Risk exchange with distorted probabilities

Topic 2: Risk finance and risk transfer

Tsanakas Andreas[1]

Centre for Quantitative Finance, Imperial College
Exhibition Road, London SW7 2BX, UK
tel: +44 20 7594 9166, fax: +44 20 7581 8809
e-mail: a.tsanakas@ic.ac.uk

Abstract

We study the equilibrium in a risk exchange, where agents' preferences are characterised by generalised (rank-dependent) expected utility, i.e. by a concave utility and a convex probability distortion (Quiggin, 1993). We obtain explicit results for the equilibrium price density, thus generalising Bühlmann's (1980, 1984) formulas. For linear utility functions, we show that the agents' preference maximisation problem is equivalent to minimisation of portfolio risk and reformulate it in an insurance context, as premium maximisation under risk capital constraints induced by a coherent risk measure. We find that equilibrium is only reached if the same risk measure is applied throughout the market. Finally, we discuss the analogy of the exchange to a pooling arrangement and show that equilibrium prices can be obtained as marginal cost prices for an agent representing the collective (pool) of market players. From that perspective we discuss the links between equilibrium pricing, cooperative games and capital allocation.

Keywords: competitive equilibrium, risk exchange, generalised expected utility, cooperative games, coherent risk measures

[1]The author gratefully acknowledges the financial support of Lloyd's of London. The views expressed in this paper are solely those of the author, and do not necessarily reflect the views of Lloyd's of London.

1 Introduction

Equilibrium models of insurance markets have been extensively studied in the actuarial literature (Borch (1962), Bühlmann (1980, 1984), Aase (1993, 2002)). In these models, (re)insurance markets have been represented as exchange economies, where individual agents decide on their optimal consumption of random assets and liabilities via expected utility maximisation. Market prices are obtained by a market clearing condition, and are represented by linear pricing functionals. The pricing functional can in turn be represented as an expectation under a change of probability measure and the corresponding Radon-Nikodym derivative is called a 'price density'. A pioneering paper in this area has been by Borch (1962), while Bühlmann obtained explicit formulae for the price density, first in the case of exponential utilities (1980), and later for more general utility functions (1984). Aase (1993) offers a comprehensive discussion of equilibrium models in insurance/financial markets, including characterisations of Pareto optimal risk allocations and conditions for the existence and uniqueness of competitive equilibria. A more general review of the literature on risk sharing and a consolidation of well-known and new results is provided by Aase (2002).

The above-mentioned papers rely on the characterisation of economic agents' preferences through utility functions. However, almost since the inception of expected utility theory by von Neumann and Morgenstern (1944), serious doubts have been raised by economists as regards the validity of the expected utility hypothesis as a means to describing the preferences of economic agents (for example, Allais (1953)). The contentious issue has in general been the independence axiom of expected utility theory, which has been found to be frequently violated in practice, with characteristic examples being the Allais and Elsberg paradoxes (for a detailed discussion see Quiggin, 1993). Alternative theories of choice have been proposed, among others, by Quiggin (1982, 1993), Yaari (1987) and Schmeidler (1989). The closely related theories of choice propagated by these authors have become known under names such as generalised (or anticipated) expected utility, rank-dependent utility and Choquet expected utility. They are generally characterised by a modification or relaxation of the independence axiom. The resulting preference functionals are akin to the expected utility operator, with the difference that the additive probability measure is either substituted by a non-additive set function (or 'capacity', Schmeidler (1989)) or distorted by a nonlinear function (Yaari (1987), Quiggin (1993)). The distortion of the probability measure can be interpreted either as a transformation which (in the case of a risk averse agent) increases the probability of adverse events, or as a mechanism for producing a set of probability measures, with the expected utility being calculated with respect to the 'worst-case measure' in that set.

Even though the mathematical and economic machinery provided by generalised expected utility has been around for a number of years, the consequences for equilibrium models of using such generalised preferences have not been explored in the bibliography, with Chateauneuf et al. (2000) forming a rare exception. The first contribution of this paper is to reformulate the equilibrium models of Borch (1962) and Bühlmann (1980, 1984), for the case that preferences are consistent with generalised expected utility. Thus, each market agent's preferences are characterised by a utility and a probability distortion function. It is shown that the agents' risk allocations after the exchange are comonotonic, a fact that makes the explicit calculation of price densities possible. The technique of defining a collective risk aversion function, as the inverse of the sum of agents' risk tolerances, is extended via the additional definition of a collective uncertainty aversion function, induced by the agents' probability distortion functions. We calculate explicitly the equilibrium price density in the risk exchange, first for the case of exponential utility and distortion functions and then for more general functions. We thus extend the pricing formulae of Bühlmann, our own formulae figuring an additional term, which captures the effect of probability distortion.

A relatively recent development in the financial literature has been the emergence of a sophisticated theory of risk measures, as a means for determining capital requirements for the holders of risky positions. The axiomatic definition of coherent risk measures by Arztner et al. (1999) has by now achieved the status of a classic in this subject area. However, the functional forms and fundamental properties of risk measures have been extensively studied in the actuarial literature for more than thirty years, in the guise of premium calculation principles (e.g. Bühlmann (1970), Goovaerts et al. (1984)). A class of risk measures, termed distortion premium principles, were introduced by Denneberg (1990) and Wang (1996). These turn out to be a specific subclass of coherent risk measures, while being consistent with Yaari's (1987) dual theory of choice.

We explore the equilibrium implications of agents' economic decision-making, based on risk measures belonging to the class of distortion principles. Agents choose their optimal consumptions of random assets and liabilities by minimizing their risk measure, subject to a budget condition. An equivalent problem, if the agents correspond to (re)insurers, is to let them maximise their premium income, subject to the constraint that each insurer's risk (as measured by the distortion principle) does not exceed his surplus, net of reinsurance expenditure. The second contribution of this paper is to formulate and study these equilibrium models. It is shown that they emerge as a special cases of the general risk exchange model, when all agents' utility functions are taken to be linear. The market defined in this case achieves an equilibrium only if all agents share the same risk measure and we comment on the implications of this result for risk management and regulation.

In the equilibrium models studied in this paper, it is always found that the price density and agents' risk allocations after the exchange are functions only of the aggregate risk in the market. This, as observed by Borch (1962), invites a re-interpretation of the risk exchange as a pooling arrangement, where all agents pool their assets and liabilities and then buy any desired cashflow from the pool according to an agreed price mechanism. The third contribution of our paper is to show that this price mechanism corresponds to marginal cost pricing, where the cost function is determined via an indifference argument. Indifference is here defined through the aggregate preferences of the collective, which were derived in the context of the equilibrium model. We furthermore give an economic justification of marginal cost prices, by interpreting the pooling arrangement as a cooperative game (Aumann and Shapley, 1974), for which the marginal cost mechanism is a semi-value (Dubey et al. (1981), Samet and Tauman (1982)). For an equilibrium model with risk measures, marginal cost pricing produces a cost sharing mechanism, which is the (Aumann-Shapley) value of the game. This produces a link between equilibrium pricing and the capital allocation methodology studied by Denault (2001) and Tsanakas and Barnett (2002).

In the following section we discuss preference functionals based on distorted probabilities and their relationship to a class of coherent risk measures. In section 3 the risk exchange equilibrium model is studied and formulae for the price functional and the risk allocation to the agents after the exchange are obtained. The case of linear utility functions and the equivalent problem of equilibrium with risk measures are treated in section 4. In section 5 we elaborate on the analogy of the exchange to a pooling arrangement and discuss the links between equilibrium pricing, marginal costs and cooperative games. Finally, in section 6 we conclude with some remarks on Pareto efficiency, market completeness and topics for future research.

2 Preferences and risk measures

2.1 Individual preferences and generalised expected utility

In this paper, a one-period economy is considered. At time 0 economic agents (e.g. financial institutions, insurance companies) make decisions concerning their consumption of random assets and liabilities. At some fixed future time t the state of the world is revealed and gains and losses are realised.

We define a probability space $(\Omega, \mathbb{P}_0, \mathcal{F})$. Ω is the set of all possible states of the world at time t, $\mathbb{P}_0$ is the actuarial ('real-world') probability measure, and $\mathcal{F} \subset 2^\Omega$ is a σ-algebra representing the amount of information available to the agents at the time of economic decision-making.

We consider a set, $\mathcal{X}$, of random variables on this probability space, which represent investment positions available to the market agents. Elements of $\mathcal{X}$ are henceforth called *cashflows*. For technical reasons we will assume that $\mathcal{X}$ also contains all Bernoulli(u) ($u \in [0, 1]$) variables. We denote by $E[\cdot]$ the expectation operator under $\mathbb{P}_0$ and use the notation $S_X(x) = \mathbb{P}_0(X > x)$ for the decumulative distribution function of $X \in \mathcal{X}$.

For each market agent, a preference relation '$\succeq$' is defined on $\mathcal{X}$, '$X \succeq Y$' being shorthand for 'cashflow $X \in \mathcal{X}$ is preferable to cashflow $Y \in \mathcal{X}$'. We assume that the preference relation is consistent with the axioms of Schmeidler (1989). Then a preference functional $V : \mathcal{X} \mapsto \mathbb{R}$ exists such that:

$$V(X) \geq V(Y) \Leftrightarrow X \succeq Y. \tag{1}$$

The functional V will be given by the Choquet integral (Schmeidler, 1989):

$$V(X) = \int u(X)d\gamma = \int_{-\infty}^{0} (\gamma(u(X) > x) - 1)dx + \int_{0}^{\infty} \gamma(u(X) > x)dx, \tag{2}$$

where $\gamma : \mathcal{F} \mapsto \mathbb{R}$ is a unique increasing set function with $\gamma(\emptyset) = 0,\ \gamma(\Omega) = 1$, and u is an increasing real valued function, unique up to affine transformations. Choquet integrals are defined with respect to monotone set functions (or 'capacities') instead of additive measures (Choquet (1954), Denneberg (1994)).

It is shown in Wang et al. (1997) that, given that $\mathcal{X}$ contains all Bernoulli variables, an increasing set function γ as above can be represented by:

$$\gamma(A) = h(\mathbb{P}_0(A)),\ \forall A \in \mathcal{F}, \tag{3}$$

where $h : \mathbb{R} \mapsto \mathbb{R}$ is an increasing function with $h(0) = 0,\ h(1) = 1$. Thus the preference functional can be rewritten as:

$$V_{u,h}(X) = \int u(X)dh(\mathbb{P}_0) = \int_{-\infty}^{0} (h(S_{u(X)} > x) - 1)dx + \int_{0}^{\infty} h(S_{u(X)} > x)dx \tag{4}$$

Here u is the well-known von Neumann-Morgenstern utility function, while h is called a *probability distortion function* and $\gamma = h(\mathbb{P}_0)$ a *distorted probability*. Thus the *generalised expected utility operator* $V_{u,h}$ is akin to expected utility, with the difference that probability measure $\mathbb{P}_0$ has been replaced by the distorted probability $h(\mathbb{P}_0)$.

As is usual in the bibliography, we will call an agent *risk averse* whenever the utility function u is concave. Correspondingly, we will call an agent *uncertainty averse* whenever the distortion function h is convex. Concavity of the utility function and convexity of the distortion function are necessary and sufficient conditions for the preference functional $V_{u,h}$ to preserve second order stochastic dominance (Quiggin, 1993). Assuming that h is differentiable, it is easy to rewrite (5) as (see Lemma 1):

$$V_{u,h}(X) = E[u(X)h'(S_X(X))] \tag{5}$$

Thus for an uncertainty averse agent, since h' is increasing, the effect of the distortion function is to assign a higher probability weighting to less favourable events (i.e. when $S_X(X)$ is high, equivalently X is low).

Another interpretation of the probability distortion is as follows. It can be shown (Denneberg, 1994), that if h is convex and $\mathbb{P}_0$ is a measure, the distorted probability $h(\mathbb{P}_0)$ is a supermodular set function and can be written as (Denneberg, 1994):

$$h(\mathbb{P}_0)(A) = \inf_{\mathbb{P} \geq h(\mathbb{P}_0)} \mathbb{P}(A), \;\; A \in \mathcal{F} \tag{6}$$

The preference operator can then be written as:

$$V_{u,h}(X) = \inf_{\mathbb{P} \geq h(\mathbb{P}_0)} E_{\mathbb{P}}[X], \;\; X \in \mathcal{X} \tag{7}$$

By viewing the set of measures $\{\mathbb{P} : \mathbb{P} \geq h(\mathbb{P}_0)\}$ as a collection of alternative scenarios regarding the probability distribution of X, it is apparent that the preference operator $V_{u,h}$ represents the expected utility of X under the most adverse of those scenarios.

We assume here and in the sequel that the utility and distortion functions are continuous and twice differentiable. The Arrow-Pratt risk aversion coefficient is defined in the bibliography as:

$$\rho(x) = -\frac{u''(x)}{u'(x)} \tag{8}$$

Correspondingly, we define the *uncertainty aversion coefficient* as:

$$\tau(s) = \frac{h''(s)}{h'(s)} \tag{9}$$

Note that we can determine the utility and distortion functions from the risk and uncertainty aversion coefficients, by solving the differential equations (8) and (9) respectively. The resulting utility and distortion functions will be unique up to a normalisation of the utility function (e.g. $u(0) = 0,\; u'(0) = 1$, given also that $h(0) = 0,\; h(1) = 1$).

Example: Consider the exponential utility and distortion functions:

$$u(x) = \frac{1}{\rho}(1 - e^{-\rho x}), \quad h(s) = \frac{e^{\tau s} - 1}{e^{\tau} - 1} \tag{10}$$

It is easily seen that these utility and distortion functions have constant risk and uncertainty aversion functions respectively:

$$-\frac{u''(x)}{u'(x)} = \rho, \quad \frac{h''(s)}{h'(s)} = \tau \tag{11}$$

2.2 Coherent risk measures and distorted probabilities

A risk measure is defined as a functional $R : \mathcal{X} \mapsto \mathbb{R}$. $R(X)$ represents 'the minimum extra cash that the agent has to add to the risky position X, and to invest "prudently", to be allowed to proceed with his plans' (Artzner et al., 1999). 'Invest prudently', in this paper, is taken to mean with zero interest.

A coherent risk measure is defined by Artzner et al. (1999) as a risk measure satisfying the following properties:

Monotonicity: If $X \leq Y$ a.s. then $R(X) \geq R(Y)$

Subadditivity: $R(X+Y) \leq R(X) + R(Y)$

Positive Homogeneity: If $a \in \mathbb{R}_+$ then $R(aX) = aR(X)$

Translation Invariance: If $a \in \mathbb{R}$ then $R(X+a) = R(X) - a$

Two random variables, X, Y, are called comonotonic if there exists a random variable U and non-decreasing functions d, e such that $X = d(U)$, $Y = e(U)$ (Dhaene et al., (2002)). Comonotonicity corresponds to the strongest form of positive dependence between random variables. An economic interpretation of comonotonicity is that comonotonic risks cannot be used as hedges for each other (Yaari, 1987). In the framework of coherent risk measurement, an additional desirable property is additivity for comonotonic risks:

Comonotonic Additivity: If X, Y are comonotonic, then $R(X+Y) = R(X) + R(Y)$

It can be shown that, if and only if $R(X)$ is a coherent risk measure satisfying comonotonic additivity, it has a representation as the negative of a Choquet integral with respect to a supermodular set function or capacity, γ (Denneberg, 1994):

$$R(X) = -\int X d\gamma = -\int_{-\infty}^{0} (\gamma(X > x) - 1)dx - \int_{0}^{\infty} \gamma(X > x)dx \tag{12}$$

As in the previous section (equation (3)), we will represent the supermodular set function γ by a distorted probability $h(\mathbb{P}_0)$, where h is increasing and convex with $h(0) = 0,\ h(1) = 1$. We thus obtain the distortion risk measure:

$$R_h(X) = -\int_{-\infty}^{0} (h(S_X(x) - 1)dx - \int_{0}^{\infty} h(S_X(x))dx \tag{13}$$

Such risk measures, based on distorted probabilities, where first introduced in the context of insurance pricing by Denneberg (1990) and Wang (1996). Similarly to equation (7) we can represent the risk measure R_h via a collection of probability measures:

$$R_h(X) = -\inf_{\mathbb{P} \geq h(\mathbb{P}_0)} E_{\mathbb{P}}[X] \tag{14}$$

Consider now an agent, whose preferences are characterised by a linear utility function $u(x) = x$ and a convex probability distortion h (such preferences we introduced by Yaari (1987)). Denote the agent's preference functional as v_h, i.e.:

$$v_h(X) = \int X dh(\mathbb{P}_0) = \inf_{\mathbb{P} \geq h(\mathbb{P}_0)} E_{\mathbb{P}}[X] \tag{15}$$

It is evident that $R_h(X) = -v_h(X)$. We can actually derive the risk measure R_h from v_h via a simple indifference argument. Assume that the agent insures a liability X and that his initial wealth is w. The price $R_h(X)$ that the agent charges for insuring the liability X is obtained by solving:

$$\begin{gathered} v_h(w) = v_h(w + X + R_h(X)) \\ \inf_{\mathbb{P} \geq h(\mathbb{P}_0)} E_{\mathbb{P}}[w] = \inf_{\mathbb{P} \geq h(\mathbb{P}_0)} E_{\mathbb{P}}[w + X + R_h(X)] \Leftrightarrow \\ w = w + \inf_{\mathbb{P} \geq h(\mathbb{P}_0)} E_{\mathbb{P}}[X] + R_h(X) \Leftrightarrow \\ R_h(X) = -v_h(X) \end{gathered} \tag{16}$$

2.3 Auxiliary lemmas

Let an agent's preferences be characterised by a utility function u and a distortion function h. In this section we state some auxiliary results, concerning the operator $V_{u,h}$, that are going to be extensively used in the sequel. The assumptions that we make for the proof of these lemmas are used throughout the paper. These are:

- The functions $u(x) : \overline{\mathbb{R}} \mapsto \overline{\mathbb{R}}$ and $h(x) : [0, 1] \mapsto [0, 1]$ are differentiable and strictly increasing.
- All survival functions $S_X(x) : \overline{\mathbb{R}} \mapsto [0, 1]$ are continuous and strictly decreasing.

- All random variables in $\mathcal{X}$ are square-integrable, i.e. $\mathcal{X} \subset L^2(\Omega, \mathcal{F}, \mathbb{P}_0)$. Furthermore, elements of $\mathcal{X}$ are assumed to have continuous conditional densities, in the sense of Appendix B.

Lemma 1. *For every $X \in \mathcal{X}$:*

$$V_{u,h}(X) = E[u(X)h'(S_X(X))]$$

Proof: Appendix A.

Corollary 1.

$$v_h(X) = -R_h(X) = E[Xh'(S_X(X))]$$

Lemma 2. *Let $X, N \in \mathcal{X}$ and $\beta \in \mathbb{R}$. Then $V_{u,h}(X + \beta N)$ is differentiable with respect to β and the partial derivative equals:*

$$\frac{\partial V_{u,h}(X + \beta N)}{\partial \beta} = E[Nu'(X + \beta N)h'(S_{X+\beta N}(X + \beta N))]$$

Proof: Appendix A.

Corollary 2.

$$\frac{\partial v_h(X + \beta N)}{\partial \beta} = -\frac{\partial R_h(X + \beta N)}{\partial \beta} = E[Nh'(S_{X+\beta N}(X + \beta N))]$$

3 Risk exchange

3.1 General setup

Let n agents, standing for financial institutions ((re)insurance companies and/or banks), be participating in an exchange economy. Each holds an initial random (including cash) endowment $X_i \in \mathcal{X}, i = 1, ..., n$, which can be traded in the exchange. Let $\mathcal{F}$ be the σ-algebra generated by the initial endowments $X_i,\ i = 1, ..., n$. Agents can acquire through trading any cashflow $Y \in \mathcal{X}$ which is measurable with respect to $\mathcal{F}$. Additionally we assume that a safe asset $\mathbf{1}_\Omega$ is traded in the market. We note that this collection of available cashflows includes nonlinear functions of the initial endowments and thus is wider than the one usually defined in the financial literature, which consists of linear combinations of traded instruments, see e.g. Duffie (1988).

We assume that market prices are given by a linear functional $\pi(X) = E[\zeta X]$, where ζ is a $\mathcal{F}$-measurable random variable. The price of the safe asset is taken to be 1, hence:

$$\pi[\mathbf{1}_\Omega] = 1 \Rightarrow E[\zeta] = 1 \tag{17}$$

Preferences are taken to conform to generalised expected utility theory, as described in section 2.1, and are consistent with second order stochastic dominance. Thus each agent is equipped with a strictly increasing and concave utility function u_i and a strictly increasing and convex probability distortion $h_i,\ i = 1, 2, ..., n$. For simplicity we assume that both u_i and h_i are continuous and twice differentiable. We denote the i^{th} agent's preference functional as V_i. The i^{th} agent decides on his optimal investment by solving the optimisation problem:

$$\max_{Y_i} V_i(Y_i), \tag{18}$$

subject to the budget constraint:

$$\pi(Y_i) \leq \pi(X_i) \tag{19}$$

3.2 Conditions for equilibrium

We define the aggregate risk in the market as:

$$Z = \sum_{j=1}^{n} X_j \tag{20}$$

The economy will be at equilibrium if and when all agents have solved their preference maximisation problem (18, 19) and the market has cleared:

$$\sum_{j=1}^{n} Y_j = Z \tag{21}$$

Using the lagrangian multiplier $\lambda_i \geq 0$ we can rewrite the optimisation problem (18), (19) as:

$$\max_{Y_i, \lambda_i} V_i(Y_i) - \lambda_i(\pi(Y_i) - \pi(X_i)) \tag{22}$$

To solve this maximisation problem we proceed using some standard methodology from variational calculus. For $N \in \mathcal{X}$ we define:

$$f(\beta) = V_i(Y_i + \beta N) - \lambda_i(\pi(Y_i + \beta N) - \pi(X_i)) \tag{23}$$

In order that the objective function of (18) achieves an optimum at Y_i it must be:

$$f'(0) = 0,\ \forall N \in \mathcal{X} \tag{24}$$

From Lemma 2 we obtain:

$$f'(\beta) = E[N u_i'(Y_i + \beta N) h_i'(S_{Y_i + \beta N}(Y_i + \beta N))] - \lambda_i \pi(N) \tag{25}$$

Thus:

$$f'(0) = E[Nu_i'(Y_i)h_i'(S_{Y_i}(Y_i))] - \lambda_i \pi(N) \tag{26}$$

Equations (24), (26) yield the following condition for equilibrium:

$$u_i'(Y_i)h_i'(S_{Y_i}(Y_i)) = \lambda_i \zeta, \ \forall i = 1, ..., n \tag{27}$$

Since both u_i and h_i are strictly increasing, it will be $\lambda_i > 0$. Consider now the function:

$$\eta_i(x) = \frac{1}{\lambda_i} u_i'(x) h_i' \circ S_{Y_i}(x) \tag{28}$$

The first derivative of η_i is strictly negative:

$$\eta_i'(x) = \frac{1}{\lambda_i}(u_i''(x)h_i' \circ S_{Y_i}(x) - u_i'(x)h_i'' \circ S_{Y_i}(x) f_{Y_i}(x)) < 0, \tag{29}$$

since the functions u_i', h_i' are strictly decreasing and increasing respectively. Thus η_i is strictly decreasing. Therefore its inverse η_i^{-1} exists and is also strictly decreasing. We observe that all random variables $Y_i = \eta_i^{-1}(\zeta)$ are strictly decreasing functions of the random variable ζ, hence they are comonotonic. An obvious consequence is that the random variables Y_i, $i = 1, ..., n$ are also comonotonic to (increasing functions of) their sum Z. Let F_{Y_i} be the cumulative distribution of Y_i and $F_{Y_i}^{-1}$ the inverse thereof. Comonotonicity of the Y_i's has the following two consequences (see e.g. Dhaene et al., 2002), which will be used in the sequel:

$$F_{Y_i}(Y_i) = F_Z(Z) = U, \ \forall i = 1, ..., n, \tag{30}$$

where U is uniformly distributed on the unit interval, $U \sim U[0, 1]$. Also:

$$\sum_{j=1}^{n} F_{Y_i}^{-1}(p) = F_{\sum_{j=1}^{n} Y_j}^{-1}(p) = F_Z^{-1}(p) \tag{31}$$

An economic characterisation of comonotonic risks is that they cannot be used as hedges for each other (Yaari, 1987). The fact that the final positions Y_i are comonotonic has the interpretation that agents have ridded themselves of the individual risk embedded in their initial endowments X_i and are left only with the market's systemic risk. Thus our model is consistent with a well known tenet of capital asset pricing.

3.3 Solution for exponential utility and distortion functions

Before proceeding with the calculation of equilibrium prices for more general utility and distortion functions, we study the case of exponential utility and distortion. We thus generalise the celebrated pricing formulas of Bühlmann (1980).

Let each agent be equipped with an exponential utility function with risk aversion $\rho_i > 0$ and an exponential distortion function h_i with uncertainty aversion $\tau_i > 0$:

$$u_i(x) = \frac{1}{\rho_i}(1 - e^{-\rho_i x}) \tag{32}$$

$$h_i(s) = \frac{e^{\tau_i s} - 1}{e^{\tau_i} - 1} \tag{33}$$

The first and second derivatives of these functions are:

$$u_i'(x) = e^{-\rho_i x} > 0,\ u_i''(x) = -\rho_i e^{-\rho_i x} < 0 \tag{34}$$

$$h_i' = \frac{\tau_i e^{\tau_i s}}{e^{\tau_i} - 1} > 0,\ h_i''(s) = \frac{\tau_i^2 e^{\tau_i s}}{e^{\tau_i} - 1} > 0 \tag{35}$$

In the sequel, the following rewriting of $h_i'(S_{Y_i}(Y_i))$ will also be used:

$$h_i'(S_{Y_i}(Y_i)) = \frac{e^{-\tau_i F_{Y_i}(Y_i)}}{E[e^{-\tau_i F_{Y_i}(Y_i)}]} \tag{36}$$

The condition (27) for equilibrium derived earlier yields:

$$\begin{gathered} u_i'(Y_i) h_i'(S_{Y_i}(Y_i)) = \lambda_i \zeta \Rightarrow \\ e^{-\rho_i Y_i} h_i'(S_{Y_i}(Y_i)) = \lambda_i \zeta \Rightarrow \\ -\rho_i Y_i + \ln(h_i'(S_{Y_i}(Y_i))) = \ln(\lambda_i \zeta) \Rightarrow \\ Y_i = \tfrac{1}{\rho_i} \ln(h_i'(S_{Y_i}(Y_i))) - \tfrac{1}{\rho_i}\ln(\zeta) - \tfrac{1}{\rho_i}\ln(\lambda_i) \end{gathered} \tag{37}$$

By summing both sides of the above equation over i and taking into account the clearing condition (21) we obtain:

$$Z = \sum_{j=1}^n \frac{1}{\rho_j} \ln(h_j'(S_{Y_j}(Y_j))) - \sum_{j=1}^n \frac{1}{\rho_j}\ln(\zeta) - \sum_{j=1}^n \frac{1}{\rho_j}\ln(\lambda_j) \tag{38}$$

The first term of the right-hand side becomes:

$$\begin{aligned} &\textstyle\sum_{j=1}^n \frac{1}{\rho_j} \ln(h_j'(S_{Y_j}(Y_j))) = \\ &= \ln \textstyle\prod_{j=1}^n h_j'(S_{Y_j}(Y_j))^{\frac{1}{\rho_j}} = \\ &\overset{(36)}{=} \ln \frac{\prod_{j=1}^n e^{-\frac{\tau_j}{\rho_j} F_{Y_j}(Y_j)}}{\prod_{j=1}^n E[e^{-\tau_j F_{Y_j}(Y_j)}]^{\frac{1}{\rho_j}}} = \\ &\overset{(30)}{=} \ln \textstyle\prod_{j=1}^n e^{-\frac{\tau_j}{\rho_j} U} - \ln \prod_{j=1}^n E[e^{-\tau_j U}]^{\frac{1}{\rho_j}} \\ &= \ln e^{-\sum_{j=1}^n \frac{\tau_j}{\rho_j} U} - \ln \textstyle\prod_{j=1}^n E[e^{-\tau_j U}]^{\frac{1}{\rho_j}} \end{aligned} \tag{39}$$

Now, by the following equations we define the *collective risk aversion* ρ and *collective uncertainty aversion* τ:

$$\frac{1}{\rho} = \sum_{j=1}^n \frac{1}{\rho_j}, \quad \tau = \rho \sum_{j=1}^n \frac{\tau_j}{\rho_j} \tag{40}$$

Thus, putting together equations (38), (39) and (40) we obtain:

$$Z = -\frac{\tau}{\rho}U - \ln\prod_{j=1}^{n} E[e^{-\tau_j U}]^{\frac{1}{\rho_j}} - \frac{1}{\rho}\ln(\zeta) - \sum_{j=1}^{n}\frac{1}{\rho_j}\ln(\lambda_j) \tag{41}$$

We set the constant, K, equal to:

$$K = -\rho(\ln\prod_{j=1}^{n} E[e^{-\tau_j U}]^{\frac{1}{\rho_j}} + \sum_{j=1}^{n}\frac{1}{\rho_j}\ln(\lambda_j)) \tag{42}$$

Then equation (41) becomes:

$$\begin{gathered} \rho Z + \tau U = K - \ln\zeta \Rightarrow \\ \zeta = e^{-\rho Z - \tau U} e^{K} \end{gathered} \tag{43}$$

Since we have assumed that there exists in the market a risk-free asset $\mathbf{1}_\Omega$ with unit price, from (17) we obtain:

$$\begin{gathered} E[\zeta] = 1 \overset{(43)}{\Rightarrow} \\ E[e^{-\rho Z - \tau U} e^{K}] = 1 \Rightarrow \\ e^{K} = E[e^{-\rho Z - \tau U}]^{-1} \end{gathered} \tag{44}$$

Thus the price density is:

$$\zeta = \frac{e^{-\rho Z - \tau U}}{E[e^{-\rho Z - \tau U}]} = \frac{e^{-\rho Z - \tau F_Z(Z)}}{E[e^{-\rho Z - \tau F_Z(Z)}]} \tag{45}$$

Note that this is a generalisation of the Esscher transform, which was obtained by Bühlmann (1980), who studied a market model where agents' preferences are characterised by utility functions only. The probability weighting factor $\exp(-\rho Z)$ in the price density, associates the price of a traded cashflow with the absolute random value of the market portfolio Z. The fact that it is a decreasing function of Z has the interpretation that a cashflow, which is likely to assume a high value when Z is low, is traded at a high price because of its usefulness in hedging market risk. On the other hand, the additional probability weighting $\exp(-\tau F_Z(Z))$ that we introduce is due to collective uncertainty aversion and associates the price of a cashflow with the rank of the outcome of Z, in the set of possible outcomes. For this factor, the absolute value of Z is not of interest, but rather the ranking of its possible outcomes, induced by the application of its cumulative distribution function, F_Z. That the price density is a decreasing function of $F_Z(Z)$ has again the interpretation that a cashflow that is likely to assume a high value when $F_Z(Z)$ is low is traded at a high price because of its usefulness in hedging. However, hedging now takes place not with respect to the absolute level of market risk, but with respect to its rank among all possible outcomes; essentially this is not hedging against *losses*, but hedging against *scenarios*.

We now can explicitly calculate the agents' final positions $Y_i,\ i=1,...,n$. Equations (37), (45) yield:

$$\begin{aligned} e^{-\rho_i Y_i}\tfrac{e^{-\tau_i U}}{E[e^{-\tau_i U}]} &= \lambda_i \tfrac{e^{-\rho Z-\tau U}}{E[e^{-\rho Z-\tau U}]} \Rightarrow \\ e^{-\rho_i Y_i} &= \lambda_i e^{-\rho Z-(\tau-\tau_i)U}\tfrac{E[e^{-\tau_i U}]}{E[e^{-\rho Z-\tau U}]} \Rightarrow \\ -\rho_i Y_i &= -\rho Z-(\tau-\tau_i)U+\ln\left(\lambda_i \tfrac{E[e^{-\tau_i U}]}{E[e^{-\rho Z-\tau U}]}\right) \Rightarrow \\ Y_i &= \tfrac{\rho}{\rho_i}Z+\tfrac{\tau-\tau_i}{\rho_i}U-\tfrac{1}{\rho_i}\ln\left(\lambda_i \tfrac{E[e^{-\tau_i U}]}{E[e^{-\rho Z-\tau U}]}\right) \end{aligned} \tag{46}$$

From the budget condition (19) we obtain (the equality being a consequence of the strict positivity of the lagrangian multiplier λ_i (27)):

$$\begin{gathered} \pi(Y_i)=\pi(X_i) \Rightarrow \\ E[(\tfrac{\rho}{\rho_i}Z+\tfrac{\tau-\tau_i}{\rho_i}U)\zeta]-\tfrac{1}{\rho_i}\ln\left(\lambda_i \tfrac{E[e^{-\tau_i U}]}{E[e^{-\rho Z-\tau U}]}\right)=E[X_i\zeta] \Rightarrow \\ -\tfrac{1}{\rho_i}\ln\left(\lambda_i \tfrac{E[e^{-\tau_i U}]}{E[e^{-\rho Z-\tau U}]}\right)=-E[(\tfrac{\rho}{\rho_i}Z+\tfrac{\tau-\tau_i}{\rho_i}U)\zeta]+E[X_i\zeta] \end{gathered} \tag{47}$$

Finally, substituting (47) in (46) yields the following expression for the agents' positions:

$$\begin{aligned} Y_i &= \tfrac{\rho}{\rho_i}Z+\tfrac{\tau-\tau_i}{\rho_i}U-E[(\tfrac{\rho}{\rho_i}Z+\tfrac{\tau-\tau_i}{\rho_i}U)\zeta]+E[X_i\zeta]= \\ &= \tfrac{\rho}{\rho_i}(Z-\pi(Z))+\tfrac{\tau-\tau_i}{\rho_i}(F_Z(Z)-\pi(F_Z(Z)))+\pi(X_i) \end{aligned} \tag{48}$$

It can easily be seen that, as expected, the share of the aggregate risk that the i^{th} agent holds after the exchange decreases as his risk and uncertainty aversion coefficients increase. Specifically, Y_i depends on how they compare with the corresponding collective risk and uncertainty aversions. Note that the risk allocation Y_i consists of two terms: the first is a proportional share of the aggregate risk Z, due to risk aversion, and the second a proportional share of $F_Z(Z)$, due to uncertainty aversion.

3.4 Solution for the general case

We now proceed with the calculation of the equilibrium price density, for the case where each agent's preferences are characterised by a strictly concave utility function u_i and a strictly convex probability distortion h_i. The only additional assumption we make on u_i and h_i is that they be continuous and twice differentiable.

In section 3.2 it was shown that at equilibrium the agents' final positions Y_i will be comonotonic to each other, as well as to their sum Z. Thus for each $i=1,...,n$, Y_i can be written as an increasing function ψ_i of Z:

$$Y_i=\psi_i(Z) \tag{49}$$

From equation (27) it is then apparent that the price density ζ will also be a (decreasing) function ϕ of Z:

$$\zeta = \phi(Z) \tag{50}$$

Thus we can rewrite the condition for equilibrium (27) as:

$$u_i'(\psi_i(Z))h'(S_Z(Z)) = \lambda_i \phi(Z), \ \forall i = 1, ..., n \tag{51}$$

Taking the logarithmic derivative of both sides of equation (51) (which will exist because of our smoothness assumptions) yields:

$$\begin{gathered} \tfrac{\partial}{\partial Z} \ln\left(u_i'(\psi_i(Z))h'(S_Z(Z))\right) = \tfrac{\partial}{\partial Z} \ln\left(\lambda_i \phi(Z)\right) \Rightarrow \\ \tfrac{u_i''(\psi_i(Z))}{u_i'(\psi_i(Z))}\psi_i'(Z)) - \tfrac{h_i''(S_Z(Z)}{h_i'(S_Z(Z)}f_Z(Z) = \tfrac{\phi'(Z)}{\phi(Z)}, \end{gathered} \tag{52}$$

where f_Z is the probability density function of Z. Denoting the ith agent's risk aversion and uncertainty aversion functions by $\rho_i(x)$ and $\tau_i(s)$ respectively:

$$\rho_i(x) = -\frac{u_i''(x)}{u_i'(x)}, \ \tau_i(s) = \frac{h_i''(s)}{h_i'(s)}, \tag{53}$$

we obtain:

$$\begin{gathered} -\rho_i(\psi_i(Z))\psi_i'(Z)) - \tau_i(S_Z(Z))f_Z(Z) = \tfrac{\phi'(Z)}{\phi(Z)} \Rightarrow \\ \psi_i'(Z) = -\tfrac{1}{\rho_i(\psi_i(Z))}\tfrac{\phi'(Z)}{\phi(Z)} - \tfrac{\tau_i(S_Z(Z))}{\rho_i(\psi_i(Z))}f_Z(Z) \end{gathered} \tag{54}$$

Differentiating also the clearing condition (21) yields:

$$\frac{\partial}{\partial Z}\sum_{j=1}^{n}\psi_j(Z) = \frac{\partial}{\partial Z}Z \Rightarrow \sum_{j=1}^{n}\psi_j'(Z) = 1 \tag{55}$$

Thus, summing over i in (54) and defining the collective risk and uncertainty aversion functions as:

$$\rho(Z) = \left(\sum_{j=1}^{n}\frac{1}{\rho_j(\psi_j(Z))}\right)^{-1} \text{ and } \tau(S_Z(Z)) = \rho(Z)\sum_{j=1}^{n}\frac{\tau_j(S_Z(Z))}{\rho_j(\psi_j(Z))}, \tag{56}$$

we obtain:

$$\begin{gathered} 1 = -\tfrac{1}{\rho(Z)}\tfrac{\phi'(Z)}{\phi(Z)} - \tfrac{\tau(S_Z(Z))}{\rho(Z)}f_Z(Z) \Rightarrow \\ \tfrac{\phi'(Z)}{\phi(Z)} = -\rho(Z) - \tau(S_Z(Z))f_Z(Z) \end{gathered} \tag{57}$$

This ordinary differential equation yields the solution:

$$\phi(Z) = \phi(0)e^{-\int_0^Z \rho(x)dx - \int_0^Z \tau(S_Z(x))f_Z(x)dx} \tag{58}$$

From the condition $E[\phi(Z)] = E[\zeta] = 1$ we finally obtain the explicit solution for the price density:

$$\zeta = \phi(Z) = \frac{e^{-\int_0^Z \rho(x)dx - \int_0^{F_Z(Z)} \tau(1-y)dy}}{E[e^{-\int_0^Z \rho(x)dx - \int_0^{F_Z(Z)} \tau(1-y)dy}]} \tag{59}$$

This price density is a generalisation of the formula obtained by Bühlmann (1984). Again the market's uncertainty aversion introduces an additional weighting factor, $\exp\{-\int_0^{F_Z(Z)} \tau(1-y)dy\}$. In the case of exponential utility and distortion functions studied in the previous section, the market as well as the individual risk and uncertainty aversion functions are constant and it is easily seen that equation (59) reduces to (45).

As opposed to the case of exponential utility and distortion functions, we do not obtain an explicit formula for the risk allocations $\psi_i(Z) = Y_i$ to the agents after the exchange. Nonetheless, some insight can be gained. Recall the collective risk and uncertainty aversion functions that were defined earlier (eq. (56)). As discussed in section 2.1, from the quantities $\rho(Z)$ and $\tau(S_Z(Z))$ we can determine unique corresponding utility and distortion functions u and h (up to a normalisation of u).

We can then rewrite the price density (59) as:

$$\phi(Z) = \frac{u'(Z)h'(S_Z(Z)}{E[u'(Z)h'(S_Z(Z)]} \tag{60}$$

This yields:

$$\frac{\phi'(Z)}{\phi(Z)} = \frac{u''(Z)h'(S_Z(Z)) - u'(Z)h''(S_Z(Z))f_Z(Z)}{u'(Z)h'(S_Z(Z))} = -\rho(Z) - \tau(Z)f_Z(Z) \tag{61}$$

Substituting the term $\frac{\phi'(Z)}{\phi(Z)}$ in equation (54) and setting $Z := x \in \mathbb{R}$ results in the differential equation:

$$\psi_i'(x) = \frac{\rho(x)}{\rho_i(\psi_i(x))} + \frac{\tau(S_Z(x)) - \tau_i(S_Z(x))}{\rho_i(\psi_i(x))} f_Z(x) \tag{62}$$

The i^{th} agent's risk allocation, $Y_i = \psi_i(Z)$, can then be determined by solving this differential equation. As previously, we see that Y_i will depend on how the i^{th} agent's risk and uncertainty aversion functions compare to the collective's. Note that in this more general case the Y_i's do not consist of proportional shares of Z and $F_Z(Z)$, but are non-linear functions thereof.

3.5 A pure liability market

Here we consider the simpler case of a reinsurance market, where each agent holds a random pure liability $\bar{X}_i \geq 0$ and cash (initial surplus) $w_i \geq 0$. Thus the initial endowments are of the form:

$$X_i = w_i - \bar{X}_i \tag{63}$$

The situation is thus similar the one studied by Bülmann (1980, 1984). Define the aggregate liabilities, $\bar{Z}$, and the aggregate cash, w, in the market as:

$$\bar{Z} = \sum_{j=1}^{n} \bar{X}_j, \ w = \sum_{j=1}^{n} w_j \tag{64}$$

It can then be easily seen that the price density (59) can be rewritten as:

$$\zeta = \frac{e^{\int_w^{\bar{Z}} \rho(w-x)dx + \int_{F_{\bar{Z}}(w)}^{F_{\bar{Z}}(\bar{Z})} \tau(y)dy}}{E[e^{\int_w^{\bar{Z}} \rho(w-x)dx + \int_{F_{\bar{Z}}(w)}^{F_{\bar{Z}}(\bar{Z})} \tau(y)dy}]} \tag{65}$$

In the case that the agents have exponential utility and distortion functions, the price density (45) becomes:

$$\zeta = \frac{e^{\rho\bar{Z}+\tau F_{\bar{Z}}(\bar{Z})}}{E[e^{\rho\bar{Z}+\tau F_{\bar{Z}}(\bar{Z})}]} \tag{66}$$

These price densities can interpreted as producing an economically motivated premium principle $\Pi(X;\bar{Z})$, in the sense of Bühlmann (1980, 1984):

$$\Pi(X;\bar{Z}) = E[\zeta X], \tag{67}$$

where ζ is given by (65) or (66). Note that if we let $\tau_i \to 0\ \forall i$ in (56) and (59), then Bühlmann's (1980, 1984) formulas for the price density are replicated.

4 Equilibrium with risk measures

4.1 Equilibrium with linear utility

We now study the special case where all utility functions are linear, that is, we let $\rho_i \to 0\ \forall i$. The optimisation problem to be solved by each agent then is:

$$\max_{Y_i,\lambda_i} v_i(Y_i) - \lambda_i(\pi(Y_i) - \pi(X_i)), \tag{68}$$

where $v_i = v_{h_i}$ is the preference operator (15) of the i^{th} agent. The analogue of condition (27) in this case is:

$$h_i'(S_{Y_i}(Y_i)) = \lambda_i\zeta,\ \forall i = 1, ..., n \tag{69}$$

This yields $\forall i$:

$$Y_i = S_{Y_i}^{-1} \circ (h_i')^{-1}(\lambda_i\zeta) \tag{70}$$

and since S_{Y_i} is decreasing and h_i' is strictly increasing the random variables $Y_i,\ i = 1, ..., n$ are comonotonic as previously. Thus $S_{Y_i}(Y_i) = S_Z(Z)\ \forall i$ as before and (69) can be written as:

$$h_i'(S_Z(Z)) = \lambda_i\zeta,\ \forall i = 1, ..., n \tag{71}$$

Since $E[h_i'(S_Z(Z))] = E[\zeta] = 1$ it follows that $\lambda_i = 1\ \forall i$ and thus:

$$h_i'(S_Z(Z)) = \zeta,\ \forall i = 1, ..., n \tag{72}$$

This yields the following condition for the existence of equilibrium:

$$h_i(s) = h_j(s)\ \forall i,j \in \{1,...,n\},\ s \in [0,1] \tag{73}$$

Thus equilibrium is reached only if all agents have identical distortion functions or, equivalently, their risk preferences are derived from the same set of probability measures $\{\mathbb{P} : \mathbb{P}(A) \geq h_1(\mathbb{P}_0(A))\ \forall A \in \mathcal{F}\}$. A further interpretation of this result is given in the next section.

Note that we can obtain the market uncertainty aversion for that model, if in equation (56) we let $\tau_j(s) = \tau_1(s), \forall j$ and $\rho_j = \rho_1(\to 0)$. Then the market uncertainty aversion becomes equal to that of every agent:

$$\tau(s) = \frac{n\frac{\tau_1(s)}{\rho_1}}{n\frac{1}{\rho_1}} = \tau_1(s) \tag{74}$$

4.2 Equilibrium with distortion risk measures

Continuing from the previous section, we let all agents share the same distortion function $h_i = h$, hence their preference functionals $v_i = v$ are identical. It is straightforward that the optimisation problem (68) is equivalent to:

$$\min_{Y_i,\kappa_i} -v(Y_i) + \kappa_i(\pi(Y_i) - \pi(X_i)) \tag{75}$$

As shown in section (2.2), the functional $-v$ is a coherent risk measure. Thus the optimisation problem (75) can be interpreted as the minimisation of the retained risk, $-v(Y_i)$, to the agent, subject to the budget condition.

Let us now return to the case of the liability market of section 3.5, with $\bar{Y}_i = -Y_i$ being the liabilities retained by the i^{th} reinsurer and $X_i = w_i - \bar{X}_i$ his initial endowment (initial surplus minus random liabilities). Then it is not difficult to show that the optimisation problem (68) is equivalent to:

$$\max_{\bar{Y}_i,\mu_i} \pi(\bar{Y}_i) - \mu_i(-v(-\bar{Y}_i) - (w_i - \pi(\bar{X}_i))) \tag{76}$$

In this context we interpret $\pi(\bar{Y}_i)$ as the price that the i^{th} reinsurer receives for insuring liabilities $\bar{Y}_i$, $\pi(\bar{X}_i)$ as his expenditure for reinsuring the initial liabilities $\bar{X}_i$, and $-v(-\bar{Y}_i)$ as the retained risk. Thus optimisation problem (76) is interpreted as a maximisation of the i^{th} insurer's premium income, $\pi(\bar{Y}_i)$, subject to the condition that the retained risk, $-v(-\bar{Y}_i)$, does not exceed the insurer's initial capital net of the cost of reinsurance, $w_i - \pi(\bar{X}_i)$.

It was argued by Jaschke and Küchler (2001) that considering a market equilibrium with a coherent risk measure (set, say, by a regulator) such as the one discussed here is a less 'personal' way of introducing preferences in market models, as the risk measure will be the same for all market agents. Our result (73) makes in fact a stronger claim: equilibrium can be reached *only* if the same risk measure is used by all agents.

Condition (73) can also be interpreted in a very different light. It has been argued by Danielson et al. (2001) that regulation, imposing the use throughout the market of the same risk measure, is bound to result in an increase in systemic risk, since it would lead to homogeneity of market players' risk assessment and mitigation strategies. Equation (73) precisely reflects this concern, as it implies that the use of the same risk measure by all market agents is equivalent with the agents' making comonotonic investments.

5 Links to cooperative game theory

5.1 Equilibrium prices as marginal costs

In the equilibrium models discussed in the previous sections, agents' risk allocations after the exchange, $Y_i,\ i = 1, ..., n$, and the price density, ζ, have been found to be functions only of the aggregate market risk, Z. This invites an alternative interpretation of the risk exchange as a pooling arrangement, where agents pool their initial endowments, $X_i,\ i = 1, ..., n$, and thereafter share the aggregate risk Z by buying their final positions Y_i from the pool according to an agreed price mechanism. The analogy between risk exchange and risk pooling has already been observed by Borch (1962), who also commented on the possibility of applying cooperative game theory to the problem. The purpose of this section is to study the problem of risk exchange from such a perspective.

We defined in section 3.4, equation (56), the 'collective risk and uncertainty aversion' functions, $\rho(x)$ and $\tau(s)$, without elaborating on the interpretation of these quantities. If the risk exchange is viewed as a pooling arrangement, the collective risk and uncertainty aversions can be taken to characterise the preferences of an agent standing for the collective of agents or, equivalently, to describe the preferences of the pool. As discussed earlier, from the quantities $\rho(x)$ and $\tau(s)$ we can determine the unique corresponding utility and distortion functions u_C and h_C for the collective (up to a normalisation of u_C). Then the collective's preferences can be expressed by the generalised expected utility functional:

$$V_C(Z) = \int_{-\infty}^{0} (h_C(S_{u_C(Z)} > x) - 1)dx + \int_{0}^{\infty} h_C(S_{u_C(Z)} > x)dx \tag{77}$$

Consider an agent buying cashflow $N \in \mathcal{X}$ from the pool. We define the indifference price of N, $\pi_{ind}(N)$, as the solution of the equation:

$$V_C(Z - N + \pi_{ind}(N, Z))) = V_C(Z) \tag{78}$$

The indifference price essentially represents the cost to the collective of parting from cashflow N. We can now define the marginal cost price of cashflow N as:

$$MC(N) = \frac{\partial \pi_{ind}(\beta N, Z)}{\partial \beta}_{|\beta=0} \tag{79}$$

In the following proposition, the relationship between marginal cost and equilibrium pricing is illustrated:

Proposition 1. *Let the marginal cost price of the cashflow N be given by equations (78), (79). Then the marginal cost price is equal to the equilibrium price (59) of N:*

$$MC(N) = E\left[N \frac{u_C'(Z)h_C'(S_Z(Z)}{E[u_C'(Z)h_C'(S_Z(Z)]}\right] = E\left[N \frac{e^{-\int_0^Z \rho(x)dx - \int_0^{F_Z(Z)} \tau(1-y)dy}}{E[e^{-\int_0^Z \rho(x)dx - \int_0^{F_Z(Z)} \tau(1-y)dy}]}\right]$$

Proof: Appendix A.

Thus, if we consider a risk pooling arrangement as an analogue to the risk exchange, the price of a cashflow is determined as its marginal cost price to the collective.

5.2 Marginal costs and semivalues of cooperative games

In this section we take a view of the marginal cost price mechanism, in the context of a non-atomic cooperative game (Aumann and Shapley, 1974) . A rigorous exposition of cooperative game theory is outside the scope of this paper, so we will restrict ourselves on a rather qualitative discussion.

We consider the economic agents who pool their assets and liabilities as players in a cooperative game. Cooperation is here understood as the pooling of the players' initial endowments and the agreement on a price mechanism. If we interpret the pooling arrangement as an economy with a single producer (the pool), the products are the cashflows that each agent buys from the pool, and the cost of producing cashflow N is the indifference cost π_{ind} to the pool of parting with the cashflow. In order that the price mechanism used is acceptable to the players, it must satisfy a number of economically motivated properties. An appropriate set of axioms was proposed by Samet and Tauman (1982). This axiomatisation corresponds to the following requirements on the price mechanism:

Rescaling: Prices should be independent of units of measurement.

Consistency: Two cashflows that have the same effect on cost should have the same price.

Additivity: If the cost of a cashflow can be broken into two additive factors, then its price should be obtainable by adding the prices attributable to the two factors separately.

Monotonicity/Positivity: It the cost function is non-decreasing in a cashflow, then the price of the cashflow should be non-negative.

Then it is shown (Samet and Tauman, 1982), all price mechanisms satisfying this set of axioms are of the form:

$$\pi_{sv,\mu}(N) = \int_0^1 \frac{\partial \pi_{ind}}{\partial \beta}(\beta N, \alpha Z)_{|\beta=0} d\mu(\alpha), \tag{80}$$

where μ is a non-negative measure on $([0,1], \mathcal{B})$, $\mathcal{B}$ being the Borel σ-algebra on $[0,1]$. The same set of price mechanisms has also been derived via purely game theoretical arguments by Dubey et al. (1981), and its elements are called the *semivalues* of the game.

It can be shown (Samet and Tauman, 1982) that for a slight strengthening of the Monotonicity/Positivity axiom, the measure $\bar{\mu}$ is uniquely given by $\bar{\mu}(\alpha) = 0,\ \alpha \in [0,1)$ and $\bar{\mu}(1) = 1$. Thus, the price mechanism emerging is actually the unique semivalue corresponding the marginal cost mechanism:

$$\pi_{sv,\bar{\mu}}(N) = MC(N) = \frac{\partial \pi_{ind}(\beta N, Z)}{\partial \beta}_{|\beta=0} \tag{81}$$

5.3 Equilibrium with risk measures and capital allocation

In this section, we apply the game theoretical framework to the equilibrium model with risk measures developed in section 4. We show that in this case marginal cost pricing is a cost sharing mechanism and the equilibrium model can be interpreted as a risk capital allocation model.

We note that the semivalue (80) of a cooperative game is in general not a cost sharing mechanism, i.e. the sum of prices that the players pay do not add up to the aggregate cost to the pool. Let us now impose on the price mechanism the additional requirement:

Cost sharing: The sum of the prices charged by the pool should be equal to the aggregate cost to the pool.

Under this additional assumption, it is shown in Mirman and Tauman (1982) and Billera and Heath (1982) that the unique price mechanism satisfying the four axioms of the previous section, plus the requirement for cost sharing, is obtained when the measure μ is the Lebesgue measure on $[0,1]$, that is:

$$\pi_v(N) = \int_0^1 \frac{\partial \pi_{ind}}{\partial \beta}(\beta N, \alpha Z)_{|\beta=0} d\alpha \tag{82}$$

In game theoretical parlance, the unique functional π_v is called the *value* of the game (Aumann and Shapley, 1974).

Let us now return to the equilibrium model with linear utility functions (equivalently with distortion risk measures) that was studied in section 4. We showed that a condition for equilibrium is that all agents must share the same probability distortion function and that this is equal to the distortion function representing the collective's preferences. Let, as previously, the corresponding preference operator be denoted by v. Thus the indifference argument (78) becomes:

$$v(Z - N + \pi_{ind}(N,Z))) = v(Z) \Leftrightarrow -v(Z - N + \pi_{ind}(N,Z))) = -v(Z), \tag{83}$$

which is equivalent to saying that the risk carried by the pool should not change as a result of trading N.

As a special case of Proposition 1 we obtain:

$$\frac{\partial \pi_{ind}(\beta N, \alpha Z)}{\partial \beta}_{|\beta=0} = E[Nh'(S_{\alpha Z}(\alpha Z))] = E[Nh'(S_Z(Z))], \tag{84}$$

which is independent of α. Then, from (82) and (84) it is seen that the price mechanism π_v is given by:

$$\pi_v(N) = \frac{\partial \pi_{ind}(\beta N, Z)}{\partial \beta}_{|\beta=0} = E[Nh'(S_Z(Z))], \tag{85}$$

which is once more the marginal cost mechanism. Thus, in the case that preferences are characterised via risk measures based on distorted probabilities, marginal cost pricing provides a mechanism for cost sharing.

The value has been proposed by Denault (2001) as a cost sharing mechanism for allocating the risk capital corresponding to a risky portfolio to the different instruments that it consists of. In Tsanakas and Barnett (2002) this problem was studied in an insurance context, with risk measures based on distorted probabilities. In the latter paper a formula identical to equation (85) was obtained for the risk capital allocated to each instrument (or sub-portfolio of instruments), using the game-theoretical concept of the *core* (Aumann and Shapley (1974), Aubin (1981)). The core of a game is interpreted as the set of all price systems that do not produce for any player a disincentive to cooperate. In the context of risk capital allocation, this means that the capital allocated to each cashflow should not exceed the cost of holding that cashflow outside the pool, as determined by the risk measure. Thus, in the case of equilibrium with risk measures, the price functional calculated in this paper is the only one that would not produce an incentive for an economic agent to leave the exchange.

6 Concluding remarks

An issue that has so far not been touched upon in the paper is that of Pareto optimality. A (post-exchange) allocation of risks to market agents is said to be Pareto optimal if there is no other allocation for which all agents' preference functionals assume higher values (and for at least one agent this increase is strict), i.e. no other allocation exists for which every agent is better off. For equilibrium models with expected utility preferences, it has been shown in the bibliography (e.g. Aase, 1993) that any competitive equilibrium allocation is Pareto optimal. The proof relies on the strict increasingness of preferences and thus easily carries over to our model. Thus the equilibrium allocations calculated in this paper are also Pareto optimal.

Pareto optimality is closely related to the concept of market completeness. While Pareto optimality is a property of a specific risk allocation, completeness refers to the structure of the market itself. It is in general the case that in complete markets equilibrium allocations are Pareto optimal (Duffie, 1988). A market is usually called complete whenever the space of possible payoffs is spanned by the traded cashflows in the market. In market models such as those treated in our paper, a prerequisite for market completeness is the existence of non-linear contracts on the initial cashflows, X_i. This is because we consider a continuous probability space, while the number of the initial cashflows is finite. Note that this situation is quite different from the one usually encountered in the literature on asset pricing, where portfolios consist only of linear combinations of traded securities. If only linear portfolios (e.g. proportional reinsurance treaties) were allowed in the market treated here, it would have been incomplete (though the resulting allocation would not necessarily be inefficient; for a discussion of this issue see Aase (1993)).

Non-linear investment portfolios can be formed in both the financial and insurance markets, using for example stock options and stop-loss reinsurance treaties. Nevertheless, the requirement that *any* non-linear transformation of the initial positions should be obtainable (or replicable) in the market appears too strong, especially in the case of insurance markets. However, we note that the Pareto optimal risk allocations have all been functions only of the aggregate market risk Z. Thus optimal insurance portfolios can be approximated using tradable securities such as index-linked insurance derivatives, which are in a sense 'completing the market'. An additional issue that emerges, due to the probability distortion functions used to model preferences and to represent risk measures, is the dependence of risk allocations on the term $F_Z(Z)$. This term implies that it is necessary for the formation of optimal portfolios to consider derivatives not only contingent on the underlying cashflows themselves but also on their order statistics. Thus generalised preferences and/or regulation create a scope for financial products such as the 'quantile options' introduced by Miura (1992).

Readers familiar with competitive equilibrium models will have recognised in our definition of collective preferences the similarity to the device of the 'representative agent' often employed in the economics literature (e.g. Duffie, 1988). The definition of aggregate preferences is indeed not a novelty; it has been first proposed as a way of solving equilibrium models by, among others, Wilson (1968) and Rubinstein (1974). One additional element introduced in this paper has been the definition of aggregate preferences in the case of generalised expected utility, using what we called 'collective uncertainty aversion'. We did not refer explicitly to the representative agent paradigm, but preferred instead to offer an interpretation of aggregate preferences via the analogy of the exchange to a pooling arrangement. We found this analogy, as well as the subsequent association of competitive equilibrium with cooperative games, conceptually stimulating. As Aumann and Shapley comment in the introduction to their book (1974), 'interaction between people - as in economic or political activity - usually involves a subtle mixture of competition and cooperation'. We attempted to illustrate this point in the case of a risk exchange, where competing financial entities cooperate in order to share the risks that they are exposed to.

Finally, we note that the results in this paper were not obtained at the highest level of mathematical generality; we chose instead to make assumptions yielding explicit and transparent formulae. The strongest (and indeed the most disputable) of these assumptions has been that of continuous conditional probability distributions, which is a condition for the differentiability of the generalised preference functionals. Obtaining similar results without the continuity assumption is a topic for future research; a step into this direction has been made by Acerbi and Simonetti (2002). Other lines of future enquiry could include the generalisation of our market model to a multi-period (or time-continuous) setting, as well as the study of equilibrium in incomplete markets. Cooperative game theory and the concept of the core could provide useful insight in the incomplete market situation, as was demonstrated by Aase (2002).

References

[1] Aase, K. (1993), 'Equilibrium in a reinsurance syndicate; existence, uniqueness and characterisation', *ASTIN Bulletin, 23 (2), 185-211.*

[2] Aase, K. (2002), 'Perspectives of Risk Sharing', *Scandinavian Actuarial Journal*, 2, 73-128.

[3] Acerbi, C. and Simonetti, P. (2002), 'Portfolio Optimization with Spectral Measures of Risk', *Working Paper*, http://www.gloriamundi.org.

[4] Allais, M. (1953), 'Le comportement de l'homme rationnel devant le risque: Critique des postulates et axioms de l'ecole Americaine', *Econometrica*, 21, 503-546.

[5] Artzner, P., Delbaen, F., Eber, J-M., Heath, D. (1999), 'Coherent Measures of Risk', *Mathematical Finance*, 9 (3), 203-228.

[6] Aubin, J-P. (1981), 'Cooperative fuzzy games', *Mathematics of Operations Research*, 6 (1), 1-13.

[7] Aumann, R. J. and Shapley, L. S. (1974), *Values of Non-Atomic Games*, Princeton University Press, Princeton.

[8] Billera, L. J. and Heath, D. C. (1982), 'Allocation of shared costs: a set of axioms yielding a unique procedure', *Mathematics of Operations Research*, 7 (1), 32-39.

[9] Borch, K. (1962), 'Equilibrium in a reinsurance market', *Econometrica*, 30 (3), 424-444.

[10] Bühlmann, H. (1980), 'An economic premium principle', *ASTIN Bulletin*, 11 (1), 52-60.

[11] Bühlmann, H. (1984), 'The general economic premium principle', *ASTIN Bulletin*, 14 (1), 13-21.

[12] Chateauneuf, A, Dana, R-A. and Tallon, J-M. (2000), 'Optimal risk-sharing rules and equilbria with Choquet-expected-utility', *Journal of Mathematical Economics*, 34, 191-214.

[13] Choquet, G. (1953), 'Theory of capacities', *Annales de l'Institut Fourier*, 5, 131-295.

[14] Danielson, J., Embrechts, P., Goodhart, C., Keating, C., Muennich, F., Renault, O., Shin, H. S. (2001), 'An Academic Response to Basel II', Financial Markets Group, London School of Economics, http://risk.lse.ac.uk.

[15] Denault, M. (2001), 'Coherent allocation of risk capital', *Journal of Risk*, 4 (1), 1-34.

[16] Denneberg, D. (1990), 'Distorted probabilities and insurance premiums', *Methods of Operations Research*, 63, 3-5.

[17] Denneberg, D. (1994), *Non-additive Measure and Integral*, Kluwer Academic Publishers, Dordrecht.

[18] Dhaene, J., Denuit, M., Goovaerts, M. J., Kaas, R. and Vyncke, D. (2002), 'The concept of comonotonicity in actuarial science and finance: Theory', *Insurance: Mathematics and Economics*, 31 (1), 3-33.

[19] Dubey, P., Neyman, A. and Weber, R. J. (1981), 'Value theory without efficiency', *Mathematics of Operations Research*, 6, 122-128.

[20] Duffie, D. (1988), *Security Markets: Stochastic Models*, Academic Press, San Diego.

[21] Goovaerts, M.J., de Vylder, F., and Haezendonck, J. (1984), *Insurance premiums: theory and applications*, North-Holland, Amsterdam.

[22] Jaschke, S. and Küchler, U. (2001), 'Coherent risk measures, valuation bounds and (μ, ρ)-portfolio optimization', *Finance and Stochastics*, 5(2), 181-200.

[23] Mirman, L. and Tauman, Y. (1982), 'Demand compatible equitable cost sharing prices', *Mathematics of Operations Research*, 7 (1), 40-56.

[24] Miura, R., (1992), 'A note on a look-back option based on order statistics', *Hitosubashi Journal of Commerce and Management*, 27, 15-28.

[25] Neumann, J. von, and Morgenstern, O. (1944), *Theory of Games and Economic Behavior*, Princeton University Press, Princeton.

[26] Quiggin, J. (1982), 'A theory of anticipated utility', *Journal of Economic Behavior and Organization*, 3, 323-343.

[27] Quiggin, J. (1993), *Generalized expected utility theory: The rank-dependent model*, Kluwer Academic, Boston.

[28] Rubinstein, M. (1974), 'An aggregation theorem for securities markets', *Journal of Financial Economics*, 1, 225-244.

[29] Samet, D. and Tauman, Y. (1982), 'The determination of marginal cost prices under a set of axioms', *Econometrica*, 50 (4), 895-909.

[30] Schmeidler, D. (1989), 'Subjective Probability and Expected Utility without Additivity', *Econometrica*, 57 (3), 571-587.

[31] Tasche, D. (2000), 'Conditional expectation as quantile derivative' *Preprint*, TU Munich.

[32] Tsanakas, A. and Barnett, C. (2002), 'Risk capital allocation and cooperative pricing of insurance liabilities', *Insurance: Mathematics and Economics*, forthcoming.

[33] Wang, S. (1996), 'Premium calculation by transforming the premium layer density', *ASTIN Bulletin*, 26 (1), 71-92.

[34] Wang, S. S., Young, V. R., and Panjer, H. H. (1997), 'Axiomatic characterization of insurance prices', *Insurance: Mathematics and Economics*, 21 (2), 173-183.

[35] Wilson, R. (1968), 'The theory of syndicates', *Econometrica*, 36 (1), 119-132.

[36] Yaari, M. (1987), 'The dual theory of choice under risk', *Econometrica*, 55 (1), 95-115.

Appendix

A Proofs

A.1 Proof of Lemma 1

The Choquet integral (5) of $u(X)$ with respect to the supermodular set function $h(\mathbb{P}_0)$, admits the following quantile representation (Denneberg, 1994):

$$V_{u,h}(X) = \int_0^1 G^{-1}_{u(X)}(t)dt,$$

where $G^{-1}_{u(X)}(t)$ is the inverse of the (decumulative) distribution function of $u(X)$ under $h(\mathbb{P}_0)$:

$$G_{u(X)}(x) = h(\mathbb{P}_0(u(X) > x)) = h(S_{u(X)}(x))$$

Since the functions $S_X, h, u, S_{u(X)}$ are strictly monotonic, $G^{-1}_{u(X)}(t)$ becomes:

$$G^{-1}_{u(X)}(t) = S^{-1}_{u(X)}(h^{-1}(t)) = u(S^{-1}_X(h^{-1}(t)))$$

$V_{u,h}(X)$ can then be written as:

$$V_{u,h}(X) = \int_0^1 u(S^{-1}_X(h^{-1}(t)))dt$$

By performing the change of variable $t = h(S_X(x))$, we obtain:

$$V_{u,h}(X) = \int_{+\infty}^{-\infty} u(x))dh(S_X(x)) = \int_{+\infty}^{-\infty} u(x)h'(S_X(x))(-f_X(x))dx$$

Thus:

$$V_{u,h}(X) = E[u(X)h'(S_X(X))]$$

A.2 Proof of Lemma 2

As before, we use the quantile representation of the Choquet integral:

$$V_{u,h}(X + \beta N) = \int_0^1 u(S^{-1}_{X+\beta N}(h^{-1}(t)))dt = \int_0^1 u(S^{-1}_{X+\beta N}(s))dh(s)$$

Assuming continuity of conditional densities, Tasche (2000) has shown that (Appendix B):

$$\frac{\partial}{\partial\beta} S^{-1}_{X+\beta N}(s) = E[N|X + \beta N = S^{-1}_{X+\beta N}(s)]$$

Thus, the derivative of $V_{u,h}(X+\beta N)$ with respect to β is:

$$\begin{gathered}
\tfrac{\partial}{\partial\beta}V_{u,h}(X+\beta N)= \\
\int_0^1 u'(S^{-1}_{X+\beta N}(s))\tfrac{\partial}{\partial\beta}S^{-1}_{X+\beta N}(s)dh(s)= \\
\int_0^1 u'(S^{-1}_{X+\beta N}(s))E[N|X+\beta N=S^{-1}_{X+\beta N}(s)]dh(s)\overset{(S^{-1}_{X+\beta N}(s)=y)}{=} \\
\int_{+\infty}^{-\infty} u'(y)E[N|X+\beta N=y]dh(S_{X+\beta N}(y))= \\
\int_{+\infty}^{-\infty} u'(y)\left(\int_{-\infty}^{+\infty} n\tfrac{f_{N,X+\beta N}(n,y)}{f_{X+\beta N}(y)}dn\right)h'(S_{X+\beta N}(y))(-f_{X+\beta N}(y))dy= \\
\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} nu'(y)h'(S_{X+\beta N}(y))f_{N,X+\beta N}(n,y)dndy
\end{gathered}$$

So, we finally obtain:

$$\frac{\partial}{\partial\beta}V_{u,h}(X+\beta N)=E[Nu'(X+\beta N)h'(S_{X+\beta N}(X+\beta N))]$$

A.3 Proof of Proposition 1

The proof of Proposition 1 is a simple application of the techniques utilised in the proofs of the auxiliary lemmas. We begin with equation (78):

$$V_C(Z-\beta N+\pi(\beta N)))=V_C(Z),$$

which we differentiate with respect to β:

$$\frac{\partial}{\partial\beta}V_C(Z-\beta N+\pi(\beta N)))=0$$

The left hand side of the above equation is:

$$\begin{gathered}
\tfrac{\partial}{\partial\beta}V_C(Z-\beta N+\pi(\beta N)))= \\
\tfrac{\partial}{\partial\beta}\int_0^1 u_C(S^{-1}_{Z-\beta N}(t)+\pi(\beta N))dh(t)= \\
\int_0^1 u_C'(S^{-1}_{Z-\beta N}(t)+\pi(\beta N))\tfrac{\partial}{\partial\beta}(S^{-1}_{Z-\beta N}(t)+\pi(\beta N))dh(t)= \\
\int_0^1 u_C'(S^{-1}_{Z-\beta N}(t)+\pi(\beta N))\tfrac{\partial}{\partial\beta}(S^{-1}_{Z-\beta N}(t)+\pi(\beta N))dh(t)= \\
\int_0^1 u_C'(S^{-1}_{Z-\beta N}(t)+\pi(\beta N))(E[-N|Z-\beta N=S^{-1}_{Z-\beta N}(t)]+\tfrac{\partial}{\partial\beta}\pi(\beta N))dh(t)
\end{gathered}$$

Thus it is:

$$\frac{\partial}{\partial\beta}V_C(Z-\beta N+\pi(\beta N)))=0\Rightarrow$$

$$\begin{gathered}
\int_0^1 u_C'(S^{-1}_{Z-\beta N}(t)+\pi(\beta N))E[N|Z-\beta N=S^{-1}_{Z-\beta N}(t)]dh(t)= \\
=\int_0^1 u_C'(S^{-1}_{Z-\beta N}(t)+\pi(\beta N))\tfrac{\partial}{\partial\beta}\pi(\beta N)dh(t)\qquad\Rightarrow
\end{gathered}$$

$$\begin{gathered}
E[Nu'(Z+\pi(\beta N))h_C'(S_{Z-\beta N}(Z-\beta N))]= \\
=\tfrac{\partial\pi(\beta N)}{\partial\beta}E[u_C'(Z+\pi(\beta N))h_C'(S_{Z-\beta N}(Z-\beta N))]\qquad\Rightarrow \\
\frac{\partial\pi(\beta N)}{\partial\beta}=\frac{E[Nu'(Z+\pi(\beta N))h_C'(S_{Z-\beta N}(Z-\beta N))]}{E[u_C'(Z+\pi(\beta N))h_C'(S_{Z-\beta N}(Z-\beta N))]}
\end{gathered}$$

Finally, taking $\beta = 0$ yields the marginal cost price mechanism:

$$MC(N) = \frac{\partial \pi(\beta N)}{\partial \beta}_{|\beta=0} = E\left[N \frac{u_C'(Z))h_C'(S_Z(Z))}{E[u_C'(Z)h_C'(S_Z(Z))]}\right]$$

B Quantile derivatives

This section follows Tasche (2000b). Let X be a real valued random variable. For $a \in (0,1)$ the a-quantile of X, $Q_a(X)$ is defined as:

$$Q_a(X) = \inf\{x \in \mathbb{R} | \mathbb{P}_0(X \le x) \ge a\}$$

Now let

$$Z^u = \sum_j u_j X_j$$

be a portfolio consisting of random liabilities X_j, $j = 1, 2, ..., n$. We are interested in derivatives of the a-quantile of Z^u, with respect to the portfolio weights u_j, i.e. in expressions of the form:

$$\frac{\partial Q_a(Z^u)}{\partial u_i}$$

Such 'quantile derivatives' exist, subject to a set of technical assumptions. Let $n \ge 2$ and $(X_1, ..., X_n)$ be an $\mathbb{R}^n$-valued vector with a conditional density ϕ of X_1 given $(X_2, ..., X_n)$. ϕ satisfies the assumptions in an open set $U \subset \mathbb{R} \setminus \{0\} \times \mathbb{R}^{d-1}$ if:

(i) For fixed $x_2, ..., x_n$, the function $t \mapsto \phi(t, x_2, ..., x_n)$ is continuous in t.

(ii) The mapping

$$(t, u) \mapsto E[\phi(u_1^{-1}(t - \sum_{j=2}^{n} u_j X_j), X_2, ..., X_n)],$$
$$\mathbb{R} \times U \mapsto [0, \infty)$$

is finite-valued and continuous.

(iii) For each $i = 2, ..., n$ the mapping

$$(t, u) \mapsto E[X_i \phi(u_1^{-1}(t - \sum_{j=2}^{n} u_j X_j), X_2, ..., X_n)],$$
$$\mathbb{R} \times U \mapsto \mathbb{R}$$

is finite-valued and continuous.

If ϕ satisfies the above assumptions in some open set $U \subset \mathbb{R}$ $\{0\} \times \mathbb{R}^{d-1}$, the quantile derivative exists and is given by:

$$\frac{\partial Q_a(Z^u)}{\partial u_i} = E[X_i | \sum_j u_j X_j = Q_a(Z^u)]$$