relation

A useful geometrical relation is derived by repeating the derivation leading to (18), but with Δl_i^c replaced by l_i^c in (17). It follows that

$$\sum_{\alpha \in B} l_i^\alpha x_j^\alpha = \sum_{c \in S} l_i^c g_j^c. \tag{23}$$

From Gauss' theorem for the area of S it follows after some algebra that

$$S\delta_{ij} = -e_{jk} \sum_{\alpha \in B} l_i^\alpha x_k^\alpha, \tag{24}$$

and hence

$$\delta_{ij} = \frac{1}{S} \sum_{c \in S} l_i^c h_j^c \tag{25}$$

where δ_{ij} is the Kronecker symbol. This means that l_i^c and h_i^c are colinear on average. In terms of group averages, the continuous form of (25) is

$$\delta_{ij} = m_S \int_0^{2\pi} \overline{l_i^c h_j^c}(\varphi) E(\varphi) d\varphi. \tag{26}$$

Uniform strain and stress can now be characterized by, respectively,

$$\Delta l_i^c = \bar{\epsilon}_{ij} l_j^c \tag{27}$$

$$f_i^c = \bar{\sigma}_{ij} h_j^c \tag{28}$$

as can be verified from (8), (19), and (25).

Verification of the Micromechanical Strain Definition

The developed micromechanical strain definition for granular materials is verified by computer simulation of a biaxial test and a shear test. The average strain tensor according to the micromechanical definition is compared with the macroscopic strain tensor determined from the displacement of boundaries of the assembly.

The computer simulation is performed using the discrete element method as proposed by Cundall and Strack (1979). This essentially is an explicit time-stepping scheme for solving Newton's equations of motion. The constitutive relation at the contact as employed here is identical to that of Cundall and Strack (1979). It involves linear springs in normal and tangential direction. The tangential force is limited by dry Coulomb friction. The assembly consists of 1001 disks with various disk radii. In the biaxial test the assembly is contained in a box with height H and width B. Initial height and width are denoted by H_0 and B_0. The macroscopic (logarithmic) displacement gradient is determined from the geometry of the box by

$$\bar{\theta}_{ij} = \begin{bmatrix} \log \dfrac{B}{B_0} & 0 \\ 0 & \log \dfrac{H}{H_0} \end{bmatrix}. \tag{29}$$

In the constant volume shear test the assembly is contained in a parallelogram with angle α, constant height H, and constant width B. The macroscopic displacement gradient is determined from the geometry of the parallelogram by

$$\overline{\theta}_{ij} = \begin{bmatrix} 0 & \tan\alpha \\ 0 & 0 \end{bmatrix}. \qquad (30)$$

For both the shear test and the biaxial test the macroscopic and microscopic displacements gradients were virtually identical: the differences were of the order of the roundoff error. This result completes the verification of the micromechanical strain definition.

Elastic Moduli of Isotropic Assemblies of Bonded, Nonrotating Particles

The micromechanical expressions for the stress and strain tensors are used here to derive relations for the elastic moduli of two-dimensional isotropic systems of bonded, *nonrotating* disks. This work is an extension of Bathurst and Rothenburg (1988a, b) to a wider range of coordination numbers, but restricted to the case of nonrotating particles.

The link between contact forces and contact displacements is made through the constitutive relation at the contact. A simple linear contact model without particle rotation is assumed:

$$f_n^c = k_n \Delta l_n^c \qquad (31)$$

$$f_t^c = k_t \Delta l_t^c \qquad (32)$$

where f_n^c and f_t^c are the normal and tangential forces at the contact and Δl_n^c and Δl_t^c are the normal and tangential relative displacements at the contact. Parameters k_n and k_t are the normal and tangential contact stiffnesses. The ratio of tangential over normal stiffness k_t/k_n is denoted by λ.

In the isotropic case considered, bulk modulus K and the shear modulus G relate macroscopic stress and strain by

$$\sigma_{11} + \sigma_{22} = 2K(\epsilon_{11} + \epsilon_{22}) \qquad (33)$$

$$\sigma_{11} - \sigma_{22} = 2G(\epsilon_{11} - \epsilon_{22}) \qquad (34)$$

$$\sigma_{12} = 2G\epsilon_{12}. \qquad (35)$$

Geometrical Considerations. Isotropy of the contact distribution means that

$$E(\varphi) = \frac{1}{2\pi}. \qquad (36)$$

Contact and polygon vectors are (approximately) related to orientation by, respectively,

$$\overline{l_i^c}(\varphi) \cong l_0 n_i(\varphi) \qquad (37)$$

$$\overline{h_i^c}(\varphi) \cong h_0 n_i(\varphi) \qquad (38)$$

where l_0 and h_0 are the average lengths of the contact and polygon vectors.

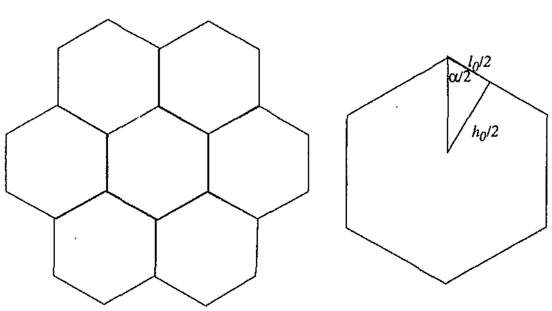

Fig. 2 (*a*) Polygons for a regular packing, (*b*) geometry of a single polygon

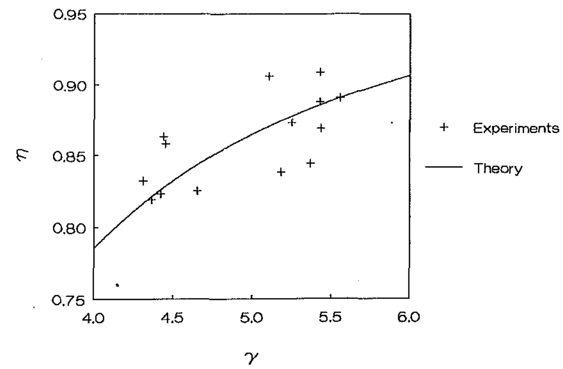

Fig. 3 Comparison of the theoretical relation for the packing density with experimental results of Quickenden and Tan (1974)

From the geometrical identity (26) it follows that

$$m_S l_0 h_0 \equiv 2. \qquad (39)$$

A simple approximate relation for the ratio of the lengths of the polygon and contact vector l_0/h_0 is derived here. Assume that the assembly is regular with coordination number γ (see Fig. 2). The average internal angle α between two sides of a polygon is then given by

$$\alpha = \frac{2\pi}{\gamma}. \qquad (40)$$

Considering the triangle formed by the center of the polygon, the midpoint of a side and a vertex it follows that

$$\tan\frac{\alpha}{2} = \frac{h_0}{l_0} \qquad (41)$$

and hence

$$\frac{h_0}{l_0} = \tan\frac{\pi}{\gamma}. \qquad (42)$$

For assemblies consisting of equal-sized disks an approximate relation for the packing density η, i.e., the fraction of area filled by the disks, can be derived with (39) and (42):

$$\eta \cong \frac{\pi}{\gamma \tan\dfrac{\pi}{\gamma}}. \qquad (43)$$

In Fig. 3 this theoretical expression is compared with experimental results from (Quickenden and Tan, 1974). The agreement is acceptable.

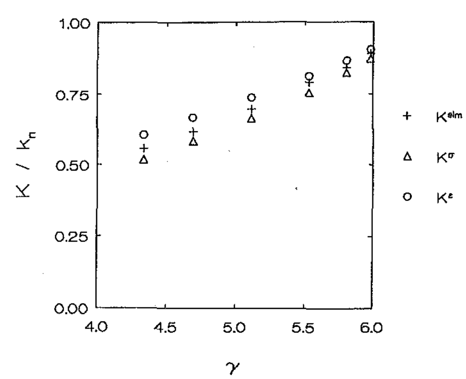

Fig. 4 Comparison between bulk moduli based on the uniform stress and strain assumptions and bulk modulus as determined from computer simulations for various coordination numbers and stiffness ratio $\lambda = 0.5$

Uniform Strain. Uniform strain is characterized by (27). Application of the constitutive relation at the contact gives

$$\sigma_{ij} = A_{ijkl}\epsilon_{kl} \tag{44}$$

where

$$A_{ijkl} = m_S l_0^2 k_n \int_0^{2\pi} (n_i n_j n_k n_l + \lambda t_i n_j t_k n_l) E(\varphi) d\varphi. \tag{45}$$

Evaluating the integrals gives the bulk modulus K^ϵ and the shear modulus G^ϵ, based on the uniform strain assumption

$$K^\epsilon = \frac{m_S l_0^2 k_n}{4} \tag{46}$$

$$G^\epsilon = \frac{m_S l_0^2 k_n}{4} \frac{1+\lambda}{2}. \tag{47}$$

Uniform Stress. Uniform stress is characterized by (28). Application of the constitutive relation at the contact gives

$$\epsilon_{ij} = B_{ijkl}\sigma_{kl} \tag{48}$$

where

$$B_{ijkl} = \frac{m_S h_0^2}{k_n} \int_0^{2\pi} \left(n_i n_j n_k n_l + \frac{1}{2\lambda} (t_i n_j + t_j n_i) t_k n_l \right) E(\varphi) d\varphi. \tag{49}$$

Evaluating the integrals gives the bulk modulus K^σ and the shear modulus G^σ, based on the uniform stress assumption

$$K^\sigma = \frac{k_n}{m_S h_0^2} \tag{50}$$

$$G^\sigma = \frac{k_n}{m_S h_0^2} \frac{2\lambda}{1+\lambda} \tag{51}$$

General Case: A Heuristic Argument. Based on the assumption of uniform strain and stress, two expressions have been derived for the bulk and shear modulus. Here a heuristic argument is given to propose expressions for the general case. Consider the case $\lambda = 0$. Conditions of static equilibrium require that $4 \le \gamma$, conditions of compatibility for the polygons require that $\gamma \le 6$. Hence it is expected that the expressions for the bulk and shear modulus based on the *uniform stress* assumption are valid for coordination number γ around 4 and that the expressions for the bulk and shear modulus based on the *uniform strain* assumption are valid for coordination number γ around 6.

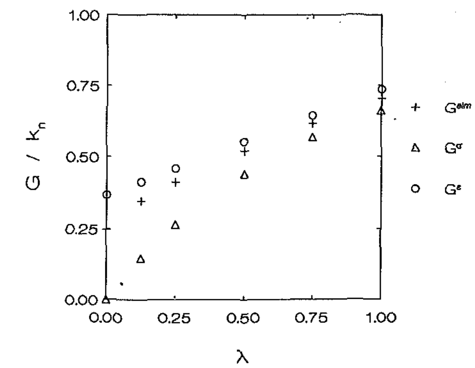

Fig. 6 Comparison between shear moduli based on the uniform stress and strain assumptions and shear modulus as determined from computer simulations for various stiffness ratios and coordination number γ = 5.12

The general heuristic expression for the bulk and shear modulus proposed here is obtained by (weighted) interpolation of the moduli obtained from the uniform strain and stress assumption:

$$K(\gamma) = \frac{\gamma - 4}{2} K^\epsilon + \frac{6-\gamma}{2} K^\sigma \tag{52}$$

$$G(\gamma) = \frac{\gamma - 4}{2} G^\epsilon + \frac{6-\gamma}{2} G^\sigma. \tag{53}$$

Using the geometrical relations (39) and (42) it follows that

$$K(\gamma) = \frac{\gamma - 4}{2} \frac{k_n}{2} \sqrt{3} + \frac{6-\gamma}{2} \frac{k_n}{2} \tag{54}$$

$$G(\gamma) = \frac{\gamma - 4}{2} \frac{k_n}{2} \sqrt{3} \frac{1+\lambda}{2} + \frac{6-\gamma}{2} \frac{k_n}{2} \frac{2\lambda}{1+\lambda}. \tag{55}$$

Computer Simulations. Computer simulations of two-dimensional isotropic assemblies of bonded, nonrotating disks have been performed using the discrete element method. A relatively narrow particle size distribution was employed for the disk radii. Simulations were performed over the range of coordination numbers $4 \le \gamma \le 6$ and stiffness ratios $0 \le \lambda \le 1$.

Figure 4 gives the comparison between the bulk moduli based on the uniform stress and strain assumptions and the bulk modulus as determined from the computer simulations for stiffness ratio $\lambda = 0.5$. A similar comparison for the shear moduli is presented in Fig. 5. For $\gamma = 5.12$, the shear moduli based on the uniform stress and strain assumptions are compared with the shear modulus as determined from the computer simulations in Fig. 6. These figures seem to indicate that the elastic moduli based on the uniform strain assumption form an upper bound

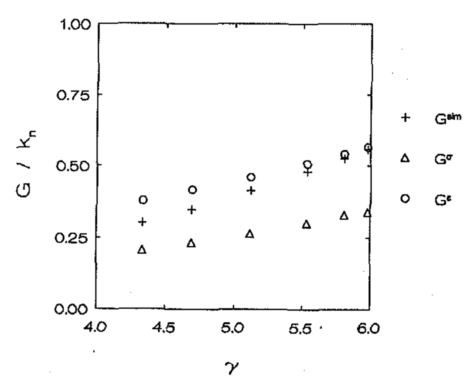

Fig. 5 Comparison between shear moduli based on the uniform stress and strain assumptions and shear modulus as determined from computer simulations for various coordination numbers and stiffness ratio λ = 0.5

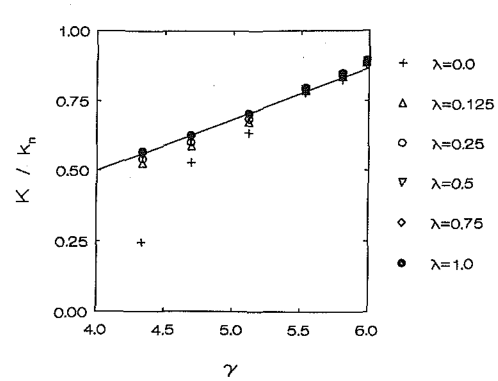

Fig. 7 Comparison between theoretical bulk modulus and bulk modulus as determined from computer simulations for various coordination numbers and stiffness ratios

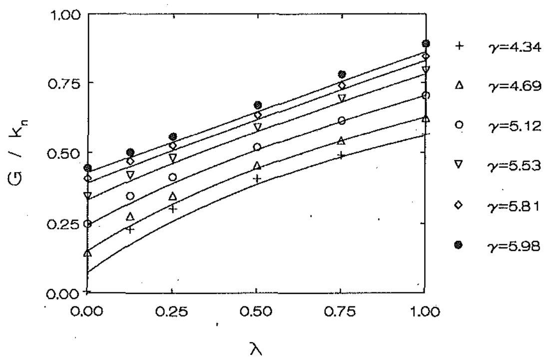

Fig. 8 Comparison between theoretical shear modulus and shear modulus as determined from computer simulations for various coordination numbers and stiffness ratios

to the elastic moduli, while the elastic moduli based on the uniform strain assumption form a lower bound to the elastic moduli. These observations are rigorously proven in Rothenburg and Kruyt (1994).

The bulk and shear modulus as determined from the computer simulations are shown in Figs. 7 and 8, together with the theoretical predictions according to (54) and (55). The agreement is fairly good, although larger deviations occur for low coordination number γ and low stiffness ratio λ.

Discussion

A micromechanical expression for the average strain tensor has been developed in the two-dimensional case, similar to an existing micromechanical expression for the average stress tensor. These relations exhibit the following duality:

$$f_i^c \leftrightarrow \Delta l_i^c$$

$$l_j^c \leftrightarrow h_j^c$$

$$\frac{1}{S}\sum_{c\in S} f_i^c l_j^c = \bar{\sigma}_{ij} \leftrightarrow \bar{\epsilon}_{ij} = \frac{1}{S}\sum_{c\in S} \Delta l_i^c h_j^c.$$

The micromechanical definition of the strain tensor has been applied to the development of (heuristic) theoretical expressions for the elastic moduli of two-dimensional, isotropic assemblies of bonded, nonrotating particles. These expressions give the moduli in terms of the coordination number γ and contact stiffnesses k_n and k_t. In contrast to Bathurst and Rothenburg (1988a, b) it has been possible to derive the dependencies of the elastic moduli K and G on coordination number γ. The developed theory has been compared with results from computer simulations; agreement is fairly good over the range of coordination numbers and stiffness ratios considered. The largest deviations occur in the case of zero tangential stiffness k_t and coordination number γ around 4. The reason for these deviations is that the system is least stable under these conditions.

The assumption of uniform strain is frequently made in continuum mechanics of heterogeneous systems (for example, Batchelor and O'Brien, 1977). This study and a related study (Rothenburg et al., 1987) show that the uniform strain assumption does not always lead to accurate prediction of the system behavior. In fact, two regimes have been distinguished here, uniform stress and uniform strain, each with its range of validity.

Future work is directed towards development of theories for the elastic moduli of bonded systems *with* particle rotations. Another subject of research is the extension of the presented micromechanical strain definition to the three-dimensional case.

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