Applied Mathematical Sciences, Vol. 8, 2014, no. 107, 5303 - 5315
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2014.46416

Inverse Domination Numbers and Disjoint Domination Numbers of Graphs under Some Binary Operations

Edward M. Kiunisala[1]

Mathematics and ICT Department
College of Arts and Sciences, Cebu Normal University
Cebu City, Philippines

Ferdinand P. Jamil[2]

Department of Mathematics and Statistics
College of Science and Mathematics
MSU-Iligan Institute of Technology
Iligan City, Philippines

Abstract

In this note, we investigate the inverse domination numbers and the disjoint pair domination numbers of graphs resulting from the join, corona and composition of graphs

Mathematics Subject Classification: 05C69

Keywords: Domination number, inverse domination number, disjoint pair domination number

[1]Research is fully supported by the Department of Science and Technology (DOST), Philippines, though the Accelerated Science and Technology Human Resource Development (ASTHRD).

[2]Corresponding author

1 Introduction

Throughout this study, G denotes a graph which is simple and undirected. The symbols $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. We write uv to denote the edge joining the vertices u and v. The *order* (resp. *size*) of G refers to the cardinality of $V(G)$ (resp. $E(G)$). In symbols, $|V(G)|$ denotes the order, while $|E(G)|$ denotes the size of G. If $E(G) = \emptyset$, G is called an *empty graph*. If $V(G)$ is a singleton, G is called a *trivial* graph.

Any graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a non-empty $S \subseteq V(G)$, $\langle S \rangle$ denotes the subgraph H of G for which $|E(H)|$ is the maximum size of a subgraph of G with vertex set S.

An edge e of G is said to be *incident* to vertex v whenever $e = uv$ for some $u \in V(G)$. We write $G - v$ to denote the resulting subgraph of G after removing from G the vertex v and all edges of G incident to v. In general, for $S \subseteq V(G)$, the symbol $G - S$ denotes the resulting subgraph of G after removing all vertices $v \in S$ from G and all edges in G incident to v. If $u, v \in V(G)$, the symbol $G + uv$ denotes the graph obtained from G by adding to G the edge uv.

Let G and H be any graphs. The *join* of G and H is the graph $G + H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* $G \circ H$ of G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H. We denote by H^v that copy of H whose vertices are adjoined with the vertex v of G. In effect, $G \circ H$ is composed of the subgraphs $H^v + v = H^v + \langle \{v\} \rangle$ joined together by the edges of G. The *composition* $G[H]$ of G and H is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G[H])$ if and only either $uu' \in E(G)$ or $u = u'$ and $vv' \in V(H)$.

Two distinct vertices u and v of G are *neighbors* in G if $uv \in E(G)$. The *closed neighborhood* $N_G[v]$ of a vertex v of G is the set consisting of v and every neighbor of v in G. Any $S \subseteq V(G)$ is a *dominating set* in G if $\cup_{v \in S} N_G[v] = V(G)$. The minimum cardinality $\gamma(G)$ of a dominating set in G is the *domination number* of G. Any dominating set in G of cardinality $\gamma(G)$ is referred to as a γ-*set* in G. A dominating set S in G is a *total dominating set* if for every $x \in S$ there exists $y \in S$ such that $xy \in E(G)$. The minimum cardinality of a total dominating set in G is the *total domination number* of G, and is denoted by $\gamma_t(G)$. The reader may refer to [3, 9, 11, 13, 14, 15] for the fundamental concepts and recent developments of the domination theory, including its various applications.

A classical result in domination theory due to Ore[3] in 1962 motivated the introduction of the concept of an inverse dominating set. It can be stated as follows:

Theorem 1.1 [3] *Let G be a graph with no isolated vertex. If $S \subseteq V(G)$ is a γ-set in G, then $V(G) \setminus S$ is also a dominating set in G.*

Let G be a graph without isolated vertices. An *inverse dominating set* in G is any dominating set S in G such that $S \subseteq V(G) \setminus D$, where D is a γ-set in G. The minimum cardinality of an inverse dominating set is called the *inverse domination number*, and is denoted by $\gamma'(G)$. Such definition was first introduced by V.R. Kulli and S.C. Sigarkanti [1] in 1991, and studied further in [2, 7, 8]. It may be noted that P.G. Bhat and S.R. Bhat in [2] made mention of its application in an Information Retrieval System. It can be readily verified that $\gamma(G) \leq \gamma'(G)$. T. Tamizh Chelvam, T. Asir and G.S. Grace Prema in [7] studied graphs G where $\gamma(G) = \gamma'(G)$.

For our purposes in this paper, any inverse dominating set S in G with $|S| = \gamma'(G)$ is called a γ'-*set* in G.

Theorem 1.1 also guarantees that any graph G with no isolated vertices contains two disjoint subsets of $V(G)$ which are both dominating sets in G. This is the motivation of the concept of disjoint domination introduced by S.M. Hedetniemi et al. in [10]. Any pair of subsets S and D of $V(G)$ is called *dd-pair* if S and D are disjoint dominating sets in G. We define

$$\gamma\gamma(G) = \min\{|S| + |D| : S, D \text{ } are \text{ } dd - pairs \text{ } in \text{ } G\}.$$

Any dd-pair (S, D) in G satisfying $|S| + |D| = \gamma\gamma(G)$ is called $\gamma\gamma$-*pair* in G. It is easy to verify that

$$2\gamma(G) \leq \gamma\gamma(G) \leq \gamma'(G) + \gamma(G). \tag{1}$$

For graphs where $\gamma'(G) = \gamma(G)$, $\gamma\gamma(G) = 2\gamma(G)$.

2 Realization Problems

Proposition 2.1 *For every pair (a, b) of positive integers with $a \leq b$, there exists a graph G such that $\gamma(G) = a$, $\gamma'(G) = b$ and $\gamma\gamma(G) = a + b$.*

Proof: If $a = 1$, then we take $G = K_{1,b}$. Suppose that $a \geq 2$. Let $n = 3a - 2$ and let the path P_n be given by $P_n = [v_1, v_2, \ldots, v_n]$. Form G by adding to $P_n, c = b - \lfloor \frac{n}{3} \rfloor$ pendant edges $u_j v_n, j = 1, 2, \ldots, c$. If $c = 1$, then $\gamma(G) = \gamma(P_{n+1}) = \lceil \frac{n+1}{3} \rceil = a$, while $\gamma'(G) = \gamma'(P_{n+1}) = b$ [1]. Suppose that $c \geq 2$. Since $D = \{v_1, v_4, ..., v_n\}$ is a dominating set in G,

$$\gamma(G) \leq \left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{3a-2}{3} \right\rceil = a.$$

On the other hand, since $\gamma(P_{n+1}) = \lceil \frac{n+1}{3} \rceil \geq \lceil \frac{n}{3} \rceil = a$, $\gamma(G) \geq \gamma(P_{n+1}) \geq a$. Therefore, $\gamma(G) = a$. Consequently, D is a γ-set in G. Note further that, in particular, the set $S = \{v_{n-2}, v_{n-5}, \ldots, v_2\} \cup \{u_i : i = 1, 2, \ldots, c\}$ is a dominating in G and $S \subseteq V(G) \setminus D$ so that $\gamma'(G) \leq \left\lfloor \frac{n}{3} \right\rfloor + c = b$. Now, let $T \subseteq V(G)$ be a γ-set in G. Since $\gamma(P_n) = a$ and $c \geq 2$, $u_i \notin T$ for all $i = 1, 2, \ldots c$. Consequently, $v_n \in T$. Let $D_0 \subseteq V(G) \setminus T$ be an inverse dominating set in G. Since $v_n \notin D_0$, $u_i \in D_0$ for all i. Similarly, since $v_1 \notin D_0$, $v_2 \in D_0$. Apparently, the definition of D_0 implies that $D_0 = S$. Therefore, $\gamma'(G) = b$. Finally, let (S, D') be a $\gamma\gamma$-pair in G. Either each $u_j \in D'$ for each j or $u_j \in S$ for each j. If $u_j \in D'$ for all j, then $D' = D$, and the conclusion follows. ■

Theorem 2.2 *For each integer $n \geq 1$, there exists a connected graph G such that $\gamma'(G) - \gamma(G) = n$ and $|V(G)| = \gamma'(G) + \gamma(G)$.*

Proof: Let $n \geq 1$, and consider the star graph $K_{1,n+2} = K_1 + \overline{K_{n+2}}$. Let $\{v\} = V(K_1)$ and let $u \in V(\overline{K_{n+2}})$. Obtain the graph G by adding to $K_{1,n+2}$ a pendant uz. Then $\gamma(G) = 2$, which is determined by the dominating set $\{v, z\}$ in G. Since $S = V(G) \setminus \{v, z\}$ is a dominating set in G, S is an inverse dominating set in G and $\gamma'(G) \leq |S| = n + 2$. But since $N_G[D] \neq V(G)$ for all proper subsets D of S, $\gamma'(G) = |S| = n + 2$. Thus, $\gamma'(G) - \gamma(G) = n$. ■

Corollary 2.3 *The difference $\gamma'(G) - \gamma(G)$ can be made arbitrarily large.*

Theorem 2.4 *For each integer $n \geq 1$, there exists a connected graph G such that $\gamma(G) + \gamma'(G) - \gamma\gamma(G) = n$.*

Proof: Consider the graph G as in Figure 1 obtained by adding to the corona

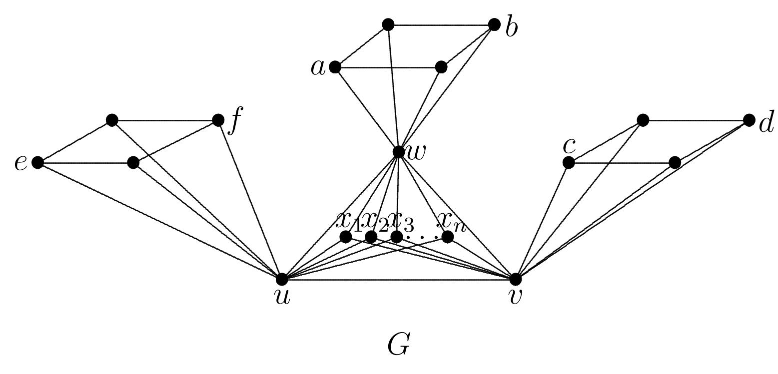

Figure 1: Graph G with $\gamma\gamma(G) < \gamma(G) + \gamma'(G)$.

$K_3 \circ C_4$ n vertices $x_1, x_2, \ldots, x_n$ and the edges x_jw, x_ju and x_jv ($j = 1, 2, \ldots, n$). The set $\{u, v, w\}$ is the unique minimum dominating set in G, and $\{a, b, c, d, e, f\} \cup \{x_1, x_2, \ldots, x_n\}$ is a γ'-set in G. Thus $\gamma(G) = 3$ and $\gamma'(G) = 6 + n$. On the other hand, the sets $S = \{u, w, c, d\}$ and $D = \{a, b, e, f, v\}$ constitute a $\gamma\gamma$-pair in G. Thus $\gamma\gamma(G) = |S|+|D| = 9$. Therefore, $\gamma(G) + \gamma'(G) - \gamma\gamma(G) = n$. ■

Corollary 2.5 *The difference* $(\gamma(G)+\gamma'(G))-\gamma\gamma(G)$ *can be made arbitrarily large.*

3 Join of graphs

Clearly, $\gamma'(G+K_1) = \gamma(G)$. In what follows, we consider $G+H$ with nontrivial graphs G and H. For any $u \in V(G)$ and $v \in V(H)$, the set $\{u, v\}$ is a dominating set in $G + H$. Thus, $\gamma(G + H) \leq 2$.

Lemma 3.1 *For nontrivial graphs* G *and* H*,* $\gamma'(G + H) \leq 2$.

Proof: Either $\gamma(G + H) = 1$ or $\gamma(G + H) = 2$. Suppose that $\gamma(G + H) = 1$, and let $D = \{v\}$ be a dominating set in $G + H$. Assume $v \in V(G)$. Take $u \in V(G) \setminus \{v\}$ and $w \in V(H)$. Then $S = \{u, w\} \subseteq V(G + H) \setminus D$ and S is a dominating set in $G + H$. Thus $\gamma'(G + H) \leq |S| = 2$. Suppose that $\gamma(G + H) = 2$. Pick any $u \in V(G)$ and $v \in V(H)$. Then $D = \{u, v\}$ is a γ-set in $G + H$. For any $x \in V(G) \setminus D$ and $y \in V(H) \setminus D$, the set $S = \{x, y\}$ is a γ'-set in $G + H$. Thus $\gamma'(G + H) = |S| = 2$. ■

Theorem 3.2 *Let* G *and* H *be nontrivial graphs. Then* $\gamma'(G + H) = 2$ *if and only if one of the following is true:*

(i) $\gamma(G) \geq 2$ *and* $\gamma(H) \geq 2$*;*

(ii) $\gamma(H) \geq 2$ *and* G *has a (unique) vertex that dominates* $V(G)$*;*

(iii) $\gamma(G) \geq 2$ *and* H *has a (unique) vertex that dominates* $V(H)$.

Proof: Suppose that $\gamma'(G+H) = 2$. Again, either $\gamma(G+H) = 1$ or $\gamma(G+H) = 2$. If $\gamma(G+H) = 2$, then $\gamma(G) \geq 2$ and $\gamma(H) \geq 2$. Suppose that $\gamma(G+H) = 1$. Then $\gamma(G) = 1$ or $\gamma(H) = 1$. Assume that $\gamma(G) = 1$. Then $G = \{v\} + \bigcup_j G_j$ for some $v \in V(G)$ and components G_j of G. Thus,

$$\gamma'(G + H) = \gamma(H + \bigcup_j G_j) = 2.$$

Necessarily, $\gamma(H) \geq 2$ and $\gamma(\bigcup_j G_j) \geq 2$. This means that v is a unique vertex of G that dominates $V(G)$. Similarly, if $\gamma(H) = 1$, then $\gamma(G) \geq 2$ and H has a unique vertex that dominates $V(H)$.

To prove the converse, first, consider the case where $\gamma(G) \geq 2$ and $\gamma(H) \geq 2$. Then $\gamma(G+H) = 2$. Now pick $u \in V(G)$ and $v \in V(H)$, and choose $x \in V(G) \setminus \{u\}$ and $y \in V(H) \setminus \{v\}$. Then $D = \{u, v\}$ and $S = \{x, y\}$ are disjoint γ-sets in $G + H$. Accordingly, $\gamma'(G+H) = 2$. Next, suppose that (ii) holds. Let $D = \{u\} \subseteq V(G)$ be a dominating set in G. Then D is a dominating set in $G + H$. Consider

$$(G+H) - u = (G-u) + H.$$

Since u is a unique vertex that dominates $V(G)$, $\gamma(G-u) \geq 2$. If $\gamma(G-u) \geq 2$ and $\gamma(H) \geq 2$, then $\gamma'(G+H) = \gamma((G-u)+H) = 2$. Similarly, if (iii) holds, then $\gamma'(G+H) = 2$. ■

Corollary 3.3 *Let G and H be nontrivial graphs. Then $\gamma'(G+H) = 1$ if and only if one of the following is true:*

(i) $\gamma(G) = 1$ *and* $\gamma(H) = 1$;

(ii) *G has two distinct vertices each of which dominates $V(G)$;*

(iii) *H has two distinct vertices each of which dominates $V(H)$.*

Corollary 3.3 asserts that for nontrivial graphs G and H, if $\gamma'(G+H) = 1$, then $\gamma(G) = 1$ or $\gamma(H) = 1$. The converse, however, is not necessarily true. To see this, consider the graph $K_{1,4} + P_5$. Note that $\gamma(K_{1,4}) = 1$ but $\gamma'(K_{1,4} + P_5) = 2$ by Theorem 3.2.

Corollary 3.4 *Let G be any graph with no isolated vertex. Then $\gamma'(G) = 1$ if and only if $G = K_p$ $(p \geq 2)$ or $G = K_2 + H$ for some noncomplete graph H.*

Proof: First, note that $\gamma'(K_p) = 1$ for all $p \geq 2$. Thus, we proceed with a noncomplete G. Suppose that $\gamma'(G) = 1$. There exist two distinct vertices u and v of G such that $\{u\}$ and $\{v\}$ are γ-sets in G. Then $\langle \{u, v\} \rangle = K_2$ and $G = K_2 + H$, where $H = G - \{u, v\}$. The converse follows immediately from Corollary 3.3. ■

Now we consider pair of disjoint dominating sets in the join of graphs. Clearly, $\gamma\gamma(G + K_1) = 1 + \gamma(G) = 1 + \gamma'(G + K_1)$ for any graphs G. In particular, $\gamma\gamma(K_{1,n}) = n + 1$ for all positive integers n.

Proposition 3.5 *Let G and H be nontrivial graphs. Then*

$$2 \leq \gamma\gamma(G+H) \leq 4. \tag{2}$$

More precisely,

(i) $\gamma\gamma(G+H) = 2$ *if and only if* $\gamma'(G+H) = 1$*;*

(ii) $\gamma\gamma(G+H) = 3$ *if and only if either* $\gamma(G) \geq 2$ *and* H *has a unique vertex that dominates* $V(H)$ *or* $\gamma(H) \geq 2$ *and* G *has a unique vertex that dominates* $V(G)$*;*

Proof: From previous discussion,

$$2 \leq \gamma(G+H) + \gamma(G+H) \leq \gamma\gamma(G+H) \leq \gamma(G+H) + \gamma'(G+H) \leq 4.$$

Statement (i) is clear. Suppose that $\gamma\gamma(G+H) = 3$. Then $\gamma(G+H) = 1$ and $\gamma'(G+H) = 2$. By Theorem 3.2, either $\gamma(G) \geq 2$ and H has a unique vertex that dominates $V(H)$ or $\gamma(H) \geq 2$ and G has a unique vertex that dominates $V(G)$. Conversely, by Theorem 3.2, the hypothesis implies that $\gamma'(G+H) = 2$ so that $\gamma\gamma(G+H) \geq 3$ by Statement (i). The same also implies that $\gamma(G+H) = 1$. Therefore, $\gamma\gamma(G+H) \leq 3$. This proves Statement (ii). ■

Corollary 3.6 *Let* G *and* H *be nontrivial graphs. Then* $\gamma\gamma(G+H) = 4$ *if and only if* $\gamma(G) \geq 2$ *and* $\gamma(H) \geq 2$

Proof: Suppose that $\gamma(G) \geq 2$ and $\gamma(H) \geq 2$. Then $\gamma(G+H) = 2$. Thus, for any dd-pair S and D in $G+H$, $|S|+|D| \geq 4$. This means that $\gamma\gamma(G+H) \geq 4$. Invoking Inequality 2, $\gamma\gamma(G+H) = 4$. The converse follows from Proposition 3.5 and Theorem 3.2. ■

4 Corona of graphs

It is worth noting that for any connected graph G and for all graphs H, $V(G)$ is a γ-set in $G \circ H$. The following theorem is found in [5].

Theorem 4.1 [5] *Let* G *be a connected graph of order* m *and* H *any graph of order* n*. Then* $C \subseteq V(G \circ H)$ *is a dominating set in* $G \circ H$ *if and only if* $C \cap V(H^v + v)$ *is a dominating set in* $H^v + v$ *for every* $v \in V(G)$*.*

Proposition 4.2 *For any connected graph* G *and for any graph* H*,* $\gamma'(G \circ H) = |V(G)|\gamma(H)$*.*

Proof: Suppose that $\gamma(H) = 1$. For each $v \in V(G)$, let $u^v \in V(H^v)$ such that $N_{H^v}[u^v] = V(H^v)$. Let $S \subseteq V(G \circ H)$ be a γ-set in $G \circ H$. Define

$$D = \{v \in V(G) : v \notin S\} \cup \{u^v : v \in S \cap V(G)\}.$$

Then D is a γ-set in $G \circ H$. Since $S \cap D = \emptyset$, S is a γ'-set in $G \circ H$. Therefore, $\gamma'(G \circ H) = \gamma(G \circ H) = |V(G)|$.

Suppose that $\gamma(H) > 1$. Let $S \subseteq V(G \circ H)$ be an inverse dominating set in $G \circ H$. For each $v \in V(G)$, let $S_v = S \cap V(H^v + v)$. Since $V(G)$ is the unique γ-set in $G \circ H$, $S \cap V(G) = \emptyset$. Consequently, $S_v \subseteq V(H^v)$ for all $v \in V(G)$. Moreover, S_v dominates $V(H^v)$. Thus,

$$\gamma'(G \circ H) = |S| = \sum_{v \in V(G)} |S_v| \geq |V(G)|\gamma(H).$$

To get the desired equality, for each $v \in V(G)$, let $S_v \subseteq V(H^v)$ be a γ-set in $V(H^v)$. Clearly, $S = \cup_{v \in V(G)} S_v$ is a dominating set in $G \circ H$. Since $S \cap V(G) = \emptyset$, S is an inverse dominating set in $G \circ H$. Therefore, $\gamma'(G \circ H) \leq |S| = |V(G)|\gamma(H)$. ■

Corollary 4.3 *For any connected graphs G and for any graph H,*

$$\gamma\gamma(G \circ H) = |V(G)|(1 + \gamma(H)).$$

Proof: Let $S, T \subseteq V(G \circ H)$ and, for each $v \in V(G)$, let $S_v = S \cap V(H^v + v)$ and $T_v = T \cap V(H^v + v)$. By Theorem 4.1, S and T are disjoint dominating sets in $G \circ H$ if and only if S_v and T_v are disjoint dominating sets in $H^v + v$. Moreover, $|S| + |T| = \gamma\gamma(G \circ H)$ if and only if $|S_v| + |T_v| = \gamma\gamma(H^v + v)$ for every $v \in V(G)$. Thus, $\gamma\gamma(G \circ H) = \sum_{v \in V(G)} \gamma\gamma(H^v + v) = |V(G)|(1 + \gamma(H))$. ■

5 Composition of graphs

Theorem 5.1 [6] *Let G and H be connected graphs. Then $C = \cup_{x \in S}(\{x\} \times T_x) \subseteq V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$, is a dominating set in $G[H]$ if and only if either*

(*i*) *S is a total dominating set in G or*

(*ii*) *S is a dominating set in G and T_x is a dominating set in H for every $x \in S \setminus N_G(S)$.*

Theorem 5.2 [6] *Let G and H be connected graphs with $\gamma(H) \geq 2$. Then $\gamma(G[H]) = \gamma_t(G)$.*

Proposition 5.3 *Let G and H be nontrivial connected graphs with $\gamma(H) \geq 2$. Then $\gamma'(G[H]) = \gamma_t(G)$. Consequently, $\gamma\gamma(G[H]) = 2\gamma_t(G)$.*

Proof: Let $A \subseteq V(G)$ be a minimum total dominating set in G, and let $u, v \in V(H)$, $u \neq v$. By Theorem 5.1, both $S = A \times \{u\}$ and $D = A \times \{v\}$ are (disjoint) dominating sets in $G[H]$. Moreover, $|S| = |D| = |A| = \gamma_t(G) = \gamma(G[H])$, by Theorem 5.2. Thus, S is an inverse dominating set in $G[H]$ so that

$$\gamma_t(G) = \gamma(G[H]) \leq \gamma'(G[H]) \leq |S| = \gamma_t(G).$$

This proves the proposition. ■

Lemma 5.4 *Let G and H be nontrivial connected graphs such that $V(H)$ is dominated by a vertex $v \in V(H)$. If $A \subseteq V(G)$ is an inverse dominating set in G, then $A \times \{v\}$ is an inverse dominating set in $G[H]$.*

Proof: Let $A, B \subseteq V(G)$ be dominating sets in G such that $A \cap B = \emptyset$ and $|B| = \gamma(G)$. By Theorem 5.1, $A \times \{v\}$ and $B \times \{v\}$ are dominating sets in $G[H]$. Now, $\gamma(G[H]) \leq |B \times \{v\}| = |B| = \gamma(G) \leq \gamma(G[H])$ so that $B \times \{v\}$ is a γ-set in $G[H]$. Since $(A \times \{v\}) \cap (B \times \{v\}) = \emptyset$, $A \times \{v\}$ is an inverse dominating set in $G[H]$. ■

For convenience, we write $S^\circ = S \setminus N_G(S)$ for any $S \subseteq V(G)$.

Theorem 5.5 *Let G and H be nontrivial connected graphs with $\gamma(H) = 1$. Then*

$$\gamma(G) \leq \gamma'(G[H]) \leq \gamma'(G). \tag{3}$$

More precisely,

(*i*) *if H has (at least) two distinct vertices each of which dominates $V(H)$, then $\gamma'(G[H]) = \gamma(G)$; and*

(*ii*) *if H has a unique vertex that dominates $V(H)$, then*

$$\gamma'(G[H]) = \min\{(|A| + |A^\circ \cap B|)(\gamma'(H) - 1) : A, B \in \Gamma(G) \quad \text{with } |B| = \gamma(G)\},$$

where $\Gamma(G)$ is the family of all dominating sets in G.

Proof: Inequality 3 follows immediately from Lemma 5.4. Suppose that H has two distinct vertices u and v such that $N_H[u] = V(H) = N_H[v]$. Let $A \subseteq V(G)$ be γ-set in G. By Theorem 5.1, $S = A \times \{u\}$ and $D = A \times \{v\}$ are γ-sets in $G[H]$. Since $S \cap D = \emptyset$, S is a γ'-set in $G[H]$. Hence, $\gamma'(G[H]) = |S| = |A| = \gamma(G)$.

Suppose that H has a unique vertex v that dominates $V(H)$. Let $\Gamma = \Gamma(G)$ denote the family of all dominating sets in G, and let

$$\alpha = \min\{(|A| + |A^\circ \cap B|)(\gamma'(H) - 1) : A, B \in \Gamma \text{ with } |B| = \gamma(G)\}.$$

Let $A, B \in \Gamma(G)$ with $|B| = \gamma(G)$, and let $v \in V(H)$ such that $N_H[v] = V(H)$. Choose $w \in V(H) \setminus \{v\}$ and a γ'-set $C \subseteq V(H)$ in H. It is worth noting that $v \notin C$. Define $D = B \times \{v\}$ and

$$S = (\cup_{u \in A \setminus B}\{(u, v)\}) \cup (\cup_{u \in A \setminus A^\circ) \cap B}\{(u, w)\} \cup (\cup_{u \in A^\circ \cap B}(\{u\} \times C).$$

By Theorem 5.1 and the fact that $|D| = |B| = \gamma(G)$, D is a γ-set in $G[H]$. Let $u \in A^\circ$. Then $T_u = \{x \in V(H) : (u, x) \in S\}$ is either C or $\{v\}$. In any case, T_u is a dominating set in H. By Theorem 5.1, S is a dominating set in $G[H]$. Since $S \cap D = \emptyset$, S is an inverse dominating set in $G[H]$. Thus,

$$\gamma'(G[H]) \leq |S| = |A| + |A^\circ \cap B|(\gamma'(H) - 1).$$

Since A and B are arbitrary, $\gamma'(G[H]) \leq \alpha$.

Let (S, D) be a dd-pair in $G[H]$ such that $|D| = \gamma(G[H])$ and $|S| = \gamma'(G[H])$. By Theorem 5.1, $S = \cup_{u \in A}(\{u\} \times T_u)$ and $D = \cup_{u \in B}(\{u\} \times T_u)$ for some dominating sets A and B in G. Since $\gamma(H) = 1$, Theorem 5.1 implies that $|B| = |D| = \gamma(G)$ and $|T_u| = 1$ for all $u \in B$. Since S is a γ'-set, $|T_u| = 1$ for all $u \in A \setminus B$, in which case, we may assume that $T_u = \{v\} \subseteq V(H)$ where $N_H[v] = V(H)$. Since $S \cap D = \emptyset$, for all $u \in A^\circ \cap B$, if $(u, w) \in D$, then $(u, w) \notin S$. Moreover, in view of Theorem 5.1(ii), for each such u, $T_u = \{x \in V(H) : (u, x) \in S\}$ is a γ'-set in H. Thus,

$$\begin{aligned} |S| &= |\cup_{u \in A \setminus B} (\{u\} \times T_u)| + |\cup_{u \in (A \setminus A^\circ) \cap B} (\{u\} \times T_u)| + \\ &\qquad |\cup_{u \in A^\circ \cap B} (\{u\} \times T_u)| \\ &\geq |A \setminus (A^\circ \cap B)| + |A^\circ \cap B|\gamma'(H) \\ &= |A| + |A^\circ \cap B|(\gamma'(H) - 1) \end{aligned}$$

so that $\gamma'(G[H]) \geq \alpha$. ■

Corollary 5.6 *Let G and H be nontrivial connected graphs. If H has a unique vertex that dominates $V(H)$, then $\gamma'(G[H]) = \gamma'(G)$ if and only if G has an inverse dominating set A_0 such that $|A_0| \leq |A| + |A^\circ \cap B|(\gamma'(H) - 1)$ for all dd-pairs A and B in G with $|B| = \gamma(G)$.*

The inequalities in Inequality 3 can be both strict. Consider, for example, the composition $G[P_3]$, where G is the graph in Figure 2. Verify that $\gamma(G) = 2$,

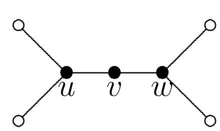

Figure 2: Graph G where $\gamma(G) < \gamma'(G[P_3]) < \gamma'(G)$

$\gamma'(G[P_3]) = 3$ and $\gamma'(G) = 5$. The set $B = \{u, w\}$ is the unique γ-set in G. Consider $A = \{u, v, w\}$, which is a total dominating set in G so that $A^\circ = \emptyset$. Applying Theorem 5.5(ii), $\gamma'(G[P_3]) = |A|$.

Inequality 3 implies that for connected graphs G and H with $\gamma(H) = 1$,

$$\gamma\gamma(G[H]) \leq \gamma(G) + \gamma'(G). \tag{4}$$

The next result is an improvement of inequality 4.

Theorem 5.7 *Let G and H be nontrivial connected graphs with $\gamma(H) = 1$. Then*

$$2\gamma(G) \leq \gamma\gamma(G[H]) \leq \gamma\gamma(G).$$

More precisely,

(*i*) *if H has (at least) two distinct vertices each of which dominates $V(H)$, then $\gamma\gamma(G[H]) = 2\gamma(G)$; and*

(*ii*) *if H has a unique vertex that dominates $V(H)$, then*

$$\gamma\gamma(G[H]) = \min\{|A| + |B| + |A^\circ \cap B^\circ|(\gamma'(H) - 1) : A, B \in \Gamma(G)\},$$

where $\Gamma(G)$ is the family of all dominating sets in G.

Proof: There exists $v \in V(H)$ such that $N_H[v] = V(H)$. Let (A, B) be a $\gamma\gamma$-pair in G. Then $(A \times \{v\}, B \times \{v\})$ is a dd-pair in $G[H]$. Thus, $\gamma\gamma(G[H]) \leq |A \times \{v\}| + |B \times \{v\}| = |A| + |B| = \gamma\gamma(G)$.

If H has two distinct vertices that both dominate $V(H)$, then Theorem 5.5(i) implies

$$2\gamma(G) \leq \gamma\gamma(G[H]) \leq \gamma(G[H]) + \gamma'(G[H]) = 2\gamma(G).$$

Suppose that H has a unique vertex v that dominates $V(H)$. Let

$$\alpha = \min\{|A| + |B| + |A^\circ \cap B^\circ|(\gamma'(H) - 1) : A, B \in \Gamma(G)\}.$$

Let $w \in V(H) \setminus \{v\}$, let $A, B \in \Gamma(G)$ and (X, Y) a dd-pair in H. Define

$$S = (\cup_{u\in(A\setminus A^\circ)\cap B^\circ}\{(u, w)\}) \cup (\cup_{u\in A\setminus B^\circ}\{(u, v)\}) \cup (\cup_{u\in A^\circ\cap B^\circ}(\{u\} \times X),$$

and $D = \cup_{u\in B}(\{u\} \times T_u)$ such that

(a) for each $u \in A^\circ \cap B^\circ$, $T_u = Y$;

(b) for each $u \in (B \setminus A) \cup ((A \setminus A^\circ) \cap B^\circ)$, $T_u = \{v\}$; and

(c) for each $u \in [(B \setminus B^\circ) \cap A^\circ] \cup [(A \setminus A^\circ) \cap (B \setminus B^\circ)]$, $T_u = \{w\}$.

By Theorem 5.1, S and D are dominating sets in $G[H]$. Moreover, $S \cap D = \emptyset$. Thus,

$$\gamma\gamma(G[H]) \leq |S| + |T| = |A| + |B| + |A^\circ \cap B^\circ|(|X| + |Y| - 2).$$

Since X and Y are arbitrary,

$$\gamma\gamma(G[H]) \leq |A|+|B|+|A^\circ\cap B^\circ|(\gamma\gamma(H)-2) = |A|+|B|+|A^\circ\cap B^\circ|(\gamma'(H)-1).$$

Since A and B are arbitrary, $\gamma\gamma(G[H]) \leq \alpha$.

To prove the converse, let (S, D) be a $\gamma\gamma$-pair in $G[H]$. There exist dominating sets A and B in G such that $S = \cup_{u\in A}(\{u\}\times T_u)$ and $D = \cup_{u\in B}(\{u\}\times T_u)$. Further, if A (resp. B) is not a total dominating set in G, then for each $u \in A^\circ$ (resp B°), T_u is a dominating set in H. In view of Theorem 5.1, since (S, D) is a $\gamma\gamma$-pair in $G[H]$, we have for each $u \in A^\circ \cap B^\circ$, $\{y \in V(H) : (u, y) \in S\}$ and $\{y \in V(H) : (u, y) \in D\}$ constitute a $\gamma\gamma$-pair in H. Thus,

$$\gamma\gamma(G[H]) = |S| + |D| \geq |A| + |B| + |A^\circ \cap B^\circ|(\gamma\gamma(H) - 2) \geq \alpha.$$

This proves Statement (ii). ■

Corollary 5.8 *Let G and H be nontrivial connected graphs. If H has a unique vertex that dominates $V(H)$, then $\gamma\gamma(G[H]) = \gamma\gamma(G)$ if and only if G has a $\gamma\gamma$-pair (A_0, B_0) such that $|A_0| + |B_0| \leq |A| + |B| + |A^\circ \cap B^\circ|(\gamma'(H) - 1)$ for all dominating sets A and B in G.*

Example 5.9 (1) For all integers $n, m \geq 3$,

$$\gamma'(K_{1,n}[K_{1,m}]) = 2 \text{ and } \gamma\gamma(K_{1,n}[K_{1,m}]) = 3.$$

(2) For noncomplete connected graphs G and integers $p \geq 2$,

$$\gamma'(G[K_p]) = \gamma(G) \text{ and } \gamma\gamma(G[K_p]) = 2\gamma(G).$$

(3) For noncomplete graphs G and integers $p \geq 2$,

$$\gamma'(K_p[G]) = \begin{cases} 1, & if\ \gamma(G) = 1, \\ 2, & if\ \gamma(G) \geq 2 \end{cases}$$

and

$$\gamma\gamma(K_p[G]) = \begin{cases} 2, & if\ \gamma(G) = 1, \\ 4, & if\ \gamma(G) \geq 2. \end{cases}$$

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Received: June 15, 2014