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Proofs and refutations in the undergraduate mathematics classroom

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Abstract In his 1976 book, *Proofs and Refutations*, Lakatos presents a collection of case studies to illustrate methods of mathematical discovery in the history of mathematics. In this paper, we reframe these methods in ways that we have found make them more amenable for use as a framework for research on learning and teaching mathematics. We present an episode from an undergraduate abstract algebra classroom to illustrate the guided reinvention of mathematics through processes that strongly parallel those described by Lakatos. Our analysis suggests that the constructs described by Lakatos can provide a useful framework for making sense of the mathematical activity in classrooms where students are actively engaged in the development of mathematical ideas and provide design heuristics for instructional approaches that support the learning of mathematics through the process of guided reinvention.

Keywords Lakatos · Guided reinvention · Abstract algebra · Realistic mathematics education · Proof · Proving · Defining refutations · Counterexamples · Monster barring · Exception barring · Proof-analysis

1 Introduction

Mathematics is often presented in what Lakatos (1976) refers to as the deductivist style. The presentation begins with carefully stated axioms and definitions, followed by carefully worded theorems and their proofs. Lakatos argues that this presentation hides the process by which the mathematics was discovered. In his book, *Proofs and Refutations*, Lakatos presents two case studies to illustrate methods of mathematical discovery in the history of mathematics. These case studies make it clear that a great deal of mathematical activity is often hidden behind the precisely worded definitions and theorems that we find in mathematical texts.

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Freudenthal (1991) makes the related observation that “Traditionally, mathematics is taught as a ready made subject. Students are given definitions, rules and algorithms, according to which they are expected to proceed” (pp. 47–48). Freudenthal argues that most students do not learn mathematics this way and that students should be permitted to reinvent mathematics for themselves. In this paper, we will explore the utility of Lakatos' (1976) description of historical methods of mathematical discovery as a framework for making sense of and supporting students' reinvention of mathematics.

Many mathematics educators have made brief reference to Lakatos' (1976) historical case studies and his description of processes of mathematical discovery (e.g., Alcock and Simpson 2004; Mariotti and Fischbein 1997; Zaslavsky and Shir 2005). Zandieh and Rasmussen (2007) note that the formulation of a definition in a classroom community can evolve through processes like those described by Lakatos, while Lampert (1990) draws a number of parallels between her 5th-grade students' mathematical activity and Lakatos' case studies.

Other researchers have drawn more extensively on Lakatos' ideas. For example, De Villiers (2000) describes an example in which the Fibonacci series is generalized through a process roughly like that described by Lakatos. In this case a conjecture and proof are produced by high school students and later become the focus of an exchange in a mathematics journal, eventually resulting in significant modifications to the conjecture and the proof.

Balacheff (1991) investigated 13–14 year old students' treatments of refutations. These treatments of counterexamples included some of the methods attributed to mathematicians in Lakatos' (1976) historical case studies. Balacheff examined students' treatments of counter-examples with an eye toward understanding what influenced their choices. Our interest in these kinds of activities is from an instructional design perspective; we are concerned with the role these processes can play in supporting the reinvention of mathematics in the classroom.

We draw this notion of reinvention from the instructional design theory of realistic mathematics education (RME), which has grown out of the desire to develop mathematics education that is consistent with Freudenthal's (1991) view of mathematics as a human activity. RME is continually under construction as researchers conduct projects to develop local (content specific) instructional theories and to elaborate emerging general design principles. Recently a number of these projects have been conducted at the undergraduate level (e.g., Gravemeijer and Doorman 1999; Larsen 2004; Rasmussen and King 2000; Zandieh and Rasmussen 2007).

A primary characteristic of RME is the guided reinvention of mathematics. Gravemeijer and Doorman (1999) explain that, “The idea is to allow learners to come to regard the knowledge that they acquire as their own private knowledge, knowledge for which they themselves are responsible” (p. 116). Products of reinvention can include algorithms, definitions, conjectures, and proofs. As RME has developed, heuristics have emerged to guide the design of instruction that supports students' reinvention of mathematics. For example, the *emergent models* heuristic provides a framework for supporting the constitution of formal mathematics based on students' informal strategies (Gravemeijer 1999). A primary goal of the research reported here is to explore the utility of recasting Lakatos' (1976) methods of mathematical discovery as heuristics for designing instruction that supports students' reinvention of mathematics.

2 The method of proofs and refutations

Lakatos (1976) presents two detailed case studies from the history of mathematics. One of these concerns the discovery of the concept of uniform convergence of sequences of functions. We draw on this case study for illustration as we describe some of the methods of mathematical discovery outlined by Lakatos. Following this description, we recast Lakatos' constructs as analytical tools, highlighting the aspects of the constructs that we viewed as most essential during our analyses.

2.1 The methods of mathematical discovery described by Lakatos

In Lakatos' (1976) case study, the process of discovery begins with an initial conjecture, referred to by Lakatos as the *primitive conjecture*. For example, throughout the eighteenth century, it was simply assumed that the limit of a convergent sequence of continuous functions was continuous. In 1821, Cauchy stated this conjecture and provided a proof. The flaw in the proof was undiscovered until 1847, in spite of the fact that Fourier's work had produced counterexamples as early as 1812.

It is important to note that, at this time, the concepts of function, limit, and continuity were still being developed, and it was only in light of the emerging modern definitions of these concepts that the examples found in Fourier's work were considered to be counterexamples. Lakatos describes a number of different ways that mathematicians responded to potential counterexamples to theorems that had been previously proved.

One method of dealing with potential counterexamples was to reject them. For example, the pathological functions of analysis were dismissed by some mathematicians as monsters, and not functions at all. This method, dubbed *monster-barring* by Lakatos, involved devising definitions that barred these examples from consideration. This method can be problematic because a definition that bars one kind of counterexample may admit another counterexample or bar examples for which the theorem is valid. However, although Lakatos describes monster-barring as primitive, it can be an important technique, since it is often the case that what is needed is to "give more precise meaning to the terms" (Pólya 1954, p. 44).

Lakatos (1976) also describes the method of *exception-barring*, which consists of enumerating the exceptions to the theorem in order to find a safe domain of validity for the theorem. In this case, counterexamples are recognized as legitimate examples, but treated as exceptions to the theorem. For example, Abel restricted the domain of suspect theorems about functions to those functions expressible as power series. The primary weakness of exception-barring is that even if one can be sure that all exceptions have been discovered, there is the risk of too severely restricting the theorem's domain of validity.

The method of *proofs and refutations* is a more sophisticated method of mathematical discovery consisting of four primary stages. The first stage is the primitive conjecture. The second stage is the *proof*, characterized by Lakatos (1976) as "a rough thought experiment or argument, decomposing the primitive conjecture into sub-conjectures or lemmas" (p. 127). The third stage is the *emergence of global counterexamples*. These counterexamples are global in the sense of applying to the primitive conjecture rather than merely one of the sub-conjectures. The fourth stage is the *analysis of the proof* to discover the lemma (perhaps hidden) to which the global counterexample is a local counterexample. The result of this stage is an improved conjecture featuring a new *proof-generated* concept.

According to Lakatos, Seidel discovered the concept of uniform convergence (a hidden assumption in Cauchy's proof) and the *method of proofs and refutations* in 1847. The characteristic that distinguishes the method of proofs and refutations from exception-barring is the proof-analysis stage. Seidel discovered the concept of uniform convergence by examining Cauchy's proof to find the hidden assumption. The result was an improved conjecture featuring this new proof-generated concept (the limit of a uniformly convergent series of continuous functions is continuous).

2.2 Recasting Lakatos' constructs for use in analyzing classroom activity

In many ways, the context of the undergraduate mathematics classroom is quite different from the historical setting in which mathematical ideas were originally developed. For example, in the classroom, the teacher or textbook author may be seen as a mathematical authority who knows the "correct" answer, while no such authority was available to mathematicians when the ideas were originally being developed. However, in a mathematics classroom community in which students see themselves as responsible for the development of the mathematical ideas, their mathematical activity may be strikingly similar to that of creative mathematicians.

The kinds of mathematical activity that Lakatos describes seem to require a particular mathematical situation. There must be a primitive conjecture as well as a proposed justification for this conjecture. This conjecture will involve one or more concepts whose definitions may or may not have been carefully formulated or fully understood by the students. Finally, there must be one or more counterexamples to the conjecture.

When categorizing students' mathematical activity in response to this situation as monster-barring, exception-barring, or proof-analysis, we primarily consider the focus of the students' attention and the outcome of their activity. In particular, we analyze whether the students' attention is focused on the counterexample(s), definition(s), the conjecture, or the proof, and whether the students' activity results in a modification to a definition or to the conjecture.

We categorize as *monster-barring* any response in which the counterexample is rejected on the grounds that it is not a true instance of the relevant concept. We include behaviors varying in sophistication from simply rejecting the example out of hand with no reason given, to carefully formulating a definition to exclude a counterexample. The focus of the students' activity in the case of monster-barring is on the counterexample and the underlying definition(s). The typical outcome of monster-barring activity is a modification or clarification of a definition.

We categorize as *exception-barring* any response that results in a modification of the conjecture to exclude a counterexample without reference to the proof. We include behaviors varying in sophistication from simply listing counterexamples as exceptions, to reformulating the conjecture by restricting its domain to exclude the counterexample. Exception-barring differs from monster-barring in that the counterexample is recognized as legitimate and thus the conjecture is modified rather than the underlying definition. The focus of the students' activity in the case of exception-barring is on the counterexample and the conjecture. The typical outcome of exception-barring activity is a modification of the conjecture.

The method of proofs and refutations is distinguished from the other methods by the central role of the proof and in particular what Lakatos calls *proof-analysis*. With proof-analysis, the resulting modification to the conjecture is intended to make the proof work rather than simply exclude the counterexample from the domain of the conjecture (as is the case with exception-barring). The focus of the students' activity in the case of proof analysis is on the proof as well as the counterexample and conjecture. The typical outcome of proof-analysis activity is a modification of the conjecture and sometimes a definition for a new proof-generated concept. Table 1 summarizes our reframing of these three constructs of Lakatos.

3 Background for the classroom episode

Before presenting the classroom episode, we briefly situate its analysis within our research program and offer specific details about the classroom environment from which the episode

Table 1 Reframing the methods of mathematical discovery described by Lakatos (1976)

Type of activity	Focus of activity	Outcome of activity
Monster-barring	Counterexample & underlying definitions	Modification or clarification of an underlying definition
Exception-barring	Counterexample & conjecture	Modification of the conjecture
Proof-analysis	The proof, the counterexample, & the conjecture	Modification of the conjecture & sometimes a definition for a new proof-generated concept

was drawn. Our research is part of a growing effort to explore the utility of the theory of realistic mathematics education for supporting the learning of undergraduate mathematics. There are ongoing RME-guided instructional design projects at the undergraduate level in the areas of geometry (see Zandieh and Rasmussen 2007; Zandieh et al. 2008), differential equations (see Rasmussen and King 2000), and abstract algebra (see Larsen 2004; Weber and Larsen 2008). Our report will focus on a classroom episode from an abstract algebra project. Although the design of the course drew extensively on some of the more recent ideas of RME (e.g. the emergent models heuristic), the analysis reported here is primarily intended to connect the ideas of Lakatos (1976) with Freudenthal's (1973) more general notion of reinvention.

The abstract algebra episode was drawn from a classroom teaching experiment (Cobb 2000; Steffe and Thompson 2000) in an introductory group theory course. The first author was the teacher for the course, a regular 10-week course taken primarily by third-year mathematics majors. Much of the course curriculum had been designed previously through a series of RME-guided teaching experiments (Larsen 2004). An important aspect of the course was its emphasis on the development of mathematical concepts through a process of guided reinvention.

A significant data corpus was collected during the teaching experiment. Two cameras were used to videotape each class session. During small group activities, these cameras were focused on two specific groups. During whole class discussions, one camera focused on the front of the classroom while the other captured the rest of the classroom activity. Many of the students' written artifacts were collected, including journal entries, exams, and homework.

The teaching experiment data was first analyzed using iterative analysis techniques modified from those described by Cobb and Whitenack (1996) and Lesh and Lehrer (2000). One outcome of this analysis was the observation that students' defining activity could be supported by their proving activity (Larsen and Zandieh 2005). This perspective on the students' activity struck us as reminiscent of the process of proofs and refutations described by Lakatos (1976). This paper is the result of our using Lakatos' constructs to re-analyze the data from this point of view and explore implications for supporting the guided reinvention of mathematics.

4 Classroom episode: A theorem about subgroups

The classroom episode described here illustrates the reinvention of an abstract algebra theorem in an undergraduate classroom through processes like those described in Lakatos' (1976) historical case studies. An introductory abstract algebra class is engaged in the task of finding the smallest set of conditions that is sufficient to ensure that a subset of a group is a subgroup. We will focus our attention on the mathematical activity of one group of students (Phil, Steve, and Mike) who generated a number of different conjectures, developed arguments in support of these conjectures, and then were faced with counter-examples (from the teacher's perspective). Thus, they found themselves in a situation that was similar in some ways to that of the mathematicians featured in Lakatos' case studies.

4.1 Definition of subgroup

Prior to this episode, the class had developed a definition for the group concept through a process of guided reinvention (see Weber and Larsen 2008). A *group* was defined to be a set, ***G***, together with a binary operation, •, that satisfies the following properties: (1) ***G*** is *closed* under the operation, (2) ***G*** has an *identity* element, (3) Each element of ***G*** has an *inverse* in ***G***, and (4) The *associative* law holds. The teacher informed the class that a *subgroup* is a subset of a group that is a group itself under the same operation and noted that a closed subset of a group inherits the associative property. Thus as the students began the task, they were aware that closure, the existence of an identity element, and the existence of inverses formed a set of sufficient conditions for a subset of a group to be a subgroup.

4.2 The primitive conjecture is proposed

The students quickly developed a conjecture: Any subset of a group that is closed under the operation is a subgroup.

> Phil: Closure, and after closure…
>
> Steve: I think it's just closure.
>
> Mike: You only need to check closure as long as you know it's a subset of a group.

4.3 A proof

Phil almost immediately offered a proof of the primitive conjecture and shared this with the teacher and his group. This proof drew on an idea that had been significant during the reinvention of the group concept—each element of a group appears exactly once in each row of the operation table. We present Phil's proof as he explained it to the teacher and then provide an analysis of this proof to explicate his way of thinking and identify the hidden lemma.

> Phil: Closure means each element appears exactly once….Well, we already proved it for the infinite case that each element will appear exactly once in each row and column. So if we know it's going to appear exactly once in each row or column then we can make x the arbitrary element which means a times the arbitrary element still guarantees that a is somewhere in there. So if we solve for x then x would have to be I. And then if we know I is in the group then we can basically say a times some arbitrary element will still give me I in the group, and then if you solve for x…

The *first step* of Phil's proof is to observe that (1) in the operation table for a group each element occurs exactly once in each row and (2) if the subset is closed under the group operation, only elements from the subset appear in the operation table for the subset. Phil deduces from this that in each row of the subset operation table, each element of the subset appears exactly once.

The *second step* of Phil's proof is to consider the row of the operation table that corresponds to the subset element a. Since a is an element of the subset, it must appear in this row of the operation table. If we let x be the element that corresponds to the column in which this a appears, we see that $a \bullet x = a$. This means that the element x must be the identity element I of the group. Thus the subset has an identity element, I.

The *third step* of Phil's proof is to again consider the row corresponding to the arbitrary subset element a. Since the identity element I is an element of the subset, it must also appear in this row of the operation table. If we let x be the element that corresponds to the column in which this I appears, we see that $a \bullet x = I$. This means that the element x must be the inverse of a. Thus the inverse of each element of the subset is also in the subset.

Given that the class had already agreed that the associative law held for any closed subset of a group, this proof establishes that a closed subset of a group possesses all four conditions necessary for it to be a subgroup under the same operation as the original group.

A hidden lemma can be found in the first step of the proof. It is true that in the operation table for a closed subset of a group only the subset elements can appear and none of these can appear more than once in a row (this second property is inherited from the original group table). However, in general it does not follow from this fact that each subset element must appear at least once in each row of the subset operation table. In the finite case, this is true (it is a consequence of the pigeon-hole principle). This does not work if the subset is infinite. So the hidden lemma is that *each element occurs at least once in each row of the subset operation table*, and one way to ensure that this lemma is true is to require the subset to be finite. (Note that there is a second hidden lemma—the subset must be non-empty. However, this issue did not come up during the discussion related to this proof.)

4.4 The emergence of a global counterexample

Immediately upon hearing this proof, the teacher (T) offered a counterexample:

Phil: I'm trying to prove you only need closure.

T: So consider the following example: real numbers under addition. Is that a group?

Phil: Yeah.

T: Now, consider the following subset: positive numbers under addition.

The positive real numbers do not form a subgroup (among other things, this subset does not include the additive identity zero). This global counterexample is also a local counterexample to the first step of the proof. For example, if we consider the row of the table corresponding to the number two, we see that one does not appear in this row. Thus it is not true that each element appears at least once in each row of the operation table of this subset.

4.5 Monster-barring and exception-barring

The students did not immediately view the example as a counterexample although the teacher provided it. The students' initial response was to reject the counterexample, to bar it as a monster in defense of their conjecture. However, unlike the monster-barring described in Lakatos' (1976) historical case studies, the students' attempt to bar the example was quite limited (and did not involve modifying a definition). This difference is likely due in part to the fact that the teacher provided the example. Nevertheless, we categorize this activity as monster-barring because the focus is strictly on the counterexample and the students did not modify the conjecture.

Phil: I forgot to say it has to have the same group operation.

T: I didn't change the operation.

Mike: It's not closed.

T: Are you sure?

After the teacher rebuffed the students' attempts to bar the counterexample as a monster, there was an immediate shift to the exception-barring method.

Phil: Not a subgroup because don't have inverses.

T: You didn't say I had to have inverses; you said I only had to be closed.

Steve: He's right.

Phil: [I'm] trying to think of a way around it.

Steve: So it's inverses and closure.

Phil's honest assessment of his approach is particularly telling. He is aware that what he is trying to do at this point is to defend his conjecture and proof. While Steve's modification results in a conjecture that is nearly correct, it is the result of exception-barring not proof analysis, because the focus of his activity is only the conjecture and counterexample. This modification does not address the flaw in Phil's proof, and would actually require a different approach. Perhaps for this reason, the group ignores Steve's conjecture.

4.6 Two more conjectures are proposed and refuted

Rather than attending to Steve's conjecture, Phil and Mike proposed two new conjectures stating necessary conditions for a subset of a group to be a subgroup. It is not clear whether these conjectures are actually meant to modify the original conjecture, but it is clear that the focus of the students' attention is on the counterexample (and not a proof) and the outcomes are conjectures. Thus the students' activity in this case is akin to exception-barring. The students quickly dismissed these two conjectures by producing counterexamples.

Phil: A subgroup has to be a finite set.

Mike: How so?

Phil: Because if you have an infinite group, one less from infinity isn't infinite anymore, it's a finite thing. I don't know.

Mike: [The real numbers] would be a subgroup of [the real numbers], both infinite.

Phil: Right. That means you'd have to include your entire group if [the real numbers] is a subgroup. Which means if you had your group as all reals and you're trying to call that a group unto itself, well, you'd still have to include all negative numbers.

Mike: So that would mean a subgroup of an infinite group has to be infinite and a subgroup of a finite group has to be finite.

Phil: Right, but if it's finite—

Mike: Wait, a subgroup of an infinite group, does it have to be infinite? A subgroup of [the real numbers] could just be $\{0\}$ if you're talking addition.

Phil: Right, if you're talking multiplication you could have a subgroup $\{1, -1\}$.

4.7 An improved conjecture is produced

After dismissing their conjectures, Mike and Phil more carefully analyzed the counterexample originally offered by the teacher. Apparently with his earlier proof in mind, Phil formulated an improved conjecture: If the group is finite then any subset that is closed under the operation is a subgroup.

Mike: He was trying to give us the example of all reals being a group and then you take a portion, all positive reals.

Phil: All non-negative reals—you'd have to include 0.

Mike: Basically how would you prove it's not a subgroup?

Phil: How would you prove it's not a subgroup? Take 3, what's the inverse of 3 in addition?

Mike: He was just saying closure is not sufficient.

Phil: If you're talking about an infinite group—you were talking about finite groups before so maybe there's a couple different cases. If it's finite then you only need closure.

Note that with this condition of finiteness added, Phil's original proof works to prove the new conjecture (if the subset is non-empty). We argue that this condition is the result of a proof-analysis rather than exception-barring because when questioned by a teaching assistant (TA), Phil immediately refers back to his original proof and notes that it works for a finite set.

TA: You proved that? You already wrote that down?…Where does x come from?

Phil: x is just an arbitrary value.

TA: Of G, or it's an element of the subset? It's an element of the subset, okay.

Phil: You're assuming that basically $ax=a$ then x is not really arbitrary, it's really I [the identity], but that's what I'm trying to prove…. It works for a finite set.

4.8 Episode summary

In this classroom episode, a standard theorem of abstract algebra was reinvented through a process very much like that described in Lakatos' (1976) historical case studies. The students began with a primitive conjecture (that closure is sufficient to guarantee a subset of a group is a subgroup). Phil described a thought-experiment that seemed to prove the conjecture. However, the teacher produced a counterexample to this conjecture. The students first tried to dismiss this counterexample (monster-barring), and then tried to exclude it from the domain of the theorem (exception-barring). Finally, Phil noticed that the proof would work in the finite case (proof-analysis). The end result of this process was a version of a standard theorem from abstract algebra that a nonempty finite closed subset of a group is a subgroup.

5 Conclusions and discussion

Our analysis of this classroom episode provides an existence proof; it illustrates that mathematics can be reinvented in undergraduate mathematics classrooms through processes that strongly parallel those described by Lakatos in his 1976 book, *Proofs and Refutations*. In this section we briefly discuss the value of Lakatos' constructs for describing and explaining students' mathematical activity and argue that Lakatos' methods of mathematical discovery can be reconceived as instructional design heuristics.

5.1 Lakatos' constructs as a lens for describing and explaining students' mathematical activity

Our recasting of Lakatos' (1976) ideas as a framework for teaching and learning provided a powerful lens for making sense of the mathematical activity in the classroom episode we analyzed. Like the mathematicians of Lakatos' case studies, the students engaged in monster-barring, exception-barring, and proof-analysis on the way to developing significant mathematical ideas. Thus this framework provided a nice structure for describing the mathematical activity in the presented episode. However, we note that the framework can be explanatory as well. For example, we can explain why the students disregarded Steve's conjecture by drawing on the differences between exception-barring and proof-analysis. Since Steve's conjecture was the result of exception-barring and not proof-analysis, it did not fulfill the students' intellectual need (Harel 2001) to make Phil's proof work. This proof seemed to make sense to the group, and although this new conjecture barred a counterexample, it did not reveal or fix a flaw in the proof.

We also note that the framework we have adapted from Lakatos (1976) provides a way to explain the overall success of the process of reinvention observed in the group theory episode. How were the students able to develop the theorem stating that every (nonempty) finite subset of a group that is closed under the group operation is a subgroup? Drawing on the framework we argue that this theorem was proof-generated. The theorem developed out of an analysis of Phil's attempt to prove that all closed subsets of groups are subgroups. This naïve conjecture probably came from the students' experiences finding subgroups of finite groups by first identifying subsets that were closed under the operation. Phil's thought experiment came very close to proving the conjecture. In order to make his proof work, Phil needed to be able to deduce that each subset element appeared in each row of the subset operation table. He eventually realized that this would work in the finite case, and this realization led to the final version of the theorem.

5.2 Methods of mathematical discovery as heuristics for instructional design

As mentioned earlier, the design of the course from which the classroom episode was drawn was guided by the theory of realistic mathematics education. However, the course was not intentionally designed to promote students' engagement in the kinds of processes described by Lakatos (1976). It was only after our second major pass through the data that we began to draw the parallels that inspired the analysis presented here. We believe that this analysis suggests that Lakatos' (1976) constructs of monster-barring, exception-barring, and proof-analysis, can serve as heuristics for designing instruction to support reinvention.

For example, if an instructional designer has the goal of supporting the reinvention of a specific concept, he or she could attempt to engage the students in the method of proofs and refutations. He or she could begin by identifying important mathematical results that depend on this particular concept (i.e., what proof might be able to generate this concept). Then instruction could be designed to evoke one or more of these results in the form of a primitive conjecture. The students could be asked to propose an argument to support the conjecture, or the teacher could propose one. The students could then be asked whether the conjecture is always true and encouraged to look for counterexamples, or the teacher could propose counterexamples. As the students respond to these counterexamples, they should be encouraged to focus on both the proof and the counterexamples so that through a proof analysis they may discover what condition is necessary to make the proof work and as a result reinvent the desired concept.

This approach to reinvention is consistent with Harel's (2001) necessity principle which states that "students are likely to learn if they see need for what we intend to teach them" (pp. 207–208). Here, the concept is seen as necessary because it is needed to make the proof work and the conjecture true. For example, in the classroom episode, the condition that the group be finite was seen as necessary to make Phil's proof work. Note that the required condition (finiteness) did not necessitate a new concept. For an example of a proof analysis that supported the development of a new concept, see Larsen and Zandieh (2005). In this case the concept of a *small triangle* on the sphere was defined by students through an analysis of a proof of the side-angle-side congruence theorem.

The method of monster-barring by itself could also be used to support students' defining activity. Here the students' attention would be focused on the counterexamples and the definition. For example, in their description of students' defining of triangle, Zandieh and Rasmussen (2007) identify the important role of non-examples of triangles during the negotiation of the definition; the students revised their definitions to bar these monsters. Similarly, exception-barring may be sufficient to produce the appropriate revision of a conjecture. Here the students' attention would be focused on the counterexample and the conjecture. For example, in the abstract algebra episode, Steve's conjecture was the result of exception-barring.

5.3 Summary

We argue that Lakatos (1976) method of proofs and refutations along with the processes of monster-barring and exception-barring can be profitably adapted and applied to mathematics education. Our analysis provides an existence proof that mathematics can be reinvented in an undergraduate mathematics classroom through mathematical activity that has strong parallels to the processes described by Lakatos. This analysis also supports our contention that these processes provide a useful framework for making sense of students' and teachers' mathematical activity in classrooms in which students are actively engaged in the development of mathematical ideas. Finally, our analysis suggests that these processes can provide useful heuristics for the design of instruction that supports the guided reinvention of mathematics.

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