Globally Evolutionarily Stable Portfolio Rules*

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Abstract

The paper examines a dynamic model of a financial market with endogenous asset prices determined by short run equilibrium of supply and demand. Assets pay dividends that are partially consumed and partially reinvested. The traders use fixed-mix investment strategies (portfolio rules), distributing their wealth between assets in fixed proportions. Our main goal is to identify globally evolutionarily stable strategies, allowing an investor to "survive," i.e. to accumulate in the long run a positive share of market wealth, regardless of the initial state of the market. It is shown that there is a unique portfolio rule with this property—an analogue of the famous Kelly (1956) rule of "betting one's beliefs." A game theoretic interpretation of this result is given.

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1 Introduction

1.1. Evolutionary finance. Price changes and dividend payments of stocks induce wealth dynamics among investors using different investment strategies (portfolio rules) in financial markets. These dynamics act as a natural selection force among the portfolio rules: some prove to be successful and "survive," the others fail and "become extinct." The purpose of the present paper is to investigate financial dynamics from this evolutionary perspective with the view to identifying evolutionarily stable (surviving) investment strategies.

Evolutionary ideas have a long history in the social sciences going back to Malthus, who played an inspirational role for Darwin (for a review of the subject see, e.g., Hodgeson [20]). A more recent stage of development of these ideas began in the 1950s with the publications of Alchian [2], Penrose [32] and others. A powerful momentum to work in this area was given by the interdisciplinary research conducted in the 1980s and 1990s under the auspices of the Santa Fe Institute in New Mexico, USA, where researchers of different backgrounds—economists, mathematicians, physicists and biologists—combined their efforts to study evolutionary dynamics in biology, economics and finance; see, e.g., Arthur, Holland, LeBaron, Palmer and Taylor [5], Farmer and Lo [16], LeBaron, Arthur and Palmer [24], Blume and Easley [7], and Blume and Durlauf [6].

Questions of survival and extinction of portfolio rules have been studied by Blume and Easley [7, 8] and Sandroni [35] in general equilibrium models with perfect foresight (e.g. Laffont [23], Ch. 6), where agents maximize discounted sums of expected utilities. The selection results in their papers are driven by the interplay between an agent's consumption and the accuracy of his subjective beliefs (i.e. the individual assessment of the probabilities of future states). Most of the positive results in that line of research pertain to the case where the markets under consideration are complete.

The approach to evolutionary finance pursued here marks a departure from the conventional general equilibrium paradigm. In the model we deal with, the asset market dynamics are determined by the dynamic interaction of strategies of the traders, rather than by the maximization of utilities of consumption. These strategies are taken as fundamental characteristics of the agents, while the optimality of individual behavior and the coordination of beliefs (or the lack of it) are not reflected in formal terms but are rather left to the interpretation of the observed behavior. A specific feature of this approach is that it rests only on model components that are observable and can be estimated empirically, which makes the theory closer to practical applications. Our modeling framework based on random dynamical systems is substantially new, but one can trace its connections to some classical ideas in economics (we revive in the new context the Marshallian [28] concept of temporary equilibrium—see the discussion in Section 2). Although one does not require notions of rationality to define the dynamics, the main result has a game theoretic interpretation (Theorem 2), which links this study to the theory of market games—Shapley and Shubik [37] and others.

1.2. Outline of the model and the results. The model we propose describes the dynamics of a financial market in which there are I investors (traders) and K traded assets (securities). Asset supply is constant over time. Each trader chooses a strategy prescribing to distribute, at the beginning of each time period $t = 1, 2, ...$, his/her investment budget between the securities according to given proportions. Assets pay dividends, that are random and depend on a discrete-time stochastic process of exogenous "states of the world."

The prices of the securities at each date are derived endogenously from the equilibrium condition: aggregate market demand of each asset is equal to its supply. Each investor's individual demand depends on his/her budget and investment proportions. The main results pertain to the case where these proportions are fixed (constant over time). The budget of each investor depends on time and random factors. It has two sources: the dividends paid by the assets and capital gains. These two sources form investor's wealth, which is partially consumed and partially reinvested at each time period. When analyzing the long-run performance of trading strategies, we assume that the investment/consumption ratio is fixed and that it is the same for all the traders.

We note that the class of fixed-mix, or constant proportions, strategies we consider in this work is quite common in financial theory and practice; see, e.g., Perold and Sharpe [33], Mulvey and Ziemba [29], Browne [10] and Dempster [12, 13]. From the theoretical standpoint, this class of strategies provides a convenient laboratory for the analysis of questions we are interested in. It makes it possible to formalize in a clear and compact way the concept of the *type* of an investor which determines the evolutionary performance of his/her portfolio rule in the long run. A similar approach is common in evolutionary game theory (e.g. Weibull [40]), and in this paper we initiate the analysis of our model by pursuing it in the context of an asset market dynamics.

The strategy profile of the investors determines the "ecology" of the market and its random dynamics over time. In the evolutionary perspective, survival or extinction of investment strategies is governed by the long-run behavior of the relative wealth of the investors, which depends on the combination of the strategies chosen. A portfolio rule (or an investor using it) is said to survive if it accumulates in the limit a positive fraction of total market wealth. It is said to become extinct if the share of market wealth corresponding to it tends to zero.

An investment strategy, or a portfolio rule, is called evolutionarily stable if the following condition holds. If a group of investors uses this rule, while all the others use different ones, those and only those investors survive who belong to the former group. If this condition holds regardless of the initial state of the market, the investment strategy is called globally evolutionarily stable. If it holds under the additional assumption that the group of investors using other portfolio rules (distinct from the one we consider) possesses a sufficiently small initial share of market wealth, then the above property of stability is termed local.

We prove that among all fixed-mix (i.e. constant proportions) investment strategies, the only globally evolutionarily stable portfolio rule is to invest according to the proportions of the expected dividends. This recipe is similar to the well-known Kelly's principle of "betting one's beliefs." The present paper contributes to that field of studies which originated from the pioneering work of Shannon[1] and Kelly [22]—see Breiman [9], Thorp [39], Algoet and Cover [3], Hakansson and Ziemba [18] and references therein. Most of the previous work was concerned with models where asset prices were given exogenously, or where the analysis was based on a reduction to such models [7]. Our aim is to obtain analogous results in a dynamic equilibrium setting, with endogenous prices. Intermediate steps towards this aim were made in the previous papers [4] and [14]. Those papers dealt with a special case of "short-lived" assets. Here, we extend the results to a model with long-lived, dividend-paying assets and thus achieve the long-sought goal of providing a natural and general framework for this class of results.

1.3. Plan of the paper. The structure of the paper is as follows. Section 2 provides a rigorous description of the model, a brief outline of which was given above. In Section 3, we formulate and discuss the main results. Section 4 develops methods needed for the analysis of the model under consideration. The Appendix contains proofs of the technical results stated in Section 4.

[1]Although Claude Shannon—the famous founder of the mathematical theory of information—did not publish on investment-related issues, his ideas, expressed in his lectures on investment problems, should apparently be regarded as the initial source of that strand of literature which we cite here. For the history of these ideas and the related discussion see Cover [11].

2 The model

2.1. Equilibrium dynamics of an asset market. In the model we deal with, there are $I \geq 2$ *investors* (*traders*) acting in a market where $K \geq 2$ risky *assets* are traded. The situation on the market at date $t = 0, 1, ...$ is characterized by the set of vectors

$$(p_t;\ x_t^1, ..., x_t^I)$$

where $p_t \in \mathbb{R}_+^K$ is the vector of market prices of the assets and $x_t^i = (x_{t,1}^i, ..., x_{t,K}^i) \in \mathbb{R}_+^K$ is the portfolio of investor i. For each $k = 1, ..., K$, the coordinate $p_{t,k}$ of the vector $p_t = (p_{t,1}, ..., p_{t,K})$ stands for the price of one unit of asset k at date t. The coordinate $x_{t,k}^i$ of the vector $x_t^i = (x_{t,1}^i, ..., x_{t,K}^i)$ indicates the amount of ("physical units" of) asset k in the portfolio x_t^i. The scalar product $\langle p_t, x_t^i \rangle = \sum_{i=1}^K p_{t,k} x_{t,k}^i$ expresses the market value of investor i's portfolio at date t.

Investor i's behavior (and implicitly his/her preferences) at date t are characterized by the *demand function* $X_t^i(p_t, x_{t-1}^i)$, assigning to each pair of vectors $p_t \in \mathbb{R}_+^K$ and $x_{t-1}^i \in \mathbb{R}_+^K$ the vector $x_t^i = X_t^i(p_t, x_{t-1}^i) \in \mathbb{R}_+^K$. If the investor possessed the portfolio x_{t-1}^i at date $t-1$ ("yesterday"), then he/she will be willing to purchase the portfolio $x_t^i = X_t^i(p_t, x_{t-1}^i)$ at date t ("today"), provided that today's asset price system is p_t. All the coordinates of $X_t^i(p_t, x_{t-1}^i)$ are non-negative: borrowing and short sales are ruled out.

Define the *aggregate demand function* for the market under consideration as

$$X_t(p_t, x_{t-1}) := \sum_{i=1}^I X_t^i(p_t, x_{t-1}^i), \tag{1}$$

where $x_{t-1} = (x_{t-1}^1, ..., x_{t-1}^I)$ is the set of portfolios of all the investors. It is supposed that the supply of each asset in each time period is constant and, for simplicity, normalized to 1. We examine the *equilibrium market dynamics*, assuming that, in each time period, the demand on each asset is equal to its supply:

$$\sum_{i=1}^I X_{t,k}^i(p_t, x_{t-1}^i) = 1,\ k = 1, ..., K.$$

By using the notation (1) and $e := (1, 1, ..., 1)$, the previous system of equations can be written in the vector form:

$$X_t(p_t, x_{t-1}) = e. \tag{2}$$

Each solution to this system, p_t, is an *equilibrium price vector*. It will follow from the assumptions we are going to impose that this vector exists, is unique and is strictly positive for each x_{t-1} satisfying

$$x_{t-1} = (x_{t-1}^1, ..., x_{t-1}^I) \in \mathbb{R}_+^{K \times I}, \ \sum_{i=1}^{I} x_{t-1}^i = e. \tag{3}$$

Equations (2) and

$$x_t^i = X_t^i(p_t, x_{t-1}^i), \ i = 1, ..., I, \tag{4}$$

define a *dynamical system* describing the evolution of the asset market. The state of this dynamical system at time $t = 1, 2, ...$ is a pair (p_t, x_t), where $x_t = (x_t^1, ..., x_t^I) \in \mathbb{R}_+^{K \times I}$ is a collection of investors' portfolios satisfying $\sum_{i=1}^{I} x_t^i = e$ and p_t is a non-negative K-dimensional vector whose coordinates are the equilibrium asset prices prevailing at date t. For date 0, the set of initial endowments (initial portfolios) $x_0 = (x_0^1, ..., x_0^I)$ satisfying (3) is given. The knowledge of the state x_{t-1} at time $t-1$ allows us to compute the equilibrium price vector p_t at time t as the solution to equation (2). Based on p_t and x_{t-1}, we can determine $x_t = (x_t^1, ..., x_t^I)$ from (4). By iterating this procedure, we can generate a *path*

$$(p_1, x_1), (p_2, x_2), ... \tag{5}$$

of the dynamical system in question.

In the model considered here, the demand functions $X_t^i(\cdot, \cdot)$ of investors depend on random factors (via random asset dividends). Therefore paths (5) are random processes, and so we will deal with a *random dynamical system*. We are interested, in particular, in the long-run behavior (as $t \to \infty$) of paths of this system in connection with questions of evolutionary market dynamics.

2.2. Investors' budgets and demand functions. In the model under consideration, investor i's *budget* at date t, on which his/her demand depends, is given by

$$B_t^i(p_t, x_{t-1}^i) := \langle d_t, x_{t-1}^i \rangle + \langle p_t, x_{t-1}^i \rangle. \tag{6}$$

There are two sources from which the budget is formed: the value $\langle p_t, x_{t-1}^i \rangle$ of yesterday's portfolio x_{t-1}^i, expressed in terms of the current prices p_t, and the *dividend*

$$\langle d_t, x_{t-1}^i \rangle = \sum_{k=1}^{K} d_{t,k} x_{t-1,k}^i$$

yielded by the portfolio x_{t-1}^i. It is supposed that (one unit of) asset k pays the dividend $d_{t,k} \geq 0$ at date $t \geq 1$. The dividends depend on random factors as described below.

Randomness is modeled as follows. There is a finite set S and a sequence $s_0, s_1, ...$ of random variables with values in S. The random parameter s_t characterizes the *state of the world* at date t. The dividends $d_{t,k}$ of the assets $k = 1, ..., K$ are supposed to be functions of the history $s^t := (s_0, ..., s_t)$ of states of the world prior to date t:

$$d_{t,k} = d_{t,k}(s^t) \geq 0 \ (k = 1, ..., K, \ t = 1, 2, ...).$$

We impose the following fundamental assumptions regarding $d_{t,k}(\cdot)$:

(**d1**) for all s^t, we have $\sum_{k=1}^K d_{t,k}(s^t) > 0$;

(**d2**) for each $k = 1, 2, ..., K$, the expectation $Ed_{t,k}(s^t)$ is strictly positive.

The first assumption means that always at least one asset pays a strictly positive dividend. According to the second condition, for each asset k the probability that it pays a strictly positive dividend is strictly positive.

We assume that the individual demand function of investor i is of the form

$$X_{t,k}^i(p_t, x_{t-1}^i) = \mu_{t,k}^i \frac{B_t^i(p_t, x_{t-1}^i)}{p_{t,k}}, \tag{7}$$

where $\mu_{t,1}^i, ..., \mu_{t,K}^i$ are nonnegative numbers satisfying

$$\mu_{t,1}^i + ... + \mu_{t,K}^i < 1. \tag{8}$$

According to (6) and (7), the trader acts in the market as follows. He/she gets the dividends $\langle d_t, x_{t-1}^i \rangle$ from the portfolio x_{t-1}^i and rebalances the portfolio (by selling some assets and buying others) at the prices p_t so that the new portfolio $x_t^i = X_t^i(p_t, x_{t-1}^i)$ satisfies

$$p_t^k X_{t,k}^i(p_t, x_{t-1}^i) = \mu_{t,k}^i B_t^i(p_t, x_{t-1}^i).$$

Thus the portfolio x_t^i is constructed by *investing the fraction* $\mu_{t,k}^i$ *of the budget* $B_t^i(p_t, x_{t-1}^i)$ (given by (6))) *into the kth position* $x_{t,k}^i$ *of* x_t^i.

If $p_{t,k} = 0$, then the expression on the right-hand side of (7) is not defined, and in this case we put $X_{t,k}^i(p_t, x_{t-1}^i) = 0$. We define the price $p_{t,k}$ of asset k as zero if this asset is not traded; in this case, the holdings of this asset in the portfolio of each investor are zero, i.e. $X_{t,k}^i(p_t, x_{t-1}^i) = 0$. This possibility, however, will be excluded by the assumptions we are going to impose. Under these assumptions all the equilibrium asset prices will be strictly positive.

According to (8), the sum $\mu^i_{t,1} + ... + \mu^i_{t,K}$ is strictly less than one. This means that not all of the investor's budget is used for purchasing assets. Some fraction of the budget is used for consumption. This fraction,

$$\mu^i_{t,0} := 1 - (\mu^i_{t,1} + ... + \mu^i_{t,K}), \tag{9}$$

is the investor i's *consumption rate.*

2.3. Investors' strategies and the random dynamical system they generate. The vectors $\mu^i_t = (\mu^i_{t,1}, ..., \mu^i_{t,K})$ describing trader i's investment decisions can depend on the observed states of the world: $\mu^i_t = \mu^i_t(s^t)$. A *trading (investment) strategy* of investor i is a sequence of non-zero non-negative vector functions

$$\mu^i_t(s^t) = (\mu^i_{t,1}(s^t), ..., \mu^i_{t,K}(s^t)), t = 1, 2, ...$$

satisfying (8) for all t and all realizations of the states of the world. As long as the vectors $d_t = (d_{t,1}, ..., d_{t,K})$, $t = 1, 2, ...$, depend on s^t, the demand functions (7) also depend on s^t, and so the state (p_t, x_t) of the random dynamical system under consideration is a function of the history s^t of the states of the world from time zero to time t. If a strategy profile $(\mu_t(s^t))_{t=1}^\infty = (\mu^1_t(s^t), ..., \mu^I_t(s^t))_{t=1}^\infty$ of all the investors is fixed, the random sate of the market $(x_t(s^t), p_t(s^t))$ (where $x_t(s^t) = (x^1_t(s^t), ..., x^I_t(s^t))$) evolves according to the following system of equations

$$p_{t,k} = \sum_{i=1}^I \mu^i_{t,k} \langle d_t + p_t, x^i_{t-1} \rangle, \ k = 1, 2, ..., K; \tag{10}$$

$$p_{t,k} x^i_{t,k} = \mu^i_{t,k} \langle d_t + p_t, x^i_{t-1} \rangle. \tag{11}$$

The vector $p_t = p_t(s^t) \in \mathbb{R}^K_+$ satisfying (10) exists and is unique. This follows from the fact that, for each s^t and each vector x^i_{t-1} satisfying (3), the operator transforming a vector $p = (p_1, ..., p_K) \in \mathbb{R}^K_+$ into the vector $q = (q_1, ..., q_K) \in \mathbb{R}^K_+$ with coordinates

$$q_k = \sum_{i=1}^I \mu^i_{t,k} \langle d_t + p, x^i_{t-1} \rangle \tag{12}$$

is contracting in the norm $|p| := \sum_k |p_k|$. Indeed,

$$|q - q'| = \sum_{k=1}^K |q_k - q'_k| = \sum_{k=1}^K |\sum_{i=1}^I \mu^i_{t,k} \langle p - p', x^i_{t-1} \rangle| \leq (\max_i \sum_{k=1}^K \mu^i_{t,k})|p - p'|, \tag{13}$$

where $\max_i \sum_{k=1}^K \mu_{t,k}^i < 1$ (see (8)) and $|\langle p - p', x_{t-1}^i \rangle| \leq |p - p'|$ because $0 \leq x_{t-1,k}^i \leq 1$ (see (3)).

The unique price vector $p_t \geq 0$ satisfying (10) is strictly positive if one of the following conditions holds:

(I) $\mu_{t,k}^i > 0$, $0 \leq x_0^i \neq 0$;

(II) $\sum_{i=1}^I \mu_k^i > 0$, $d_{t,k}(s^t) > 0$, $0 \leq x_0^i \neq 0$.

Indeed, if (I) is valid, then we have

$$p_{t,k} = \sum_{i=1}^I \mu_{t,k}^i \langle d_t + p_t, x_{t-1}^i \rangle \geq \min_j(\mu_{t,k}^j) \sum_{i=1}^I \langle d_t + p_t, x_{t-1}^i \rangle \geq$$

$$\min_j(\mu_{t,k}^j) \sum_{i=1}^I \langle d_t, x_{t-1}^i \rangle = \min_j(\mu_{t,k}^j) \langle d_t, e \rangle > 0$$

because $\langle d_t, e \rangle = \sum_{k=1}^K d_{t,k} > 0$ by virtue of (**d1**). Once $p_t > 0$, the budget of each investor $\langle d_t + p_t, x_{t-1}^i \rangle$ is strictly positive, provided that $0 \leq x_{t-1}^i \neq 0$. Hence, $x_t^i > 0$ for each $t \geq 1$. (All inequalities between vectors are understood coordinatewise.)

Suppose (II) holds. Then we have

$$p_{t,k} = \sum_{i=1}^I \mu_{t,k}^i \langle d_t + p_t, x_{t-1}^i \rangle \geq (\sum_{i=1}^I \mu_{t,k}^i) \min_i \langle d_t + p_t, x_{t-1}^i \rangle.$$

If $0 \leq x_{t-1}^i \neq 0$, then $\langle d_t, x_{t-1}^i \rangle > 0$, and so $p_t > 0$. Furthermore, $x_{t,k}^i \neq 0$ for some k because $x_{t,k}^i = (p_{t,k})^{-1} \sum_{i=1}^I \mu_{t,k}^i \langle d_t + p_t, x_{t-1}^i \rangle$ and $\mu_{t,k}^i > 0$ for some k.

The validity of at least one of conditions (I) or (II) is essentially necessary for the dynamical system in question to be non-degenerate in the following sense. If neither condition is supposed to hold, then it might happen that $p_{t,k} = 0$ for some t and k. Indeed, denote by e_i the vector whose coordinates are equal to 0 except the ith coordinate which is equal to 1 and put $K = I = 2$, $x_0^i = e_i$, $d_1 = e_1$ and $\mu_1^i = e_i/2$. Then $p_{1,2} = \mu_{1,2}^1 \langle d_1, x_0^1 \rangle + \mu_{1,2}^2 \langle d_1, x_0^2 \rangle = 0 \cdot 1 + (1/2) \cdot 0 = 0$. Throughout the paper, we are going to deal with a model in which condition (I) holds; condition (II) is provided here only for the sake of completeness.

2.4. Market evolution and Marshallian temporary equilibrium. In the model we deal with, the dynamics of an asset market is modeled in terms of a sequence of temporary equilibria. At each date t the investors' strategies $\mu_{t,k}^i$, the asset dividends $d_{t,k}$ and the portfolios x_{t-1}^i determine—in accordance with (10)—the asset prices $p_t = (p_t^1, ..., p_t^K)$ equilibrating aggregate asset demand and supply. The asset holdings $x_{t-1}^i = (x_{t-1,1}^i, ..., x_{t-1,K}^i)$

play the role of initial endowments available at the beginning of date t. The portfolios x_t^i constructed according to (11) are transferred to date $t+1$ and then in turn serve as initial endowments for the investors.

The dynamics of the asset market described above are similar to the dynamics of the commodity market as outlined in the classical treatise by Alfred Marshall [28] (Book V, Chapter II "Temporary Equilibrium of Demand and Supply"). Marshall's ideas were introduced into formal economics by Samuelson [34], pp. 321–323. Equations analogous to (10), (11) were derived in continuous time by Lotka [26]; they became the classics of mathematical evolutionary biology. In rigorous terms, the Marshallian concept of temporary equilibrium, as it is understood in the present work, was in much detail analyzed in an economics context by Schlicht [36]. The equations on pp. 29–30 in the monograph [36] are direct continuous-time, deterministic counterparts of our equations (10) and (11). They underlie the Marshallian temporary equilibrium approach.

As it was noticed by Samuelson [34] and emphasized by Schlicht [36], in order to study the process of market dynamics by using the Marshallian "moving equilibrium method," one needs to distinguish between at least two sets of economic variables changing with different speeds. Then the set of variables changing slower (in our case, the set $x_t = (x_t^1, ..., x_t^I)$ of investors' portfolios) can be temporarily fixed, while the other (in our case, the asset prices p_t) can be assumed to rapidly reach the unique state of partial equilibrium. Samuelson [34] writes about this approach:

> I, myself, find it convenient to visualize equilibrium processes of quite different speed, some very slow compared to others. Within each long run there is a shorter run, and within each shorter run there is a still shorter run, and so forth in an infinite regression. For analytic purposes it is often convenient to treat slow processes as data and concentrate upon the processes of interest. For example, in a short run study of the level of investment, income, and employment, it is often convenient to assume that the stock of capital is perfectly or sensibly fixed.

As it follows from the above citation, Samuelson thinks about a hierarchy of various equilibrium processes with different speeds. In our model, it is sufficient to deal with only two levels of such a hierarchy. We leave the price adjustment process leading to the solution of the partial equilibrium problem (10) beyond the scope of the model. It can be shown, however, that this equilibrium will be reached at an exponential rate in the course of any naturally defined *tâtonnement* procedure (cf. (12) and (13)). Results yielding a rigorous justification of the above approach, involving "rapid" and "slow" variables, are provided in continuous time by the theory of *singular perturbations*—e.g. Smith [38].

The concept of a temporary, or moving, equilibrium was introduced and analyzed apparently for the first time by Marshall. However, in the last four decades this term has been by and large associated with a different notion, going back to Lindahl [25] and Hicks [19]. This notion was developed in formal settings by Grandmont and others (see, e.g., the volume [17]). The characteristic feature of the Lindahl-Hicks temporary equilibrium is the idea of *forecasts* or *beliefs* about the future states of the world, which all the market participants possess and which are formalized in terms of stochastic kernels (transition functions) conditioning the distributions of future states of the world upon the agents' private information. A discussion of the modern state of this direction of research is provided by Magill and Quinzii [27].

In this work, we pursue a completely different approach. Our model might indirectly take into account agents' forecasts or beliefs, but they can be only implicitly incorporated into the agents' investment strategies. These strategies are the only agents' characteristics we rely upon in our modeling. Such characteristics can be observed and estimated based on the real market data, and we formulate our results in the form of recommendations for an investor what strategies to follow.

To distinguish the above approach to temporary equilibrium from the one based on the Hicks–Lindahl concepts, we suggest the terms "*Marshallian*" or "*evolutionary* temporary equilibrium." The former term is motivated by the already cited Marshall's [28] ideas. The latter is justified not only by the main focus of the model on questions of survival and extinction of portfolio rules, but also by deep analogies between the dynamic processes governed by the evolutionary equations (10), (11) and similarly described processes in evolving complex systems in physics, mechanics, chemical kinetics, evolutionary biology, ecology and other sciences—see, e.g., Hofbauer and Sigmund [21].

3 The main results

3.1. Dynamics of market shares. Our further analysis will be based on the following assumptions:

(i) the states of the world $s_0, s_1, s_2, ...$ form a sequence of independent identically distributed (i.i.d.) elements in S such that the probability $P\{s_t = s\}$ is strictly positive for each $s \in S$;

(ii) the asset dividends $d_{t,k}(s^t)$ are functions of the current state s_t of the

world:

$$d_{t,k}(s^t) = D_k(s_t),$$

where the functions $D_k(s)$ ($s \in S$, $k = 1, 2, ..., K$) do not depend on t, are non-negative and satisfy

(**D1**) $\sum_{k=1}^{K} D_k(s) > 0,\ s \in S$;

(**D2**) $ED_k(s_t) > 0,\ k = 1, ..., K$.

Each investor i chooses an investment strategy (portfolio rule) characterized by a fixed non-negative vector $\mu^i = (\mu_1^i, ..., \mu_K^i)$ such that $0 < \mu_1^i + ... + \mu_K^i < 1$. The numbers μ_k^i indicate the fractions of investor i's budget according to which he/she distributes wealth between the assets $k = 1, ..., K$. These fractions remain the same over time, so that we deal here with *simple*, or *fixed-mix*, investment strategies. In the remainder of the paper we will consider only those portfolio rules $(\mu_1^i, ..., \mu_K^i)$ which are *completely mixed*, i.e., $\mu_k^i > 0$ for each $k = 1, ...K$. Furthermore, we will assume that the consumption rate $\mu_0^i = 1 - \sum_{k=1}^{K} \mu_k^i$ is *the same* for all the investors and it is equal to $1 - \rho$, where ρ is some given number in $(0, 1)$. We suppose that the consumption rate is the same for all the market traders because we are mainly interested in comparing the long-run performance of investment strategies. This can be done only for a group of traders having the same consumption rate. Otherwise, a seemingly lower performance of a strategy may be simply due to a higher consumption rate of the investor.

As long as the sum $\sum_{k=1}^{K} \mu_k^i$ does not depend on i and is equal to some given number $\rho \in (0, 1)$, it is convenient to characterise the investment decisions of each investor i in terms of the vector of *investment proportions*

$$\lambda^i = (\lambda_1^i, ..., \lambda_K^i),\ \lambda_k^i := \mu_k^i / \rho.$$

The numbers λ_k^i are strictly positive and $\lambda_1^i + ... + \lambda_K^i = 1$. The set of vectors whose coordinates satisfy these conditions will be denoted by Δ_+^K. *From now on, we will associate the terms "investment strategy" or "portfolio rule" with such vectors of investment proportions.*

If each investor $i = 1, ..., I$ selects a portfolio rule $\lambda^i = (\lambda_1^i, ..., \lambda_K^i) \in \Delta_+^K$, the strategy profile $(\lambda^1, ..., \lambda^I)$ determines, in accordance with equations

$$p_{t,k} = \rho \sum_{i=1}^{I} \lambda_k^i \langle d_t + p_t, x_{t-1}^i \rangle,\ k = 1, 2, ..., K, \tag{14}$$

$$p_{t,k} x_{t,k}^i = \rho \lambda_k^i \langle d_t + p_t, x_{t-1}^i \rangle,\ k = 1, 2, ..., K, \tag{15}$$

equivalent to (10) and (11), the random path of market dynamics (p_t, x_t), $t = 1, 2, ...$. Both the prices $p_t = (p_{t,1}, ..., p_{t,K})$ and the asset holdings $x_t^i =$

$(x^i_{t,1}, ..., x^i_{t,K})$ depend on s^t—the history of the states of the world prior to time t.

Denote by

$$w^i_t := \langle p_t + d_t, x^i_{t-1} \rangle \tag{16}$$

investor i's *wealth* available for consumption and investment at date $t \geq 1$. The *total market wealth* is equal to $W_t = \sum_{i=1}^I w^i_t$. (From formulas (18) and (19) below, it follows that $W_t > 0$.) We are primarily interested in the long-run behavior of the *relative wealth,* or *market shares,* $r^i_t := w^i_t/W_t$ of the traders, i.e. in the asymptotic properties of the sequence of vectors $r_t = (r^1_t, ..., r^I_t)$ as $t \to \infty$. To analyze these properties, we will derive equations allowing to compute the vector r_{t+1} based on the knowledge of the vector r_t and the state of the world s_{t+1} realized at date $t+1$.

From (14) we get

$$p_{t+1,k} = \rho \sum_{i=1}^I \lambda^i_k \langle d_{t+1} + p_{t+1}, x^i_t \rangle = \rho \sum_{i=1}^I \lambda^i_k w^i_{t+1} = \rho \langle \lambda_k, w_{t+1} \rangle,$$

$$x^i_{t,k} = \frac{\lambda^i_k w^i_t}{\langle \lambda_k, w_t \rangle},\ k = 1, 2, ..., K,$$

where $\lambda_k := (\lambda^1_k, ..., \lambda^I_k)$ and $w_t := (w^1_t, ..., w^I_t)$. Consequently,

$$w^i_{t+1} := \sum_{k=1}^K [p_{t+1,k} + D_k(s_{t+1})] x^i_t =$$

$$\sum_{k=1}^K [\rho \langle \lambda_k, w_{t+1} \rangle + D_k(s_{t+1})] \frac{\lambda^i_k w^i_t}{\langle \lambda_k, w_t \rangle}. \tag{17}$$

By summing up these equations over $i = 1, ..., I$, we obtain

$$W_{t+1} = \sum_{k=1}^K [\rho \langle \lambda_k, w_{t+1} \rangle + D_k(s_{t+1})] \frac{\sum_{i=1}^I \lambda^i_k w^i_t}{\langle \lambda_k, w_t \rangle} = \rho W_{t+1} + \sum_{k=1}^K D_k(s_{t+1}).$$

This leads to the formula

$$W_{t+1} = \frac{D(s_{t+1})}{1-\rho}, \tag{18}$$

where

$$D(s_{t+1}) = \sum_{k=1}^K D_k(s_{t+1})\ (> 0) \tag{19}$$

is the sum of the dividends of all the assets. Dividing both sides of equation (17) by W_{t+1} and using formula (18), we find

$$r^i_{t+1} = \sum_{k=1}^{K} [\rho\langle\lambda_k, r_{t+1}\rangle + (1-\rho)\frac{D_k(s_{t+1})}{D(s_{t+1})}] \frac{\lambda^i_k w^i_t / W_t}{\langle\lambda_k, w_t\rangle / W_t}.$$

Thus we arrive at the system of equations:

$$r^i_{t+1} = \sum_{k=1}^{K} [\rho\langle\lambda_k, r_{t+1}\rangle + (1-\rho)R_k(s_{t+1})] \frac{\lambda^i_k r^i_t}{\langle\lambda_k, r_t\rangle}, \; i = 1, ..., I, \tag{20}$$

where

$$R_k(s_{t+1}) = \frac{D_k(s_{t+1})}{D(s_{t+1})}, \; k = 1, ..., K,$$

are the *relative dividends* of the assets $k = 1, ..., K$.

Let Δ^I denote the set of vectors $r = (r^1, ..., r^I) \geq 0$ whose norm $|r| := \sum |r^i|$ is equal to one. It can be shown (see Section 4 below) that, for any $r_t = (r^1_t, ..., r^I_t) \in \Delta^I$ and any $s_{t+1} \in S$, the system of equations (20) has a unique solution $r_{t+1} \geq 0$. We have $r_{t+1} \in \Delta^I$, which can be verified by summing up equations (20) and using the fact that $\sum_{k=1}^{K} R_k(s) = 1$, $s \in S$.

We will denote the solution r_{t+1} to system (20) (as a function of s_{t+1} and r_t) by $F(s_{t+1}, r_t)$. The mapping $F(s_{t+1}, \cdot)$ transforms Δ^I into Δ^I. Thus we deal here with a random dynamical system

$$r_{t+1} = F(s_{t+1}, r_t) \tag{21}$$

on the unit simplex Δ^I. We will assume that a strictly positive non-random vector $r_0 \in \Delta^I$ is fixed. Starting from this initial state, we can generate a path (trajectory) $r_0, r_1(s^1), r_2(s^2), ...$ of the random system (21) according to the formula

$$r_{t+1}(s^{t+1}) = F(s_{t+1}, r_t(s^t)), \; t = 0, 1, ...,$$

(If $t = 0$, we formally write $r_0 = r_0(s^0)$ having in mind that r_0 is a constant.)

Remark. The model for the dynamics of investors' market shares we have described above was proposed in [15]. Its presentation in this paper is slightly different from that in [15]. (In particular, we here write ρ in place of $1 - \lambda_0$ and λ^i_k instead of $\lambda^i_k/(1 - \lambda_0)$ in [15].) In the limit as $\rho \to 0$, the model reduces to the one studied in [14]. In particular, if $\rho = 0$, the random dynamical system described by equations (20) coincides with that examined in [14].

3.2. Survival and extinction of portfolio rules. We examine the dynamics of the relative wealth r^i_t, governed by equations (20), from an evolutionary perspective. We are interested in questions of "survival and extinction" of portfolio rules. We say that a portfolio rule $\lambda^i = (\lambda^i_1, ..., \lambda^i_K)$ (or investor i using it) *survives* with probability one in the market selection process (20) if, for the relative wealth r^i_t of investor i, we have $\lim_{t\to\infty} r^i_t > 0$ almost surely. We say that λ^i *becomes extinct* with probability one if $\lim_{t\to\infty} r^i_t = 0$ almost surely.

A central role in this work is played by the following definition.

Definition 1 *A portfolio rule $\lambda = (\lambda_1, ..., \lambda_K)$ is called* globally evolutionarily stable *if the following condition holds. Suppose, in a group of investors $i = 1, 2, ..., J$ $(1 \le J < I)$, all use the portfolio rule λ, while all the others, $i = J+1, ..., I$ use portfolio rules $\hat{\lambda}^i$ distinct from λ. Then those investors who belong to the former group $(i = 1, ..., J)$ survive with probability one, whereas those who belong to the latter $(i = J+1, ..., I)$ become extinct with probability one.*

In the above definition, it is supposed that the initial state r_0 in the market selection process governed by equations (20) is *any* strictly positive vector $r_0 \in \Delta^I$. This is reflected in the term "*global* evolutionary stability." An analogous local concept (cf. [15]) is defined similarly, but in the definition of local evolutionary stability, the initial market share $r^{J+1}_0 + ... + r^I_0$ of the group of investors who use strategies $\hat{\lambda}^i$ distinct from λ is supposed to be small enough.

Our main goal is to identify that portfolio rule which is globally evolutionarily stable. Clearly, if it exists it must be unique. Indeed if there are two such rules, $\lambda \neq \lambda'$, we can divide the population of investors into two groups assuming that the first uses λ and the second λ'. Then, according to the definition of global evolutionary stability, both groups must become extinct with probability one, which is impossible since the sum of the relative wealth of all the investors is equal to one.

3.3. Central result. Define

$$\lambda^* = (\lambda^*_1, ..., \lambda^*_K),\ \lambda^*_k = ER_k(s_t),\ k = 1, ..., K,$$

so that $\lambda^*_1, ..., \lambda^*_K$ are the *expected relative dividends* of assets $k = 1, ..., K$. The portfolio rule (investment strategy) λ^* is called the *Kelly portfolio rule*. It prescribes to invest in accordance with the principle of "betting one's beliefs," as formulated in the pioneering paper by Kelly [22], for further studies in this direction see Breiman [9], Thorp [39], Algoet and Cover [3] and Hakansson and Ziemba [18].

Recall that, according to a convention made in Section 2, we consider in this paper only completely mixed portfolio rules. Therefore the vectors

λ and $\hat{\lambda}^i$ involved in Definition 1 are supposed to be strictly positive. The Kelly rule is completely mixed by virtue of assumptions (**D1**) and (**D2**).

Throughout the paper, we will assume that the functions $R_1(s), ..., R_K(s)$ are linearly independent (there are *no redundant assets*).

The main result of this paper is as follows.

Theorem 1 *The Kelly rule is globally evolutionarily stable.*

In order to prove this theorem we have to consider a group of investors $i = 1, ..., J$ using the portfolio rule λ^*, assume that all the other investors $i = J+1, ..., I$ use portfolio rules $\lambda^i \neq \lambda^*$ and show that the former group survives, while the latter becomes extinct. In general, J should be any number between $1 \leq J < I$. We note, however, that it is sufficient to prove the theorem assuming that $J = 1$, in which case the result reduces to the assertion that $r_t^1 \to 1$ almost surely. To perform the reduction of the case $J > 1$ to the case $J = 1$, we "aggregate" the group of investors $i = 1, 2, ..., J$ into one by setting

$$\bar{r}_t^1 = r_t^1 + ... + r_t^J.$$

By adding up equations (20) over $i = 1, ..., J$, we obtain:

$$\bar{r}_{t+1}^1 = \sum_{k=1}^{K} [\rho\langle\lambda_k, r_{t+1}\rangle + (1-\rho)R_k(s_{t+1})] \frac{\lambda_k^* \bar{r}_t^1}{\langle\lambda_k, r_t\rangle},$$

where

$$\langle\lambda_k, r\rangle = \lambda_k^* \bar{r}^1 + \sum_{i=J+1}^{I} \lambda_k^i r^i.$$

Thus the original model reduces to the analogous one in which there are $I - J + 1$ investors ($i = 1, J+1, ..., I$) so that investor 1 uses the Kelly strategy λ^* and all the others, $i = J+1, ..., I$, use strategies distinct from λ^*. If we have proved Theorem 1 in the special case $J = 1$, we know that $r_t^i \to 0$ almost surely for all $i = J+1, ..., I$. Consequently, $\bar{r}_t^1 \to 1$, which means that the group of investors $i = 1, ..., I$ (which we treat as a single, "aggregate," investor) accumulates in the limit all market wealth. It remains to observe that in the original model, the proportions between the relative wealth of investors i, j who belong to the group $1, ..., J$ using the Kelly rule do not change in time. This is so because for all such investors, the growth rates of their relative wealth are the same:

$$\frac{r_{t+1}^i}{r_t^i} = \sum_{k=1}^{K} [\rho\langle\lambda_k, r_{t+1}\rangle + (1-\rho)R_k(s_{t+1})] \frac{\lambda_k^*}{\langle\lambda_k, r_t\rangle}, \ i = 1, ..., J.$$

Consequently,

$$\frac{r_{t+1}^i}{r_t^i} = \frac{r_{t+1}^j}{r_t^j},\ i, j = 1, ..., J,$$

and so

$$\frac{r_{t+1}^i}{r_{t+1}^1} = \frac{r_t^i}{r_t^1} = \frac{r_0^i}{r_0^1},\ i = 1, ..., J.$$

Therefore $r_t^i = \beta^i r_t^1$ $(i = 1, ..., J)$ for all t, where $\beta^i = r_0^i / r_0^1$ is a strictly positive constant. Since

$$\bar{r}_t^1 = \sum_{i=1}^J r_t^i = (\sum_{i=1}^J \beta^i) r_t^1 \to 1 \text{ (a.s.)},$$

we obtain that $r_t^i \to \beta^i (\sum_{i=1}^J \beta^i)^{-1} > 0$ (a.s.) for all $i = 1, ..., J$, which means that all the "Kelly investors" $i = 1, ..., J$ survive.

Thus in order to prove Theorem 1 it is sufficient to establish the following fact: if investor 1 uses the Kelly rule, while all the others use strategies distinct from the Kelly rule, investor 1 is almost surely the *single survivor* in the market selection process. We will prove this assertion in Section 4 based on a number of auxiliary results. These results provide methods needed for the analysis of the model at hand, and some of them are of independent interest.

3.4. Random asset market game. We would like to discuss our main result from a game-theoretic point of view. Consider a non-cooperative game with I players (investors), whose strategies are completely mixed simple portfolio rules $\lambda^i \in \Delta_+^K$. A strategy profile $\Lambda := (\lambda^1, ..., \lambda^I)$ defines according to (20) a random dynamical system generating for each i the random process $r_0^i, r_1^i, r_2^i, ...$ of the market shares of trader $i = 1, ..., I$. Define the random payoff function of investor i as $\Phi_i(\Lambda) := \limsup_{t\to\infty} r_t^i$. The random variable $\Phi_i(\Lambda)$ is always well-defined and takes values in [0,1]. These data define a non-cooperative game with random payoffs, which we will call the *random asset market game*. Consider the strategy profile $\Lambda^* := (\lambda^{*1}, ..., \lambda^{*I})$ for which $\lambda^{*i} := \lambda^*$ (the *Kelly strategy profile*).

Theorem 2 *The Kelly strategy profile forms with probability one a symmetric dominant strategy Nash equilibrium in the random asset market game.*

The assertion of the theorem means that if one of the players i, say player $i = 1$, employs the strategy λ^* and all the other players use *any* strategies $\lambda^2, ..., \lambda^I \in \Delta_+^K$ then

$$\Phi_1(\lambda^*, \lambda^2, ..., \lambda^I) \geq \Phi_1(\lambda^1, \lambda^2, ..., \lambda^I)$$

for *any* $\lambda^1 \in \Delta_+^K$. In other words, investor 1 cannot be better off by deviating from λ^*, regardless of what the other investors' strategies are.

Theorem 2 is an immediate consequence of Theorem 1. Indeed, if at least one of investors $i = 2, ..., I$ uses λ^*, then any deviation of investor 1 from λ^* will imply $\Phi_1(\lambda^1, \lambda^2, ..., \lambda^I) = 0$, since in the limit all the market wealth will be gathered almost surely by those who use λ^*. If only player 1 employs λ^*, then $\Phi_1(\lambda^*, \lambda^2, ..., \lambda^I) = 1$, which is not less than $\Phi_1(\lambda^1, \lambda^2, ..., \lambda^I)$ because this function takes values in $[0, 1]$.

A static (one-period) version of the above game in which the payoff functions were defined in terms of expected payoffs was considered in the paper by Alós-Ferrer and Ania [1]. That paper dealt with a model involving short lived assets (the case $\rho = 0$). The definition of the game in [1] does not involve the asymptotic performance of investment strategies in the long run, and in this connection, the results in [1] are different from ours.

4 Techniques for the analysis of evolutionary market dynamics

4.1. The mapping defining the random dynamical system. In this section we develop methods needed for the study of the model under consideration. We here provide only the statements and discussions of the results; their proofs are relegated to the Appendix. We begin with the analysis of the mapping defining the random dynamical system (20).

Let ρ be a number satisfying $0 \le \rho < 1$. For each $s \in S$, consider the mapping $F(s, r) = (F^1(s, r), ..., F^I(s, r))$ of the unit simplex Δ^I into itself defined by

$$F^i(s, r) = \sum_{k=1}^{K} [\rho \langle \lambda_k, F(s, r) \rangle + (1 - \rho) R_k(s)] \frac{\lambda_k^i r^i}{\langle \lambda_k, r \rangle}, \ i = 1, ..., I. \qquad (22)$$

The fact that the mapping under consideration is well-defined is established in Proposition 1 below. Fix some element s of the state space S and a vector $r = (r^1, ..., r^I) \in \Delta^I$. Consider the affine operator $B : \mathbb{R}_+^I \to \mathbb{R}_+^I$ transforming a vector $x = (x^1, ..., x^I) \in \mathbb{R}_+^I$ into the vector $y = (y^1, ..., y^I) \in \mathbb{R}_+^I$ defined by

$$y^i = \sum_{k=1}^{K} [\rho \langle \lambda_k, x \rangle + (1 - \rho) R_k] \frac{\lambda_k^i r^i}{\langle \lambda_k, r \rangle},$$

where $R_k = R_k(s)$.

Proposition 1 *The operator B possesses a unique fixed point in $\mathbb{R}^I_+$. This point belongs to the unit simplex Δ^I.*

Proposition 2 *We have*

$$\sum_{i=1}^{I}|F^i(s,r)\rangle - F^i(s,\bar{r})| \leq \frac{1}{(1-\rho)}\sum_{i=1}^{I}\sum_{k=1}^{K}|\frac{\lambda_k^i r^i}{\langle\lambda_k, r\rangle} - \frac{\lambda_k^i \bar{r}^i}{\langle\lambda_k, \bar{r}\rangle}|,\ r,\bar{r}\in\Delta^I. \quad (23)$$

It follows from (23) and the inequalities

$$\langle\lambda_k, r\rangle = \sum_{i=1}^{I}\lambda_k^i r^i > 0,\ \langle\lambda_k, \bar{r}\rangle = \sum_{i=1}^{I}\lambda_k^i \bar{r}^i > 0,$$

(holding because $\lambda_k^i > 0$) that the mapping $F(s,r)$ is continuous in $r \in \Delta^I$.

For each $s \in S$ and $r = (r^1, ..., r^I) \in \Delta^I$, define

$$g^i(s,r) = \sum_{k=1}^{K}[\rho\langle\lambda_k, F(s,r)\rangle + (1-\rho)R_k(s)]\frac{\lambda_k^i}{\langle\lambda_k, r\rangle},\ i = 1, ..., I. \quad (24)$$

It follows from (22) that if $r^i > 0$, then

$$g^i(s,r) = \frac{F^i(s,r)}{r^i}$$

so that $g^i(s,r)$ is the *growth rate* of the ith coordinate under the mapping F. Define

$$\mu_* = \min_{i,k}\lambda_k^i,\ \mu^* = \max_{i,k}\lambda_k^i,\ H = \mu^*/\mu_*.$$

The proposition below shows that the growth rate is uniformly bounded away from zero and infinity.

Proposition 3 *For each $r \in \Delta^I$ and each $i = 1, ..., I$, we have*

$$H^{-1} \leq g^i(s,r) \leq H,\ s \in S. \quad (25)$$

The function $g^i(s,r)$ is continuous in $r \in \Delta^I$.

4.2. Return on the Kelly portfolio. Define

$$f(s,r) = \sum_{k=1}^{K}[\rho\langle\lambda_k, F(s,r)\rangle + (1-\rho)R_k(s)]\frac{\lambda_k^*}{\langle\lambda_k, r\rangle}, \quad (26)$$

where $\lambda_k^* = ER_k(s),\ k = 1, ..., K$. Suppose, at date t, the relative wealth of the investors $i = 1, ..., I$ are given by the vector $r = (r^1, ..., r^I) \in \Delta^I$. Then the (relative) asset prices at dates t and $t+1$ are

$$p_k = \langle \lambda_k, r \rangle,\ q_k(s) = \langle \lambda_k, F(s, r) \rangle, \tag{27}$$

provided the state of the world realized at date $t+1$ is s. An investor's portfolio in which unit wealth is distributed between the assets according to the proportions $\lambda_k^*,\ k = 1, ..., K$, is called the *Kelly portfolio.* The (gross) return on this portfolio, taking into account the dividends and consumption, is given by the function $f(s, r)$ defined by (26), which can be written as

$$f(s, r) = \sum_{k=1}^{K} [\rho q_k(s) + (1-\rho) R_k(s)] \frac{\lambda_k^*}{p_k}.$$

If one of the investors $i = 1, ..., I$, say investor 1, employs the investment strategy $\lambda^* = (\lambda_1^*, ..., \lambda_K^*)$ (i.e., $\lambda_k^1 = \lambda_k^*,\ k = 1, ..., K$), then the growth rate of his/her market share is equal to $f(s, r)$:

$$g^1(s, r) = f(s, r)$$

(see (24) and (26)).

An important result on which the analysis of the model at hand is based is contained in the following theorem.

Theorem 3 *For each $r \in \Delta^I$, we have*

$$E \ln f(s, r) \geq 0. \tag{28}$$

This inequality is strict if and only if

$$\langle \lambda_k, r \rangle \neq \lambda_k^* \textit{ for at least one } k = 1, ..., K. \tag{29}$$

This result means that the expected logarithmic return on the Kelly portfolio $(\lambda_1^*, ..., \lambda_K^*)$ is non-negative. It is strictly positive if and only if $(\lambda_1^*, ..., \lambda_K^*)$ does not coincide with the market portfolio $(p_1, ..., p_K)$. Recall that the total amount of each asset is normalized to 1, so that the total wealth invested into asset k is $p_k = \langle \lambda_k, r \rangle$. We emphasize that, in Theorem 3, it is not assumed that any of the investors $i = 1, ..., I$ uses the Kelly strategy. The result is applicable without this assumption.

4.3. The Kelly portfolio and the market portfolio. According to Theorem 3, the expected logarithmic return on the Kelly portfolio $(\lambda_1^*, ..., \lambda_K^*)$ is non-negative. It is strictly positive if and only if the market portfolio

$(p_1, ..., p_K)$ does not coincide with $(\lambda_1^*, ..., \lambda_K^*)$. Of course it can happen at some moment of time that $(\lambda_1^*, ..., \lambda_K^*) = (p_1, ..., p_K)$. But can it happen that the market portfolio coincides with $(\lambda_1^*, ..., \lambda_K^*)$ at *two consecutive moments of time*? In other words, can the system of equalities

$$q_k(s) = p_k = \lambda_k^* \ (k = 1, ..., K, \ s \in S) \tag{30}$$

hold? Recall that we denote by p_k the price of the asset k corresponding to the vector $r = (r^1, ..., r^I)$ of relative wealth at some fixed moment of time,

$$p_k = \langle \lambda_k, r \rangle = \sum_{i=1}^{I} \lambda_k^i r^i$$

and by $q_k(s)$ the price of the asset at the next moment of time, when the state of the world realized is s:

$$q_k(s) = \langle \lambda_k, F(s, r) \rangle = \sum_{i=1}^{I} \lambda_k^i F^i(s, r).$$

The question we formulated is important for the analysis of the asymptotic behavior of the relative wealth of an investor using the Kelly rule. As Proposition 4 below shows, the answer to this question (under the assumptions we impose) is negative.

Recall that we assume that there are no redundant assets, i.e., the functions $R_1(s), ..., R_K(s)$ are linearly independent. This assumption will be used in the following proposition.

Proposition 4 *Suppose one of the following assumptions is fulfilled.*

(a) All the investors $i = 1, 2, ..., I$ *use portfolio rules* $\lambda^i = (\lambda_1^i, ..., \lambda_K^i)$ *distinct from the Kelly rule* $\lambda^* = (\lambda_1^*, ..., \lambda_K^*)$.

(b) All the investors $i = 2, 3, ..., I$ *use portfolio rules* $\lambda^i = (\lambda_1^i, ..., \lambda_K^i)$ *distinct from the Kelly rule* $\lambda^* = (\lambda_1^*, ..., \lambda_K^*)$*, and the wealth share* r^1 *of investor* 1 *is less than one.*

Then equations (30) cannot hold.

4.4. Limiting behavior of the Kelly investor's relative wealth. Let r_0 be a strictly positive vector in Δ^I. Define recursively the sequence of random vectors $r_0, \ r_1(s^1), \ r_2(s^2), ...$ by the formula $r_t = F(s_t, r_{t-1})$. Then $r_t = (r_t^1, ..., r_t^I)$ is the vector of relative wealths of the investors $i = 1, ..., I$ at date t, depending on the realization $s^t = (s_1, ..., s_t)$ of states of the world. It follows from Proposition 3 that $r_t > 0$ as long as $r_{t-1} > 0$ and so all the vectors $r_t(s^t)$ are strictly positive for all t and s^t. Consequently, the random variables

$$\ln r_t^i = \ln r_t^i(s^t),\ i = 1, ..., I,\ t = 0, 1, ...$$

are well-defined and finite. Clearly, they have finite expectations because each of them takes on a finite number of values (since the set S is finite).

Suppose investor 1 uses the Kelly rule

$$\lambda^* = (\lambda_1^*, ..., \lambda_K^*) = (ER_1(s), ..., ER_K(s)).$$

Consider the growth rate r_{t+1}^1/r_t^1 of investor 1's relative wealth. It can be expressed as follows:

$$\frac{r_{t+1}^1}{r_t^1} = g^1(s_{t+1}, r_t) = \frac{F^1(s_{t+1}, r_t)}{r_t^1}\ [r_t = r_t(s^t)]$$

(see (24)), and since the strategy λ^1 of investor 1 coincides with the Kelly rule λ^*, we have

$$\frac{r_{t+1}^1}{r_t^1} = g^1(s_{t+1}, r_t) = f(s_{t+1}, r_t), \tag{31}$$

where $f(s, r)$ is the function defined by (26).

Denote by $\xi_t = \xi_t(s^t)$ the logarithm of the relative wealth of investor 1, $\xi_t = \ln r_t^1$. We claim that the sequence ξ_t is a submartingale:

$$E(\xi_{t+1}|s^t) \geq \xi_t. \tag{32}$$

Indeed, we have

$$E(\xi_{t+1}|s^t) - \xi_t = E[(\xi_{t+1} - \xi_t)|s^t] = E[(\ln r_{t+1}^1 - \ln r_t^1)|s^t] = E[(\ln \frac{r_{t+1}^1}{r_t^1})|s^t]$$

$$= E[\ln f(s_{t+1}, r_t)|s^t] = E[\ln f(s, r_t)]|_{r_t = r_t(s^t)} = \sum_{s \in S} \pi(s) \ln f(s, r_t(s^t)),$$

where $\pi(s) > 0$ is the probability that $s_{t+1} = s$. The last two equalities in the above chain of relations follow from the fact that the random variables $s_1, s_2, ...$ are independent and identically distributed. By virtue of Theorem 3, $\sum_{s \in S} \pi(s) \ln f(s, r_t(s^t)) \geq 0$, which proves (32). Since $0 < r_t^1 \leq 1$, we have $\xi_t \leq 0$, and so $\xi_t, t = 0, 1, ...,$ is a non-positive submartingale. As is well-known, a non-positive submartingale converges almost surely (a.s.) $\xi_t \to \xi_\infty$ (a.s.) as $t \to \infty$ (see, e.g., [31], Section IV.5). This implies $r_t^1 = e^{\xi_t} \to e^{\xi_\infty} > 0$ (a.s.). This leads to the following result.

Theorem 4 *The relative wealth of a Kelly investor converges a.s., and the limit is strictly positive.*

It follows from Theorem 4 that an investor using the Kelly strategy survives with probability one. A key result of this study is Theorem 5 below, asserting that if one of the investors uses the Kelly rule and all the others use other strategies, distinct from the Kelly one, then the Kelly investor is the only survivor in the market selection process. As has been noticed in **3.3,** this result immediately implies Theorem 1.

Theorem 5 *Let the strategy of investor* 1 *coincide with the Kelly rule:* $\lambda_k^1 = \lambda_k^*$, $k = 1, ..., K$. *Let the strategies of investors* $i = 2, ..., I$ *be distinct from the Kelly rule:*

$$(\lambda_1^i, ..., \lambda_K^i) \neq (\lambda_1^*, ..., \lambda_K^*).$$

Then the relative wealth r_t^1 *of investor* 1 *converges to one almost surely.*

We note that if $\rho = 0$, Theorem 5 follows from the main result of the paper [14]. Methods developed in this work are different in some respects from those in [14]. Although they are applicable to a substantially more complex model, they do not give exponential estimates for the convergence of the relative wealth process.

Appendix

The Appendix contains proofs of the results presented in the previous section. *Proof of Proposition 1.* Consider any $x, \bar{x} \in \mathbb{R}_+^I$ and put $y = B(x)$, $\bar{y} = B(\bar{x})$. We have

$$|y - \bar{y}| = \sum_{i=1}^I |y^i - \bar{y}^i| = \rho \sum_{i=1}^I |\sum_{k=1}^K \langle \lambda_k, x - \bar{x} \rangle \frac{\lambda_k^i r^i}{\langle \lambda_k, r \rangle}|$$
$$\leq \rho \sum_{k=1}^K \sum_{i=1}^I |\langle \lambda_k, x - \bar{x} \rangle| \frac{\lambda_k^i r^i}{\langle \lambda_k, r \rangle} = \rho \sum_{k=1}^K |\langle \lambda_k, x - \bar{x} \rangle|$$
$$\leq \rho \sum_{k=1}^K \sum_{j=1}^I \lambda_k^j |x^j - \bar{x}^j| = \rho \sum_{j=1}^I |x^j - \bar{x}^j| = \rho |x - \bar{x}|.$$

Thus the operator $B : \mathbb{R}_+^I \to \mathbb{R}_+^I$ is contracting and hence it contains a unique fixed point $x \in \mathbb{R}_+^I$. To show that $x \in \Delta^I$ we sum up the equations

$$x^i = \sum_{k=1}^K [\rho \langle \lambda_k, x \rangle + (1 - \rho) R_k] \frac{\lambda_k^i r^i}{\langle \lambda_k, r \rangle}$$

over $i = 1, ..., I$ and obtain

$$|x| = \sum_{k=1}^K [\rho \langle \lambda_k, x \rangle + (1 - \rho) R_k] \frac{\sum_{i=1}^I \lambda_k^i r^i}{\langle \lambda_k, r \rangle} = \sum_{k=1}^K [\rho \langle \lambda_k, x \rangle + (1 - \rho) R_k]$$

$$= \rho|x| + (1-\rho),$$

which yields $|x| = 1$. $\square$

Proof of Proposition 2. For any $r, \bar{r} \in \Delta^I$ and $i = 1, ..., I$, we have

$$|F^i(s,r) - F^i(s,\bar{r})| =$$

$$|\sum_{k=1}^{K}[\rho\langle\lambda_k, F(s,r)\rangle + (1-\rho)R_k(s)]\frac{\lambda_k^i r^i}{\langle\lambda_k, r\rangle} - [\rho\langle\lambda_k, F(s,\bar{r})\rangle + (1-\rho)R_k(s)]\frac{\lambda_k^i \bar{r}^i}{\langle\lambda_k, \bar{r}\rangle}|$$

$$\leq \rho\sum_{k=1}^{K}|\langle\lambda_k, F(s,r)\rangle\frac{\lambda_k^i r^i}{\langle\lambda_k, r\rangle} - \langle\lambda_k, F(s,\bar{r})\rangle\frac{\lambda_k^i \bar{r}^i}{\langle\lambda_k, \bar{r}\rangle}| + (1-\rho)\sum_{k=1}^{K}|\frac{\lambda_k^i r^i}{\langle\lambda_k, r\rangle} - \frac{\lambda_k^i \bar{r}^i}{\langle\lambda_k, \bar{r}\rangle}|$$

$$\leq \rho\sum_{k=1}^{K}\langle\lambda_k, F(s,r)\rangle|\frac{\lambda_k^i r^i}{\langle\lambda_k, r\rangle} - \frac{\lambda_k^i \bar{r}^i}{\langle\lambda_k, \bar{r}\rangle}|$$

$$+\rho\sum_{k=1}^{K}|\langle\lambda_k, F(s,r)\rangle - \langle\lambda_k, F(s,\bar{r})\rangle|\frac{\lambda_k^i \bar{r}^i}{\langle\lambda_k, \bar{r}\rangle} + (1-\rho)\sum_{k=1}^{K}|\frac{\lambda_k^i r^i}{\langle\lambda_k, r\rangle} - \frac{\lambda_k^i \bar{r}^i}{\langle\lambda_k, \bar{r}\rangle}|$$

$$\leq \rho\sum_{k=1}^{K}|\langle\lambda_k, F(s,r)\rangle - \langle\lambda_k, F(s,\bar{r})\rangle|\frac{\lambda_k^i \bar{r}^i}{\langle\lambda_k, \bar{r}\rangle} + \sum_{k=1}^{K}|\frac{\lambda_k^i r^i}{\langle\lambda_k, r\rangle} - \frac{\lambda_k^i \bar{r}^i}{\langle\lambda_k, \bar{r}\rangle}|.$$

By summing up these inequalities over $i = 1, ..., I$, we obtain

$$\sum_{i=1}^{I}|F^i(s,r) - F^i(s,\bar{r})|$$

$$\leq \rho\sum_{k=1}^{K}|\langle\lambda_k, F(s,r)\rangle - \langle\lambda_k, F(s,\bar{r})\rangle| + \sum_{i=1}^{I}\sum_{k=1}^{K}|\frac{\lambda_k^i r^i}{\langle\lambda_k, r\rangle} - \frac{\lambda_k^i \bar{r}^i}{\langle\lambda_k, \bar{r}\rangle}|$$

$$\leq \rho\sum_{i=1}^{I}|F^i(s,r) - F^i(s,\bar{r})| + \sum_{i=1}^{I}\sum_{k=1}^{K}|\frac{\lambda_k^i r^i}{\langle\lambda_k, r\rangle} - \frac{\lambda_k^i \bar{r}^i}{\langle\lambda_k, \bar{r}\rangle}|,$$

which yields (23). $\square$

Proof of Proposition 3. Since $\mu_* \leq \langle\lambda_k, r\rangle \leq \mu^*$, we obtain

$$H^{-1} = \frac{\mu_*}{\mu^*} \leq \frac{\lambda_k^i}{\langle\lambda_k, r\rangle} \leq \frac{\mu^*}{\mu_*} = H,$$

which yields (25) because

$$\sum_{k=1}^{K}[\rho\langle\lambda_k, F(s,r)\rangle + (1-\rho)R_k(s)] = \rho\sum_{k=1}^{K}\langle\lambda_k, F(s,r)\rangle + (1-\rho)\sum_{k=1}^{K}R_k(s) = 1.$$

The function $g^i(s,r)$ is continuous in $r \in \Delta^I$, because $F(s,r)$ is continuous in r and $\langle \lambda_k, r\rangle \geq \mu_* > 0$ (see (24)). $\square$

Proof of Theorem 3. 1st step. Multiplying both sides of (22) by λ^i_m and summing up over $i = 1, ..., I$, we get

$$\langle \lambda_m, F(s,r)\rangle = \sum_{k=1}^{K} [\rho \langle \lambda_k, F(s,r)\rangle + (1-\rho) R_k(s)] \frac{\sum_{i=1}^{I} \lambda^i_k \lambda^i_m r^i}{\langle \lambda_k, r\rangle} \tag{33}$$

($m = 1, ..., K$). By using the notation introduced in (27), equations (33) and inequality (28) can be written as

$$q_m(s) = \sum_{k=1}^{K} [\rho q_k(s) + (1-\rho) R_k(s)] \frac{\sum_{i=1}^{I} \lambda^i_m \lambda^i_k r^i}{p_k},\ m = 1, ..., K, \tag{34}$$

and

$$E \ln \sum_{k=1}^{K} [\rho q_k(s) + (1-\rho) R_k(s)] \frac{\lambda^*_k}{p_k} \geq 0. \tag{35}$$

Condition (29) is necessary for this inequality to be strict (the "only if" part in (29)) because $p_k = \lambda^*_k$ for all $k = 1, ..., K$ implies that the left-hand side of (35) is zero.

2nd step. We fix the argument s and omit it in the notation. Consider the $K \times K$ matrix

$$A = (a_{mk}),\ a_{mk} = \delta_{mk} - \rho \frac{\sum_{i=1}^{I} \lambda^i_m \lambda^i_k r^i}{p_k},$$

where $\delta_{mk} = 1$ if $m = k$ and $\delta_{mk} = 0$ if $m \neq k$. Put

$$b = (b_1, ..., b_K),\ b_m = (1-\rho) \sum_{k=1}^{K} R_k \frac{\sum_{i=1}^{I} \lambda^i_m \lambda^i_k r^i}{p_k}. \tag{36}$$

Then (for the fixed s) the system of equations (34) can be written

$$Aq = b \tag{37}$$

[$q = q(s)$]. Indeed, the mth coordinate $(Aq - b)_m$ of the vector $Aq - b$ can be expressed as follows

$$(Aq - b)_m = \sum_{k=1}^{K} a_{mk} q_k - b_m$$

$$= q_m - \rho \sum_{k=1}^{K} q_k \frac{\sum_{i=1}^{I} \lambda^i_m \lambda^i_k r^i}{p_k} - (1-\rho) \sum_{k=1}^{K} R_k \frac{\sum_{i=1}^{I} \lambda^i_m \lambda^i_k r^i}{p_k}$$

$$= q_m - \sum_{k=1}^{K} [\rho q_k + (1-\rho) R_k] \frac{\sum_{i=1}^{I} \lambda^i_m \lambda^i_k r^i}{p_k},$$

which is equal to the difference between the left-hand side and the right-hand side of (34).

We can represent the matrix A as $A = \mathrm{Id} - \rho C$, where Id is the identity matrix and

$$C = (c_{mk}),\ c_{mk} = \frac{\sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i}{p_k}.$$

The norm of the linear operator C is not greater than one, because

$$|Cx| = \sum_{m=1}^{K} |\sum_{k=1}^{K} c_{mk} x_k| \le \sum_{m=1}^{K} \sum_{k=1}^{K} c_{mk} |x_k| = |x|, \tag{38}$$

where

$$\sum_{m=1}^{K} c_{mk} = \sum_{m=1}^{K} \frac{\sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i}{p_k} = \frac{\sum_{i=1}^{I} \lambda_k^i r^i}{p_k} = 1. \tag{39}$$

Consequently, the operator ρC is contracting, and so each of the equivalent equations $Aq = 0$ and $q = \rho C q$ has a unique solution. Thus the matrix A is non-degenerate, and the solution to the linear system (37) can be represented as $q = A^{-1} b$.

3rd step. Define

$$c_k = \rho \frac{\lambda_k^*}{p_k},\ d = (1-\rho) \sum_{k=1}^{K} R_k \frac{\lambda_k^*}{p_k}. \tag{40}$$

Then we have

$$\langle c, q \rangle + d = \sum_{k=1}^{K} [\rho q_k + (1-\rho) R_k] \frac{\lambda_k^*}{p_k}. \tag{41}$$

This expression appears in (35), and our goal is to estimate the expected logarithm of it (when R_k and q depend on s). To this end we write

$$\langle c, q \rangle = \langle c, A^{-1} b \rangle = \langle (A^{-1})' c, b \rangle = \langle (A')^{-1} c, b \rangle, \tag{42}$$

where A' denotes the conjugate matrix. In (42), we use the identity $(A^{-1})' = (A')^{-1}$, holding for each invertible linear operator A.

By virtue of (42),

$$\langle c, q \rangle = \langle b, l \rangle, \tag{43}$$

where $l = (A')^{-1} c$, i.e., the vector l is the solution to the linear system $A' l = c$. The matrix A' is given by

$$A' = (a'_{km}),\ a'_{km} = a_{mk} = \delta_{mk} - \rho \frac{\sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i}{p_k},$$

and the linear system $A' l = c$ can be written

$$\sum_{m=1}^{K} (\delta_{mk} - \rho \frac{\sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i}{p_k}) l_m = \rho \frac{\lambda_k^*}{p_k},\ k = 1, ..., K,$$

(see (40)) or equivalently,

$$l_k = \rho(\sum_{m=1}^{K} \frac{\sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i}{p_k} l_m + \frac{\lambda_k^*}{p_k}),\ k = 1, ..., K. \tag{44}$$

Further, in view of (40) and (36), we obtain

$$d + \langle l, b\rangle = (1-\rho)\sum_{k=1}^{K} R_k \frac{\lambda_k^*}{p_k} + (1-\rho)\sum_{m=1}^{K} l_m \sum_{k=1}^{K} R_k \frac{\sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i}{p_k}$$

$$= (1-\rho)\sum_{k=1}^{K} R_k \frac{\lambda_k^*}{p_k} + (1-\rho)\sum_{k=1}^{K} R_k \sum_{m=1}^{K} l_m \frac{\sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i}{p_k}$$

$$= (1-\rho)\sum_{k=1}^{K} R_k [\sum_{m=1}^{K} l_m \frac{\sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i}{p_k} + \frac{\lambda_k^*}{p_k}] = \frac{(1-\rho)}{\rho}\sum_{k=1}^{K} R_k l_k, \tag{45}$$

where the last equality follows from (44). Consequently,

$$\langle c, q\rangle + d = \langle l, b\rangle + d = \frac{(1-\rho)}{\rho}\sum_{k=1}^{K} R_k l_k \tag{46}$$

(see (41) and (43)).

4th step. According to Step 1 of the proof, we have to establish inequality (35) for every solution $q(s),\ s \in S$, of system (34) and show that this inequality is strict if

$$(p_1, ..., p_K) \neq (\lambda_1^*, ..., \lambda_K^*). \tag{47}$$

The considerations presented in Steps 2 and 3, allow to reduce this problem to the following one: for the solution $l = (l_1, ..., l_K)$ to system (44), show that

$$E \ln[\frac{(1-\rho)}{\rho}\sum_{k=1}^{K} R_k(s) l_k] \geq 0$$

(see (46) and (43)). Additionally, it has to be shown that the last inequality is strict if assumption (47) holds. The advantage of the new problem comparative to the original one lies in the fact that system (44), in contrast with (34), does not depend on s.

We write (44) equivalently as

$$\frac{(1-\rho)}{\rho} p_k l_k = \rho[\sum_{m=1}^{K} p_m \frac{(1-\rho)}{\rho} l_m \sum_{i=1}^{I} \frac{\lambda_m^i \lambda_k^i r^i}{p_m} + \frac{(1-\rho)}{\rho}\lambda_k^*],$$

and, by changing variables

$$f_k = \frac{(1-\rho)}{\rho} l_k p_k,$$

we transform (44) to

$$f_k = \rho \sum_{m=1}^{K} f_m \frac{\sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i}{p_m} + (1-\rho)\lambda_k^*,\ k = 1, ..., K. \tag{48}$$

Then

$$\frac{(1-\rho)}{\rho} \sum_{k=1}^{K} R_k l_k = \sum_{k=1}^{K} R_k \frac{f_k}{p_k},$$

and the problem reduces to the following one: given the solution $(f_1, ..., f_K)$ to system (48), show that

$$E \ln \sum_{k=1}^{K} R_k(s) \frac{f_k}{p_k} \geq 0 \tag{49}$$

with strict inequality if $(p_1, ..., p_K) \neq (\lambda_1^*, ..., \lambda_K^*)$.

Note that the affine operator defined by the right-hand side of (48) is contracting (see (38) and (39)) and leaves the non-negative cone $\mathbb{R}_+^K$ invariant. Therefore there exists a unique vector $f = (f_1, ..., f_K)$ solving (48). Furthermore, this vector is strictly positive (which follows from the strict positivity of $\lambda^* = (\lambda_1^*, ..., \lambda_K^*)$) and satisfies $\sum_{k=1}^{K} f_k = 1$. The last equality can be obtained by summing up equations (48) over $k = 1, ..., K$.

By virtue of Jensen's inequality, applied to the concave function $\ln(\cdot)$, we have

$$E \ln \sum_{k=1}^{K} R_k(s) \frac{f_k}{p_k} \geq E \sum_{k=1}^{K} R_k(s) \ln \frac{f_k}{p_k} = \sum_{k=1}^{K} \lambda_k^* \ln \frac{f_k}{p_k}.$$

(We use here the fact that $\sum_{k=1}^{K} R_k(s) = 1$ for all s.) Thus it is sufficient to prove that if a vector $(f_1, ..., f_K)$ satisfies (48), then

$$\sum_{k=1}^{K} \lambda_k^* \ln \frac{f_k}{p_k} \geq 0, \tag{50}$$

and inequality (50) is strict when assumption (47) is fulfilled. This problem is purely deterministic: no random parameter s is involved either in (48) or in (50).

5th step. Put $g_k = f_k/p_k,\ k = 1, ..., K$. Then, from (48), we get

$$p_k g_k = \rho \sum_{m=1}^{K} g_m \sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i + (1-\rho)\lambda_k^*,\ k = 1, ..., K. \tag{51}$$

Let us multiply both sides of these equations by $\ln g_k$ and sum up over $k = 1, ..., K$:

$$\sum_{k=1}^{K} p_k g_k \ln g_k = \rho \sum_{k=1}^{K} (\ln g_k) \sum_{m=1}^{K} g_m \sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i + (1-\rho) \sum_{k=1}^{K} \lambda_k^* \ln g_k.$$

This yields

$$\sum_{k=1}^{K}\lambda_k^*\ln g_k = \frac{1}{(1-\rho)}[\sum_{k=1}^{K}p_k g_k\ln g_k - \rho\sum_{k=1}^{K}(\ln g_k)\sum_{m=1}^{K}g_m\sum_{i=1}^{I}\lambda_m^i\lambda_k^i r^i].$$

Further, we have

$$\sum_{k=1}^{K}\lambda_k^*\ln\frac{f_k}{p_k} = \sum_{k=1}^{K}\lambda_k^*\ln g_k$$

(recall that $g_k = f_k/p_k$). Thus, in order to prove the desired inequality (50) it is sufficient to verify the relation

$$\sum_{k=1}^{K}p_k g_k\ln g_k - \rho\sum_{k=1}^{K}(\ln g_k)\sum_{m=1}^{K}g_m\sum_{i=1}^{I}\lambda_m^i\lambda_k^i r^i \geq 0. \tag{52}$$

If inequality (52) is strict, then (50) is strict as well.

We have

$$\sum_{k=1}^{K}p_k g_k\ln g_k = \sum_{k=1}^{K}f_k\ln\frac{f_k}{p_k} \geq 0, \tag{53}$$

by virtue of the well-known inequality (recall that $f, p \in \Delta^K$)

$$\sum_{k=1}^{K}f_k\ln f_k \geq \sum_{k=1}^{K}f_k\ln p_k, \tag{54}$$

which is strict if

$$(f_1, ..., f_K) \neq (p_1, ..., p_K). \tag{55}$$

Therefore relation (52) is valid if

$$\sum_{k=1}^{K}(\ln g_k)\sum_{m=1}^{K}g_m\sum_{i=1}^{I}\lambda_m^i\lambda_k^i r^i \leq 0. \tag{56}$$

In the rest of the proof, we will assume that the opposite inequality holds:

$$\sum_{k=1}^{K}(\ln g_k)\sum_{m=1}^{K}g_m\sum_{i=1}^{I}\lambda_m^i\lambda_k^i r^i > 0. \tag{57}$$

Then (52) will be obtained if we establish that

$$\sum_{k=1}^{K}p_k g_k\ln g_k \geq \sum_{k=1}^{K}(\ln g_k)\sum_{m=1}^{K}g_m\sum_{i=1}^{I}\lambda_m^i\lambda_k^i r^i. \tag{58}$$

Indeed, then we have

$$\sum_{k=1}^{K} p_k g_k \ln g_k \geq \sum_{k=1}^{K} (\ln g_k) \sum_{m=1}^{K} g_m \sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i \geq \rho \sum_{k=1}^{K} (\ln g_k) \sum_{m=1}^{K} g_m \sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i,$$

which yields (52). In the above chain of relations, the last equality holds by virtue of (57).

To verify inequality (58) we write

$$\sum_{k=1}^{K} p_k g_k \ln g_k = \sum_{k=1}^{K} \sum_{i=1}^{I} \lambda_k^i r^i g_k \ln g_k = \sum_{i=1}^{I} r^i \sum_{k=1}^{K} \lambda_k^i g_k \ln g_k,$$

and

$$\sum_{k=1}^{K} (\ln g_k) \sum_{m=1}^{K} g_m \sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i = \sum_{i=1}^{I} r^i \sum_{k=1}^{K} \sum_{m=1}^{K} (\lambda_k^i \ln g_k)(g_m \lambda_m^i).$$

Thus to prove (58) it remains to check that

$$\sum_{k=1}^{K} \lambda_k^i g_k \ln g_k \geq (\sum_{k=1}^{K} \lambda_k^i \ln g_k)(\sum_{k=1}^{K} g_k \lambda_k^i) \tag{59}$$

for each $i = 1, ..., I$.

Let us fix i and put $\lambda_k = \lambda_k^i$. Inequality (59) can be written

$$\mathbf{E}[g \ln g] \geq (\mathbf{E} \ln g)\mathbf{E}g,$$

where "**E**" stands for the weighted average

$$\mathbf{E}g = \sum_{k=1}^{K} g_k \lambda_k \quad [\lambda_k > 0, \ \sum_{k=1}^{K} \lambda_k = 1].$$

Observe that the function $\phi(g) = g \ln g$ is strictly convex. Consequently,

$$\mathbf{E}\phi(g) \geq \phi(\mathbf{E}g), \tag{60}$$

and the inequality is strict if $g_k \neq g_m$ for some k and m. Thus

$$\mathbf{E}[g \ln g] \geq (\mathbf{E}g) \ln \mathbf{E}g \geq (\mathbf{E}g)(\mathbf{E} \ln g), \tag{61}$$

where the former inequality in this chain of relations coincides with (60) and the latter is a consequence of the concavity of the function $\ln(\cdot)$. Furthermore, both inequalities in (61) are strict provided that $g_k \neq g_m$ for some k. If the last condition does not hold, then $f_k/p_k = c$ for some constant c, which must necessarily be equal to one because $\sum f_k = \sum p_k = 1$. Thus if $g_k = g_m$ for all k, m, then $f_k = p_k$, $k = 1, 2, ..., K$, which implies (see below) that $p_k = \lambda_k^*$ for all k.

6th step. At the previous step of the proof, we established inequality (52) and hence (50). Moreover, the arguments conducted show that inequality (52) (and hence (50)) is strict if condition (55) is fulfilled. Indeed, if relation (56) holds then, under assumption (55), we have a strict inequality in (53), which implies a strict inequality in (52). Alternatively, if relation (57), opposite to (56), holds, then strict inequalities in (61) and (59) imply strict inequalities in (58) and (52).

Thus to complete the proof it suffices to show that if $f_k = p_k$, $k = 1, 2, ..., K$, then $p_k = \lambda_k^*$, $k = 1, 2, ..., K$. Indeed, if $f_k = p_k$, then we have

$$p_k = \rho \sum_{m=1}^{K} p_m \frac{\sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i}{p_m} + (1-\rho)\lambda_k^*, \ k = 1, ..., K,$$

which implies

$$p_k = \rho \sum_{m=1}^{K} \sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i + (1-\rho)\lambda_k^* =$$

$$\rho \sum_{i=1}^{I} \lambda_k^i r^i + (1-\rho)\lambda_k^* = \rho p_k + (1-\rho)\lambda_k^*, \ k = 1, ..., K.$$

Thus $(1-\rho)p_k = (1-\rho)\lambda_k^*$, and so $p_k = \lambda_k^*$. $\square$

Proof of Proposition 4. The variables $q_k(s)$, p_k and r_k $(k = 1, ..., K)$ are related to each other by the system of equations (34). Suppose equations (30) hold. Then, from (34), we obtain:

$$\lambda_m^* = \sum_{k=1}^{K} [\rho\lambda_k^* + (1-\rho)R_k(s)] \frac{\sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i}{\lambda_k^*}, \ m = 1, ..., K,$$

or equivalently,

$$\lambda_m^* = \sum_{k=1}^{K} \bar{R}_k(s) \sum_{i=1}^{I} \frac{\lambda_m^i \lambda_k^i r^i}{\lambda_k^*}, \ m = 1, ..., K, \tag{62}$$

where

$$\lambda_k^* = E\bar{R}_k(s), \ \bar{R}_k(s) = \rho\lambda_k^* + (1-\rho)R_k(s).$$

Observe that if there are no redundant assets, the relation $\sum_{k=1}^{K} \gamma_k \bar{R}_k(s) = 0$ implies $\gamma_1 = ... = \gamma_K = 0$. Indeed, suppose that

$$\sum_{k=1}^{K} \gamma_k [\rho\lambda_k^* + (1-\rho)R_k(s)] = 0. \tag{63}$$

Then we have

$$0 = E\sum_{k=1}^{K} \gamma_k [\rho\lambda_k^* + (1-\rho)R_k(s)] = \sum_{k=1}^{K} \gamma_k [\rho\lambda_k^* + (1-\rho)\lambda_k^*] = \sum_{k=1}^{K} \gamma_k \lambda_k^*,$$

which in view of (63) yields

$$\sum_{k=1}^{K} \gamma_k R_k(s) = -\frac{\rho}{1-\rho} \sum_{k=1}^{K} \gamma_k \lambda_k^* = 0,$$

and so $\gamma_1 = ... = \gamma_K = 0$ because the functions $R_k(\cdot),\ k = 1, ..., K$, are linearly independent.

From formula (62) and the relation $\lambda_m^* = p_m = \sum_{i=1}^{I} \lambda_m^i r^i$, we obtain

$$\sum_{i=1}^{I} \lambda_m^i r^i = \sum_{k=1}^{K} \bar{R}_k(s) \sum_{i=1}^{I} \frac{\lambda_m^i \lambda_k^i r^i}{\lambda_k^*},\ m = 1, ..., K. \tag{64}$$

We have $\sum_{k=1}^{K} \bar{R}_k(s) = 1$, and so equations (64) imply

$$\sum_{k=1}^{K} \bar{R}_k(s) \gamma_k^m = 0,\ m = 1, ..., K,$$

where

$$\gamma_k^m = \frac{1}{\lambda_k^*} \sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i - \sum_{i=1}^{I} \lambda_m^i r^i.$$

Since there are no redundant assets, we have $\gamma_k^m = 0$ for each m and k. This gives

$$\sum_{i=1}^{I} \lambda_m^i \lambda_k^i r^i - \lambda_k^* \sum_{i=1}^{I} \lambda_m^i r^i = 0,\ k, m = 1, ..., K,$$

which can be written as

$$\sum_{i=1}^{I} \lambda_m^i (\lambda_k^i - \lambda_k^*) r^i = 0,\ k, m = 1, ..., K.$$

We derive two expressions from this equation. The first by setting $k = m$ in the foregoing formula. The second by adding up over $m = 1, ..., K$. We find

$$\sum_{i=1}^{I} \lambda_k^i (\lambda_k^i - \lambda_k^*) r^i = 0,\ k = 1, ..., K,\ \text{and}\ \sum_{i=1}^{I} (\lambda_k^i - \lambda_k^*) r^i = 0.$$

Multiplying the second equation by $-\lambda_k^*$ and adding it up with the first, we obtain

$$0 = \sum_{i=1}^{I} [\lambda_k^i (\lambda_k^i - \lambda_k^*) r^i - \lambda_k^* (\lambda_k^i - \lambda_k^*) r^i] = \sum_{i=1}^{I} (\lambda_k^i - \lambda_k^*)^2 r^i.$$

Consequently, we have

$$(\lambda_k^i - \lambda_k^*)^2 r^i = 0,\ i = 1, ..., I,\ k = 1, ..., K. \tag{65}$$

Suppose condition (a) holds. Since $\sum_{i=1}^{I} r^i = 1$ and $r^i \geq 0$, we have $r^j > 0$ for some $j = 1, ..., I$. Then, from (65), we get

$$\lambda_k^j - \lambda_k^* = 0, \ k = 1, ..., K, \tag{66}$$

which is a contradiction.

If condition (b) is fulfilled, then $\sum_{i=2}^{I} r^i = 1 - r^1 > 0$, and so $r^j > 0$ for some $j = 2, ..., I$. This implies (66), and the contradiction obtained completes the proof. □

Proof of Theorem 5. By virtue of Theorem 4, the limit $r_\infty^1 := \lim r_t^1$ exists a.s. and is strictly positive. Suppose the assertion we wish to prove is not valid. Then we have

$$P\{0 < \lim r_t^1 < 1\} > 0. \tag{67}$$

Let us write for shortness $E_t(\cdot)$ in place of $E(\cdot|s^t)$. If ξ_t is a non-positive submartingale, then $E_{t-1}\xi_{t+1} - \xi_{t-1} \to 0$ a.s. (see the Lemma below). By applying this fact to the non-positive submartingale $\xi_t = \ln r_t^1$, we obtain

$$E_{t-1}(\ln \frac{r_t^1}{r_{t-1}^1} + \ln \frac{r_{t+1}^1}{r_t^1}) = E_{t-1} \ln \frac{r_{t+1}^1}{r_{t-1}^1} = E_{t-1}\xi_{t+1} - \xi_{t-1} \to 0 \text{ (a.s.)}. \tag{68}$$

By using the fact that the random elements s^{t-1}, s_t and s_{t+1} are independent and representing the histories s^t, s^{t+1} as

$$s^t = (s^{t-1}, s_t), \ s^{t+1} = (s^{t-1}, s_t, s_{t+1}),$$

we get

$$E_{t-1}(\ln \frac{r_t^1}{r_{t-1}^1} + \ln \frac{r_{t+1}^1}{r_t^1}) = E[(\ln \frac{r_t^1}{r_{t-1}^1})|s^{t-1}] + E[\ln(\frac{r_{t+1}^1}{r_t^1})|s^{t-1}] =$$

$$\sum_{s \in S} P\{s_t = s\} \ln \frac{r_t^1(s^{t-1}, s)}{r_{t-1}^1(s^{t-1})} + \sum_{s \in S} P\{s_t = s\} \sum_{\sigma \in S} P\{s_{t+1} = \sigma\} \ln \frac{r_{t+1}^1(s^{t-1}, s, \sigma)}{r_t^1(s^{t-1}, s)} =$$

$$\sum_{s \in S} \pi(s) \ln \frac{r_t^1(s^{t-1}, s)}{r_{t-1}^1(s^{t-1})} + \sum_{s \in S} \pi(s) \sum_{\sigma \in S} \pi(\sigma) \ln \frac{r_{t+1}^1(s^{t-1}, s, \sigma)}{r_t^1(s^{t-1}, s)} =$$

$$\sum_{s \in S} \pi(s) \ln f(s, r_{t-1}(s^{t-1})) + \sum_{s \in S} \pi(s) \sum_{\sigma \in S} \pi(\sigma) \ln f(\sigma, r_t(s^{t-1}, s)) =$$

$$\sum_{s \in S} \pi(s) \ln f(s, r_{t-1}(s^{t-1})) + \sum_{s \in S} \pi(s) \sum_{\sigma \in S} \pi(\sigma) \ln f(\sigma, F(s, r_{t-1}(s^{t-1})). \tag{69}$$

The last but one equality in the above chain of relations is valid because the strategy of investor 1 coincides with the Kelly rule ($\lambda^1 = \lambda^*$) and the last equality holds because $r_t(s^{t-1}, s) = F(s, r_{t-1}(s^{t-1}))$.

By virtue of (67), (68) and (69), there exists a realization $(s_1, ..., s_t, ...)$ of the process of states of the world such that, for the sequence of vectors $r_{t-1} = r_{t-1}(s^{t-1}) \in \Delta^I$, we have

$$0 < \lim r_{t-1}^1 < 1, \tag{70}$$

$$\sum_{s\in S} \pi(s) \ln f(s, r_{t-1}) + \sum_{s\in S} \pi(s) \sum_{\sigma\in S} \pi(\sigma) \ln f(\sigma, F(s, r_{t-1})) \to 0. \tag{71}$$

In the rest of the proof, we will fix such a realization $(s_1, ..., s_t, ...)$ and write r_{t-1} in place of $r_{t-1}(s^{t-1})$.

Since the simplex Δ^I is compact, there exists a sequence $t_1 < t_2 < ...$ and a vector $r \in \Delta^I$ such that

$$r_{t_n-1} \to r \in \Delta^I. \tag{72}$$

It follows from (70) and (72) that the first coordinate r^1 if the vector $r = (r^1, ..., r^I)$ satisfies

$$0 < r^1 < 1. \tag{73}$$

Relations (71) and (72) imply

$$\sum_{s\in S} \pi(s) \ln f(s, r) + \sum_{s\in S} \pi(s) \sum_{\sigma\in S} \pi(\sigma) \ln f(\sigma, F(s, r)) = 0 \tag{74}$$

because the function $\ln f(s, r) = \ln g^1(s, r)$ is continuous in $r \in \Delta^I$ (see Proposition 3).

By virtue of Theorem 3,

$$\sum_{s\in S} \pi(s) \ln f(s, r) \geq 0, \ \sum_{\sigma\in S} \pi(\sigma) \ln f(\sigma, F(s, r)) \geq 0 \text{ (for all } s \in S).$$

Consequently, it follows from (74) that

$$\sum_{s\in S} \pi(s) \ln f(s, r) = 0, \tag{75}$$

and

$$\sum_{\sigma\in S} \pi(\sigma) \ln f(\sigma, F(s, r)) = 0 \text{ for all } s \in S. \tag{76}$$

According to Theorem 2, relation (75) can hold only if

$$\langle \lambda_k, r \rangle = \lambda_k^*, \ k = 1, ..., K, \tag{77}$$

and equations (76) imply

$$\langle \lambda_k, F(s, r) \rangle = \lambda_k^*, \ k = 1, ..., K, \ s \in S. \tag{78}$$

By virtue of Proposition 4, relations (73), (77) and (78) cannot hold simultaneously. This is a contradiction. $\square$

In the proof of Theorem 5 we used the following fact.

Lemma. *Let ξ_t be a non-positive submartingale. Then the sequence of non-negative random variables $\zeta_t = E_{t-1}\xi_{t+1} - \xi_{t-1}$ converges to zero a.s.*

Proof. We have $\zeta_t \geq 0$ by the definition of a submartingale. Further, $E\zeta_t = (E\xi_{t+1} - E\xi_t) + (E\xi_t - E\xi_{t-1})$, and so

$$\sum_{t=1}^{N} E\zeta_t = \sum_{t=1}^{N}(E\xi_{t+1} - E\xi_t) + \sum_{t=1}^{N}(E\xi_t - E\xi_{t-1})$$

$$= E\xi_{N+1} - E\xi_1 + E\xi_N - E\xi_0 \leq -E\xi_1 - E\xi_0$$

because $E\xi_t \leq 0$ for each t. Therefore the series of the expectations $\sum_{t=1}^{\infty} E\zeta_t$ of the non-negative random variables ζ_t converges, which implies (see, e.g., Corollary to Theorem 11, in Chapter VI in [30]) that $\zeta_t \to 0$ (a.s.). $\square$

References

[1] C. Alós-Ferrer and A. B. Ania, The asset market game, Journal of Mathematical Economics, 2005, **41**, 67–90.

[2] A. Alchian, Uncertainty, evolution and economic theory, Journal of Political Economy, 1950, **58**, 211-221.

[3] P. H. Algoet and T. M. Cover, Asymptotic optimality and asymptotic equipartition properties of log-optimum investment, Annals of Probability, 1988, **16**, 876-898.

[4] R. Amir, I. V. Evstigneev, T. Hens and K. R. Schenk-Hoppé, Market selection and survival of investment strategies, Journal of Mathematical Economics, 2005, **41**, 105-122.

[5] W. B. Arthur, J. H. Holland, B. LeBaron, R. G. Palmer and P. Taylor, Asset pricing under endogenous expectations in an artificial stock market, in: W. B. Arthur, S. Durlauf and D. Lane (eds.), The Economy as an Evolving Complex System II, Reading, MA, Addison Wesley, 1997, pp. 15-44.

[6] L. Blume and S. Durlauf, eds., The Economy as an Evolving Complex System III, Oxford, Oxford University Press, 2005.

[7] L. Blume and D. Easley, Evolution and market behavior, Journal of Economic Theory, 1992, **58**, 9-40.

[8] L. Blume and D. Easley, If you're so smart, why aren't you rich? Belief selection in complete and incomplete markets, Econometrica, 2006, **74**, 929-966.

[9] L. Breiman, Optimal gambling systems for favorable games, Fourth Berkeley Symposium on Math. Statist. and Probability, 1961, v. 1, pp. 65-78.

[10] S. Browne, The return on investment from proportional portfolio strategies, Advances in Applied Probability, 1998, **30**, 216-238.

[11] T. M. Cover, Shannon and investment, IEEE Information Theory Society Newsletter, Summer 1998, Special Golden Jubilee Issue, 10-11.

[12] M. A. H. Dempster (ed.): Risk Management: Value at Risk and Beyond. Cambridge, UK, Cambridge University Press, 2002.

[13] M. A. H. Dempster, M. Germano, E. A. Medova and M. Villaverde, Global asset liability management, British Actuarial Journal, 2003, **9**, 137-216.

[14] I. V. Evstigneev, T. Hens and K. R. Schenk-Hoppé, Market selection of financial trading strategies: Global stability, Mathematical Finance, 2002, **12**, 329-339.

[15] I. V. Evstigneev, T. Hens and K. R. Schenk-Hoppé, Evolutionary stable stock markets, Economic Theory, 2006, **27**, 449-468.

[16] J. D. Farmer and A. W. Lo, Frontiers of finance: Evolution and efficient markets, Proceedings of the National Academy of Sciences, 1999, **96**, 9991-9992.

[17] J-M. Grandmont (ed.), Temporary Equilibrium, San Diego, Academic Press, 1988.

[18] N. H. Hakansson and W. T. Ziemba, Capital growth theory, in: R. A. Jarrow, V. Maksimovic, and W. T. Ziemba (eds.), Handbooks in Operations Research and Management Science, Volume 9, Finance, Chapter 3, Amsterdam, Elsevier, 1995, pp. 65-86.

[19] J. R. Hicks, Value and Capital, Clarendon Press, Oxford, 2nd edition, 1946.

[20] G. M. Hodgeson, Economics and Evolution: Bringing Life Back into Economics, Cambridge, Polity Press (Blackwell Publishers), 1993.

[21] J. Hofbauer and K. Sigmund, The Theory of Evolution and Dynamical Systems, Cambridge, Cambridge University Press, 1992.

[22] J. L. Kelly, A new interpretation of information rate, Bell System Technical Journal, 1956, **35**, 917-926.

[23] J.-J. Laffont, The Economics of Uncertainty and Information, Cambridge MA, MIT Press, 1989.

[24] B. LeBaron, W. B. Arthur and R. Palmer, Time series properties of an artificial stock market, Journal of Economic Dynamics and Control, 1999, **23**, 1487-1516.

[25] E. Lindahl, Theory of Money and Capital, London, Allen and Unwin, 1939.

[26] A. J. Lotka, Elements of Mathematical Biology, Dover, New York, 1956.

[27] M. Magill and M. Quinzii, Incentives and the stock market in general equilibrium, in: F. Hahn and F. Petri (eds.), General Equilibrium: Problems, Prospects and Alternatives, New-York: Routledge, 2003.

[28] A. Marshall, Principles of Economics, 8th ed., London, Macmillan, 1949.

[29] J. M. Mulvey and W. T. Ziemba (eds.), Worldwide Asset and Liability Modeling, Cambridge, UK, Cambridge University Press, 1998.

[30] I. P. Natanson, Theory of Functions of a Real Variable, New York, Ungar, 1961.

[31] J. Neveu, Mathematical Foundations of the Calculus of Probability Theory, San Francisco, Holden Day, 1965.

[32] E. T. Penrose, Biological analogies in the theory of the firm, The American Economic Review, 1952, **42**, 804–819.

[33] A. F. Perold and W. F. Sharpe, Dynamic strategies for asset allocation, Financial Analysts Journal, 1988, **44**, 16-27.

[34] P. A. Samuelson, Foundations of Economic Analysis, Cambridge MA, Harward University Press, 1947.

[35] A. Sandroni, Do markets favor agents able to make accurate predictions? Econometrica, 2000, **68**, 1303–1341.

[36] E. Schlicht, Isolation and Aggregation in Economics, Berlin, Springer, 1985.

[37] L. Shapley and M. Shubik, Trade using one commodity as a means of payment, Journal of Political Economy, 1977, **85**, 937–968.

[38] D. R. Smith, Singular Perturbation Theory: An Introduction with Applications, Cambridge, Cambridge University Press, 1985.

[39] E. O. Thorp, Portfolio choice and the Kelly criterion, in: W. T. Ziemba and R. G. Vickson (eds.), Stochastic Models in Finance, New York, Academic Press, 1971, pp. 599-619.

[40] J. Weibull, Evolutionary Game Theory, Cambridge, MA, MIT Press, 1995.