Derivatives and the Price of Risk

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INTRODUCTION

Risk-neutral derivative valuation often requires the input of an unobservable market price of risk. Fortunately, the risk-neutral approach provides two useful results that can help identify the price of risk. First, the price of risk is equal across all derivatives contingent on the same stochastic variable. This allows one to extract information from traded securities and to use the information to value other assets, though any inference about the price of risk requires an assumption about its specification. Second, the approach provides a partial-differential equation that includes the price of risk, the solution of which is the value of the derivative under investigation. However, the precise form of the equation, and therefore the calculated derivative value, is affected by the assumed specification of the price of risk. This article illustrates the potential for valuation error when the price of risk is specified incorrectly.

Researchers have assumed a variety of risk specifications. Vasicek (1977), for example, assumes that the price of interest-rate risk is constant. Similarly, Gibson and Schwartz (1990) assume that the price of crude oil convenience yield risk is constant. Hull and White (1990) assume that the price of interest-rate risk is specified as in the Cox, Ingersoll, and Ross (1985) model. This article provides two examples that demonstrate that these seemingly innocuous assumptions can have a dramatic impact on derivative valuation. First, several widely used specifications of

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the price of interest-rate risk are calibrated with a common data set of discount bond prices. Coupon bonds and bond options are then valued, and it is found that values vary significantly across risk specifications. Second, risk specifications are calibrated with the use of Eurodollar futures options data are calibrated. It is found that a term structure of risk tightens in-sample fit and reduces out-of-sample valuation error. These results highlight the importance of testing assumptions about the price of risk when risk-neutral valuation is used.

Some researchers have avoided the problem of risk misspecification. Cox et al. (1985) develop a general equilibrium model of interest rates that yields a fully specified market price of risk and analytic derivative values; however, their model requires layers of assumptions regarding the economy and its participants. Heath, Jarrow, and Morton (1990) and Stapleton and Subrahmanyam (1993) develop alternative techniques that avoid both the price of risk and the numerous assumptions necessary for a general equilibrium model. Risk-neutral valuation remains alluring, though, due to its intuitive appeal.

The rest of this article is organized as follows. Section 1 derives the familiar partial differential equation, that forms the basis of the risk-neutral approach to valuing derivative securities. The derivation includes a discussion of the price of risk. Section 2 shows how several popular risk specifications can result in dramatically different derivative values. Section 3 tests a term structure of risk with the use of Eurodollar futures options data. Section 4 concludes the article.

RISK-NEUTRAL DERIVATIVE VALUATION

This section summarizes the risk-neutral approach to derivative valuation. Brennan and Schwartz (1979) presented a similar discussion over 15 years ago, and Hull (1989) reviews it in his textbook. Because the market price of risk is central to this article, a review of its derivation is warranted. For purposes of exposition, a model with one stochastic variable, and hence one market price of risk, is used. The derivation could be expanded to the multifactor case with little difficulty.

Suppose a series of derivatives have payoffs contingent on a random variable, θ, which is governed by the following stochastic differential equation:

$$d\theta = \mu(\theta, t)\, \theta\, dt + \sigma(\theta, t)\, \theta\, dz \tag{1}$$

where dz is an increment to a Wiener process. When μ and σ are constant, eq. (1) describes geometric Brownian motion, the familiar Black and Scholes (1973) process for stock prices. Consider two derivatives with payoffs contingent on θ. Let f_i denote the value of derivative i. With the

use of Ito's lemma, the stochastic process governing changes in the value of each derivative is

$$df_i = \left[\frac{\partial f_i}{\partial t} + \frac{\partial f_i}{\partial \theta} \mu\theta + \frac{1}{2}\frac{\partial^2 f_i}{\partial \theta^2}\sigma^2\theta^2\right] dt + \frac{\partial f_i}{\partial \theta}\sigma\theta \, dz \qquad (2)$$

where the arguments of μ and σ are suppressed for brevity. Let m_i and s_i denote the expected change in the value of derivative, i, and the volatility of changes in the value of derivative, i, respectively. With the use of eq. (2), these two functions are defined as

$$m_i = \left[\frac{\partial f_i}{\partial t} + \frac{\partial f_i}{\partial \theta}\mu\theta + \frac{1}{2}\frac{\partial^2 f_i}{\partial \theta^2}\sigma^2\theta^2\right] \qquad (3)$$

and

$$s_i = \frac{\partial f_i}{\partial \theta}\sigma\theta \qquad (4)$$

With the use of these definitions, eq. (2) can be rewritten as follows:

$$df_i = m_i \, dt + s_i \, dz \qquad (5)$$

Let ϕ_i equal the natural logarithm of f_i, so that changes in ϕ_i correspond to the continuous return of the derivative. Ito's lemma can be applied to (5) to yield the stochastic process governing changes in ϕ_i:

$$d\phi_i = \mu_i \, dt + \sigma_i \, dz \qquad (6)$$

where

$$\mu_i = \frac{1}{f_i} m_i - \frac{1}{2f_i^2} s_i^2 \quad \text{and} \quad \sigma_i = \frac{1}{f_i} s_i. \qquad (7)$$

Now consider a portfolio, P, formed by combining the two derivatives. Let x_1 and x_2 denote the relative weights of the derivatives in the portfolio, such that the sum of x_1 and x_2 equals unity. Let ϕ_P denote the natural logarithm of the portfolio value. The stochastic process governing ϕ_P can be expressed as a linear combination of the two derivatives' processes:

$$d\phi_P = [x_1\mu_1 + x_2\mu_2] \, dt + [x_1\sigma_1 + x_2\sigma_2] \, dz \qquad (8)$$

If $x_1 = -\sigma_2/(\sigma_1 - \sigma_2)$, then some algebra shows that the portfolio's volatility is always 0, in which case the return of P is certain and the

portfolio is riskless. The drift of the natural logarithm of the portfolio at any instant must then equal the riskless rate of interest:

$$x_1\mu_1 + x_2\mu_2 = r \tag{9}$$

where r is the riskless rate of interest. After substituting the particular portfolio weights, which are required for eq. (9) to hold [$x_1 = -\sigma_2/(\sigma_1 - \sigma_2)$ and $x_2 = \sigma_1/(\sigma_1 - \sigma_2)$], eq. (9) can be rewritten as

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} \tag{10}$$

One interpretation of eq. (10) is that the derivatives' expected premia per unit of risk (often referred to as Sharpe ratios) are equal. This result is intuitive, because the derivatives' underlying source of risk is the same: The random variable, θ. The ratio is the same for all derivatives contingent on θ. The ratio is called the market price of risk of θ, and is usually denoted by λ. The equality in eq. (10) holds only instantaneously, because the portfolio is riskless only instantaneously; hence, λ is a function with a value that changes constantly. The arguments of λ are usually limited to θ and t, though they need not be. In fact, the price of risk can depend on anything that might affect investor attitudes. The only restriction implied by the statistical model is that the price of risk is equal at any point in time across all derivatives contingent on the same stochastic variable.

The relation between the risk premium demanded by investors for holding derivative i and the price of risk can be expressed as

$$\mu_i - r = \lambda\sigma_i \tag{11}$$

Thus, the derivative's risk premium can be interpreted as the market price of risk, λ, times the amount of risk derivative i holds, σ_i.

The definition of the market price of risk implies that an investor will accept a reduction in the return of derivative i equal to $\lambda\sigma_i$ in exchange for eliminating the volatility of the derivative. With the use of (7), this is equivalent to a reduction of λs_i in the drift of the derivative value. The drift will then equal the riskless rate of interest, because the derivative is now riskless. This implies the following relation between the adjusted drift of the value of the derivative and the riskless rate:

$$\frac{\partial f_i}{\partial t} + \frac{\partial f_i}{\partial \theta}\theta(\mu - \lambda\sigma) + \frac{1}{2}\frac{\partial^2 f_i}{\partial \theta^2}\sigma^2\theta^2 = rf_i \tag{12}$$

where (4) is used to establish the relation between σ and s_i. This partial

differential equation, along with the definition of the market price of risk, forms the basis of the risk-neutral approach to derivative valuation. The value of all derivatives with payoffs contingent on θ must satisfy this equation.

When the underlying variable is a traded asset, the Black and Scholes (1973) argument eliminates the need for the market price of risk. Consider a portfolio consisting of one unit of a derivative, f, and $-\partial f/\partial \theta$ units of the underlying variable, θ. The value of the portfolio is given by

$$P = -\frac{\partial f}{\partial \theta}\, \theta + f \tag{13}$$

and the stochastic equation describing the evolution of the value of the portfolio is given by

$$dP = \left[\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial \theta^2}\, \sigma^2\theta^2\right] dt \tag{14}$$

Because the portfolio weights eliminate uncertain returns, the drift of the portfolio must equal the riskless rate of interest times the value of the portfolio. Thus, the following equation must be satisfied by all derivatives:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial \theta^2}\, \sigma^2\theta^2 = r\left(f - \frac{\partial f}{\partial \theta}\, \theta\right) \tag{15}$$

In addition, because the drift of θ does not appear in eq. (15), any risk preference will yield the same solution. Therefore, risk-neutral valuation is justified without introducing the market price of risk.

When the underlying variable is not a traded asset, the risk-neutral approach to derivative valuation requires a risk adjustment to the stochastic process, as expressed in eq. (12). The solution to the differential equation is the value of the derivative. The solution to the equation depends on the assumed specification for the price of risk, λ, however. Recall that the only restriction implied by the model is that λ is equal at any instant across derivatives on the same variable. As shown in the next section, different assumed risk specifications can result in widely varying derivative values.

SPECIFICATION ERROR

Much recent literature[1] addresses the problem of pricing interest-rate derivatives assuming a one-factor model of interest rates where the short

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[1]See, for example, Black, Derman, and Toy (1990), Heath et al. (1990), Ho and Lee (1986), and Hull and White (1990, 1993). This list is a small subset of the existing literature.

rate is the factor. If the stochastic process governing changes in the short rate is one of several simple forms, analytic solutions for derivatives with simple payoffs exist. To allow for a more flexible short rate process, the Hull and White (1990) methodology is used for numerically valuing derivatives. It is assumed that the Cox et al. (1985) square-root process governs interest rates:

$$dr = a(b - r) \, dt + \sigma \sqrt{r} \, dz \qquad (16)$$

Derivative valuation within a Cox et al. (1985) model has been studied extensively,[2] though the impact of misspecifying the price of risk has been given little attention. The most common approach is to specify the price of risk as in the Cox et al. (1985) general equilibrium:

$$\lambda = \eta \sqrt{r} \qquad (17)$$

This section illustrates the impact of misspecifying the price of risk. A common data set of discount bond prices is used to calibrate alternatives to (17). A series of coupon bonds and coupon bond options are valued with the use of the competing specifications to gauge the impact risk specification has on derivative values.

In this example, the parameters of the short rate process used in Hull and White (1990) are assumed. This allows direct comparison to their results. Thus, (16) is parametrized with $a = 0.4$, $b = 0.1$, and $\sigma = 0.06$. The value of a implies a half-life of disturbances of 1.7 years, the value of b implies a mean rate of 10%, and the value of σ implies a standard deviation of about 2% when the short rate is at its mean. To value derivatives, the following four specifications of the price of risk are considered:

$$\lambda = \eta r^{\kappa}, \, \kappa = 0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4} \qquad (18)$$

This setup nests the Vasicek (1977) assumption, $\kappa = 0$, and the Cox et al. (1985) model, $\kappa = \frac{1}{2}$. To value interest-rate derivatives, the only unknown parameter is η.

To estimate the remaining parameter, η, a series of discount bond prices is assumed. The risk parameter can then be inferred by finding the value that satisfies the differential equation (12) for each risk specification. These parameters are used to value coupon bonds and options on coupon bonds.[3] There are several ways of inferring market risk parame-

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[2]See, for example, Chen and Scott (1992) and Pearson and Sun (1994).

[3]This calibration approach is a popular method of estimating the price of risk. For example, Gibson and Schwartz (1990) assume that the market price of crude oil convenience yield risk is an intertemporal constant. They then use traded futures contracts to estimate the value of the price of risk.

TABLE I

Risk Parameters Implied by the Yield Curve

The panels below list risk parameters implied from 5-year zero-coupon yields assuming a one-factor model of interest rates where short rate is the factor. The following Cox et al. (1985) square-root process governs the evolution of the short rate:

$$dr = 0.4(0.1 - r)\, dt + .06 \sqrt{r}\, dz.$$

The initial short rate is 10%. The market price of risk is of the form:

$$\lambda = \eta r^\kappa.$$

Parameters are estimated using numerical procedures as in Hull and White (1990).

Panel A: Implied η

	κ			
5-Year Yield	*0.00*	*0.25*	*0.50*	*0.75*
6%	1.85	3.67	7.27	14.36
10%	−0.03	−0.05	−0.08	−0.14
14%	−1.33	−2.22	−3.67	−6.07

Panel B: Implied λ

	κ			
5-Year Yield	*0.00*	*0.25*	*0.50*	*0.75*
6%	1.85	2.06	2.30	2.55
10%	−0.03	−0.03	−0.03	−0.03
14%	−1.33	−1.25	−1.16	−1.08

ters from existing prices. In a sense, they all involve solving the partial differential equation (12), except instead of solving for the value of the derivative, one solves for the market risk parameters. Hull and White (1990) use a trinomial lattice to model the Cox et al. (1985) square-root process. This study extends the Hull and White approach by allowing the market price of risk to vary as specified in eq. (18).

Panel A of Table I shows the inferred η values for a range of 5-year discount bond prices and the four values of κ. The bond prices are chosen to show the relation between the zero-coupon yield curve and the market price of risk. Recall for this example an initial short rate of 10% is assumed. The bond prices correspond to 5-year zero-coupon yields of 6%, 10%, and 14%, corresponding to downward-sloping, flat, and upward-sloping yield curves. The identity used is simply

$$P = e^{-\mu T} \tag{19}$$

Note in Panel A that η decreases as the 5-year yield increases, and is close

to 0 when the 5-year yield equals the current short rate.[4] Because η dictates the sign of the market price of risk, this means that the price of risk is positive for downward-sloping yield curves and negative for upward-sloping yield curves. Implied market prices of risk are listed in Panel B of Table I.

To interpret the economic meaning of these results, recall the relation between the risk premium and the market price of risk for derivative i:

$$\mu_i - r = \lambda \sigma_i \tag{20}$$

From Ito's lemma, one can replace the volatility of the return of derivative i in (20) with the partial derivative of the derivative security with respect to the stochastic variable times the variable's volatility:

$$\mu_i - r = \lambda \frac{\partial f_i}{\partial \theta} \sigma \theta \tag{21}$$

For upward sloping yield curves, the LHS of (21) is positive. For bonds, the partial derivative is negative, σ and θ are both positive, so λ must be negative. For interest-rate-sensitive securities with values that are positively related to changes in the short rate, this implies a negative risk premium. For further discussion of why the market price of interest-rate risk is negative, see Hull (1989).

Inferred values of η are used to value coupon bonds and options on coupon bonds to illustrate the impact of different specifications of λ on derivative values. Assume the bonds pay coupons of 10% continuously and have 5 years to maturity. Assume the options are American style and have 3 years to maturity. The options have three exercise prices. The middle exercise price is equal to the bond price when $\kappa = \frac{1}{2}$ rounded to the nearest dollar. The low exercise price is $4.00 less, and the high exercise price is $4.00 more. These exercise prices are chosen much as an exchange would choose exercise prices: one at-the-money and others in a range around the at-the-money.

Table II lists implied bond and bond option values. The securities are valued for each initial 5-year yield of 6%, 10%, and 14%, corresponding to downward-sloping, flat, and upward-sloping zero-coupon yield curves. The columns show the values for different specifications of the market price of risk, that is, different choices for κ. For the bonds, the specification of the market price of risk does not appear to matter much.

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[4]The price of risk is not equal to zero exactly, because the bond price is calculated with the use of continuous compounding, whereas the lattice discounts discretely.

TABLE II

Values of Coupon Bonds and Call Options under Alternative Risk Specifications

The panels below list the prices of coupon bonds and call options when the following Cox et al, (1985) square-root process governs the evolution of the short rate:

$$dr = 0.4(0.1 - r) \, dt + 0.06\sqrt{r} \, dz.$$

The initial short rate is 10%. The market price of risk is of the form

$$\lambda = \eta r^\kappa.$$

Bonds pay coupons continuously at a rate of 10% annually and have five years to maturity. Options have three years to maturity and are American style. Securities are valued numerically as in Hull and White (1990).

Panel A: Bond Prices

	κ			
5-Year Yield	*0.00*	*0.25*	*0.50*	*0.75*
6%	116.34	116.40	116.46	116.51
10%	99.90	99.90	99.90	99.90
14%	86.46	86.50	86.55	86.59

Panel B: Option Prices
5-Year Yield: 6%

	κ			
Ex	*0.00*	*0.25*	*0.50*	*0.75*
112	4.77	4.77	4.78	4.79
116	1.41	1.29	1.20	1.13
120	0.04	0.01	0.00	0.00

5-Year Yield: 10%

	κ			
Ex	*0.00*	*0.25*	*0.50*	*0.75*
96	4.56	4.56	4.56	4.56
100	1.63	1.63	1.63	1.64
104	0.20	0.20	0.20	0.20

5-Year Yield: 14%

	κ			
Ex	*0.00*	*0.25*	*0.50*	*0.75*
83	5.04	5.03	5.03	5.06
87	2.40	2.42	2.47	2.55
91	0.62	0.69	0.77	0.88

When the 5-year yield is 10%, for example, the specification does not affect the bond value at all. The bond prices are close to par, as expected for bonds with coupon rates equal to the discount rate. As the 5-year rate diverges from 10%, the specification matters more, but in no case is there more than a $0.17 difference among the specifications on the $100 par bond. For the options, the specification can matter significantly. When the 5-year yield is 14%, for example, the out-of-money option values vary by 42% across risk specifications. When the 5-year yield is 6%, the at-the-money options vary by 25% across risk specifications. When the 5-year yield is 10%, the option values do not vary significantly across risk specifications.

The results show that variation in option values resulting from alternative specifications of the market price is minimal when the yield curve is flat, but is significant otherwise. Because the yield curve is usually not flat, the variation means that the assumed risk specification can have an important impact on derivative valuation. When assumptions about risk are necessary, they need to be tested when possible. The next section illustrates an approach to testing competing specifications with the use of Eurodollar futures options prices.

A TERM STRUCTURE OF RISK

Just as parameters of an assumed specification for the price of risk can be estimated with the use of market data, the specification itself can be tested with the use of existing derivative prices. This section tests the ability of different specifications of the price of interest-rate risk to fit market prices in sample and to value options accurately out of sample. In this experiment, cross sections of Eurodollar futures options prices are used to estimate the parameters of the stochastic process governing interest rates and the parameters of a variety of risk specifications. The parameters are chosen to minimize the sum of squared valuation errors between the fitted option values and the market prices. The pricing error gives an indication of the ability of different specifications of the price of risk to fit market data. To control for overfitting, the estimated parameters are used to value the same options 1 month later. The discrepancy between the predicted option values and the market prices measures the ability of different specifications to accurately capture the risk preferences of investors. This out-of-sample performance is relevant for hedging interest-rate risk.

Data

Monthly observations of Eurodollar futures options prices and 3-month Eurodollar deposit rates are from the *Wall Street Journal*. Thirty-six sets of observations are recorded from January, 1993 to December, 1995. To reduce measurement error, prices of less than 0.10 were discarded, because the *Journal* only records two digits after the decimal. For purposes of estimation, options with less than 20 or more than 200 days to maturity are also discarded. The number of remaining futures options prices is 493, and the number of prices each month range from 7 to 20. Weekly observations of 3-month Eurodollar deposit rates are from *Datastream* for the period January, 1976–September, 1995. These are used to test the validity of alternative interest-rate processes.

Interest-Rate Process

In this example derivative prices are used to estimate the parameters of the stochastic process governing interest-rate changes as well as the parameters of the market price of risk. Because the number of derivatives used is limited, parsimony of specification is desirable. Chan, Karolyi, Longstaff, and Sanders (1992) present a framework for estimating parameters for alternative specifications of the short rate's stochastic process. They nest competing specifications in the following general expression:

$$dr = a(b - r)\, dt + \sigma r^{\gamma} \tag{22}$$

Different specifications of the process restrict subsets of the four free parameters a, b, σ, and γ. Competing specifications can be tested in a generalized method-of-moments system. With the use of the Chan et al. framework and weekly observations of 3-month Eurodollar deposit rates, Dothan's model is tested:

$$dr = \sigma r\, dz \tag{23}$$

The Dothan process implies that changes in interest rates are distributed mean zero, but with volatility proportional to the level of interest rates. Because it has only one parameter, it achieves the desired level of parsimony.

The discretized version of Dothan's model implies the following moment restrictions:

$$E_{t-1} \begin{bmatrix} r_t - r_{t-1} \\ (r_t - r_{t-1})r_{t-1} \\ (r_t - r_{t-1})^2 - \sigma^2 r_{t-1}^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{24}$$

TABLE III

Estimates of the Dothan Interest-Rate Model

Listed below are parameter estimates and test statistics for the Dothan model of interest-rate changes:

$$dr = \sigma r\, dz.$$

The data are 3-month Eurodollar deposits rates from Datastream. Parameters are estimated with the use of the generalized method of moments. The test for parameter significance uses the asymptotic normal distribution of parameter estimates. The parameters are estimated from the following discrete-time moment conditions:

$$r_{t+1} - r_t = \varepsilon_{t+1}$$

$$E[\varepsilon_{t+1}] = 0,\ E[\varepsilon_{t+1} r_t] = 0,\ E[\varepsilon_{t+1}^2] = \sigma^2 r_t^2.$$

Period	1/76–1292	1/90–9/95
No. of observations	887	300
σ estimate	0.312	0.147
p value	0.000	0.000
χ^2 test	1.601	3.284
p value	0.449	0.194
d.f.	2	2

With the use of the results of Hansen (1982), the asymptotic normal distribution of parameter estimates can be used to test for their significance and for the performance of the model. Table III shows the results of the estimation for two periods. The first spans January, 1976–December, 1992, a total of 887 weekly observations of 3-month Eurodollar deposit rates. The second spans January, 1990–September, 1995, a total of 300 observations. The second period is used because it overlaps with the futures options prices, so it will serve as a benchmark for the parameters inferred from the derivative prices. For both periods, the overidentifying statistic fails to reject the model. The volatility parameter estimate is 0.3116 for the larger period and 0.1472 for the other, reflecting the recent history of interest-rate volatility.[5]

Inferring Parameters from Futures Options Prices

The goal of this experiment is to test alternative risk specifications by fitting futures options prices in sample, then testing valuation performance out of sample. In-sample fit is achieved by choosing parameter values that minimize the squared pricing error between fitted values and market data. Thus, a means of valuing the derivatives, given a vector of

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[5]The volatility estimate is annualized by multiplying by $52^{1/2}$.

parameter values, is needed. This is accomplished by estimating the solution to the partial differential equation in (12), subject to appropriate boundary conditions. The solution is estimated with the use of an explicit finite-difference method. The boundary conditions for the futures options prices include terminal payoffs, where the option is worth the maximum of zero and exercise proceeds, and the value of rational early exercise at each point in time over the option's life, because Eurodollar futures options are American style. The expectations used in the option valuation reflect a risk adjustment consistent with the risk specification in question, such that discounting can occur at the riskless rate.

The risk-neutral Dothan model is

$$dr = -\lambda\sigma r \, dt + \sigma r \, dz, \qquad \text{where } \lambda = \eta r^\kappa \qquad (25)$$

The natural logarithm of interest rates is modelled to ease estimation, because the volatility is then constant. Let ϕ denote the natural logarithm of r. Applying Ito's lemma to (25) yields

$$d\phi = -\left[\lambda\sigma + \frac{\sigma^2}{2}\right] dt + \sigma \, dz \qquad (26)$$

A grid is constructed consistent with the process in (26). Note that though the volatility is constant, the mean varies with the level of the interest rate through the risk adjustment. For this reason a trinomial lattice is used to allow for sufficient flexibility. Terminal nodes in the lattice give a range of terminal interest rates. Futures prices are calculated directly from these, the futures price is simply the present value of the par value of the deposit, discounted at the current rate. Option payoffs are then calculated as the maximum of zero and exercise proceeds at each node. By folding back through the lattice, the expected discounted option payoff at each node is calculated by taking expectations consistent with (26) and discounting at the current rate. Early exercise payoffs are calculated by setting the forward rate used to determine the futures price equal to the expected terminal interest rate. The expectations take into account a risk adjustment, again through (26).

Results

Two sets of risk specifications are tested. The restricted set uses the same parameter values for all options on a given day. The unrestricted set uses

one risk parameter for options with less than 100 days to maturity, and another for options with longer times to expiration. The unrestricted model thus allows for a term structure of risk. For the restricted model, the average volatility parameter estimate is about 0.19; whereas the unrestricted model's average volatility parameter is about .18. These values are roughly consistent with the estimate from the GMM procedure with the use of Eurodollar deposit rates from January, 1990 to September, 1995.

The ability of the different risk specifications to fit Eurodollar futures options prices in sample is illustrated in Panel A of Table IV. Of course, one expects the greater degree of freedom in the unrestricted model to enhance the in-sample fit. The root-mean-squared valuation error is about $0.02 for the restricted estimation and about $0.007 for the unrestricted models over the entire sample, a reduction of about 65%. Though the performance does not vary much across specifications, it appears that a term structure of risk fits the options prices much tighter than a restricted model.

To determine whether the term structure of risk overfits the data, an out-of-sample experiment is performed. Starting with the second month of options prices, the in-sample parameters from the prior month are used to value those options that are still listed in the current month. Panel B of Table IV lists the results. Again, the restricted risk specifications do not differ much in their ability to value the options, but the term structure of risk enhances the fit significantly. The restricted specifications have a root-mean-squared valuation error of about $0.035 versus about $0.031 for the unrestricted set, a reduction of about 10%. A term structure of risk thus could be useful for improving hedging effectiveness.

CONCLUSION

Risk-neutral valuation is an alluring technique for valuing derivative securities, because it is based on the observed stochastic process of the underlying variable, which is readily estimated by using historical data. However, unless the underlying asset is traded, another input to the risk-neutral approach is the market price of risk of the underlying variable. The price of risk is actually an unobservable function with a value that changes constantly. Researchers often assume a specification: Vasicek (1977), for example, assumes it is a constant, and Hull and White (1990) assume the Cox et al. (1985) specification. This article shows that an incorrect specification can have dramatic consequences for derivative valuation.

TABLE IV

Performance of Alternative Risk Specifications

For the in-sample test, the interest-rate volatility and risk parameters are chosen once per month to minimize the sum of squared valuation errors between fitted values and market prices of Eurodollar futures options. Monthly observations of the cross section of futures options prices are used from January, 1993 to December, 1995. For the out-of-sample test, parameters from the prior month are used to value options that are still listed 1 month later. The Dothan model governs interest-arate changes:

$$dr = \sigma r \, dz.$$

The market price of risk is assumed to be of the following form:

$$\lambda = \eta r^\kappa$$

The restricted model uses one value of η for each month's options. the unrestricted model uses two, one for options with less than 100 days to maturity, the other for options with longer maturities. Listed in both panels are the root-mean-squared valuation errors for competing specifications, and the percentage reduction achieved with the unrestricted models.

| | Panel A: In Sample | | |
κ	*Restricted*	*Unrestricted*	*% Reduction*
0.00	0.0192	0.0069	64.08%
0.25	0.0197	0.0069	65.01%
0.50	0.0203	0.0070	65.69%
0.75	0.0209	0.0071	66.10%
	Panel B: Out of Sample		
κ	*Restricted*	*Unrestricted*	*% Reduction*
0.00	0.0348	0.0314	9.67%
0.25	0.0348	0.0314	9.82%
0.50	0.0350	0.0315	9.87%
0.75	0.0352	0.0318	9.78%

In an example extending the work of Hull and White (1990), calculated derivative values are shown to vary dramatically with the choice of alternative, plausible specifications of the market price of risk. Even when the functional form of the price of risk is restricted to a certain class, varying one of the unknown parameters can cause option values to vary significantly. In addition, in an experiment with the use of Eurodollar futures options prices, a term structure of risk is shown to outperform specifications that restrict the price of risk to be time invariant. These results clearly have implications for the way investors value derivatives. When the risk-neutral approach described in the article is used, the assumed specification of the market price of risk must be justified. When

possible, different specifications can be tested with the use of existing derivative prices.

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