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Multiple Diffractions of Elastic Shear Waves by a Rigid Rectangular Foundation Embedded in an Elastic Half Space[1]

A rigid rectangular foundation, embedded in an elastic half space, is subjected to a plane, transient, horizontally polarized shear (SH) wave. Embedment depth of the foundation and the angle of the incidence of the plane wave are assumed to be arbitrary. The problem considered is of the antiplane-strain type. The Laplace and Kontorovich-Lebedev transforms are employed to derive the equation of motion for the foundation during the period of time required for an SH-wave to traverse the base width of the obstacle twice. Therefore this solution includes the process of multiple diffractions at the corners of the foundation.

Introduction

The problems of soil structure interaction have been attracting many researchers lately. However, for the case of buried foundations there are still only few theoretical results available. A detailed review of these results is given in reference [1].[3]

Thau and Umek [2] derived a transient response of a rigid rectangular foundation embedded in an elastic half space and subjected to an incident plane SH-wave. However, the response calculated in [2] was exact only within the first unit of time. In other words, the process of multiple diffractions at the corners of the foundation has been neglected. (A unit of time is defined as the interval of time required for a SH-wave to traverse the base of the foundation once.) A question arose as to whether the solution of [2] can be used as an approximate one beyond the time where it is exact. This question is the topic discussed in this paper.

In the subsequent analysis one could distinguish two types of solutions: a solution that was exact from zero up to one unit of time, called the *zeroth-order solution*, and a solution that was exact from zero up to two units of time, called the *first-order solution*. The difference between these two was found to be relatively small. Both solutions approached the predictable long-time limits for the incident wave field in the form of the Dirac delta function and the Heaviside step function. Therefore, by neglecting the multiple diffractions of elastic waves at the corners of the foundation it was found that the corresponding error for the response was relatively small.

Throughout the analysis, the method of integral transforms was employed. In particular, the Laplace transform and the Kontorovich-Lebedev (K-L) transform [3, 4] were used.

Statement of Problem

The problem model, shown in Fig. 1, consists of a rigid foundation block which extends to infinity along the z-axis. It is embedded in a homogenous isotropic elastic half space. An incident plane SH-wave, which propagates at an angle α with respect to positive x-axis, strikes the foundation and causes it to move. The problem just presented is of the antiplane-strain type in which the displacement field is given by $u_x = u_y = 0$ and $u_z = w(x, y, t)$. The incident wave field is expressed as

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[1] This work was supported by the RANN Division of the National Science Foundation through a Grant GI-34781 to the Illinois Institute of Technology.

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[3] Numbers in brackets designate References at the end of paper.

Contributed by the Applied Mechanics Division and presented at the Applied Mechanics National Conference, University of Utah, Salt Lake City, Utah, June 15-17, 1976, of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS.

Discussion on this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N. Y. 10017, and will be accepted until September 1, 1976. Readers who need more time to prepare a discussion should request an extension of the deadline from the Editorial Department. Manuscript received by ASME Applied Mechanics Division, August, 1975; final revision, December, 1975. Paper No. 76-APM-37.

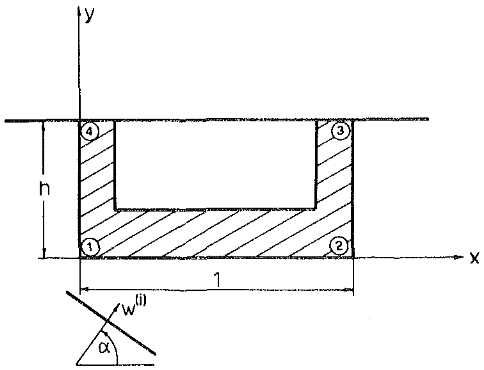

Fig. 1 Problem geometry

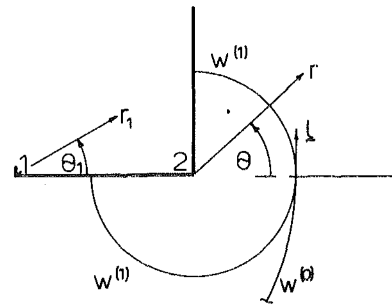

Fig. 2 Primary and secondary waves at corner 2

$$w^{(i)} = 1/2 \cdot f(t - x \cos \alpha - y \sin \alpha); \qquad 0 \le \alpha \le \pi/2, \tag{1}$$

where f is a causal function.

After the introduction of dimensionless variables, wave motions in the half space are governed by

$$\nabla^2 w - \ddot{w} = 0, \qquad \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2, \tag{2}$$

where $w(x, y, t)$ represents the total displacement field in the z-direction, which consists of incident and scattered wave fields, i.e., $w = w^{(i)} + w^{(s)}$. The nonvanishing stress field is given by $\sigma_{xz} = \partial w/\partial x$, $\sigma_{yz} = \partial w/\partial y$. Since the surface of the half space is stress-free, this implies $\partial w(x, h, t)/\partial y = 0$. Along the interface between the foundation and elastic media one requires perfect bonding, i.e., $w = W(t)$, where W is the resultant rigid-body motion of the foundation caused by the incident wave. The initial conditions are given by $w^{(s)}(x, y, 0) = \dot{w}^{(s)}(x, y, 0) = 0$. To insure the unique solution, it is required that the scattered wave field $w^{(s)}$ represents outward traveling waves and that the displacement field w remains finite at the corners of the foundation. It was shown in reference [2] that the foregoing problem can be simplified by introducing an equivalent, full-space model. This is accomplished by adding its mirror picture to the original model, Fig. 1, about $y = h$.

Solution of Problem

Throughout the analysis, a method used by Thau will be followed [5]. He showed that the problem of scattering of elastic waves by a rigid obstacle can be separated in two physically meaningful subproblems:

1 The *radiation* of waves by the foundation vibrating as a rigid body with $W(t)$.

2 The *diffraction* of incident waves by an immobile foundation.

Therefore, the boundary condition in the radiation subproblem becomes $w^{(r)} = W(t)$ along the foundation boundary, where $w^{(r)}$ represents outgoing waves. In the diffraction subproblem the boundary condition along the foundation is given by $w^{(i)} + w^{(i')} + w^{(d)} = 0$, where the prime denotes the image incident wave in the equivalent problem model and $w^{(d)}$ again represents outgoing waves. The total solution of the original problem is given as a sum $w^{(i)} + w^{(i')} + w^{(r)} + w^{(d)}$.

In order to solve the problem, the Laplace transform in time is applied first to the governing equations defined by

$$\bar{f}(s) = \int_0^\infty f(t) e^{-st} dt. \tag{3}$$

Then, the K-L transform is applied to the variable r of the plane polar coordinate system (r, θ) according to

$$\bar{f}^*(\nu) = \int_0^\infty r^{-1} \bar{f}(r) K_{i\nu}(sr) dr. \tag{4a}$$

The inverse K-L transform is given by

$$\bar{f}(r) = 2\pi^{-2} \int_0^\infty \nu \sinh(\pi\nu) \bar{f}^*(\nu) K_{i\nu}(sr) d\nu, \tag{4b}$$

where $K_{i\nu}(sr)$ is the Macdonald function. A useful integral representation of this function for applying and inverting the K-L transform is given by

$$K_{i\nu(sr)} = \int_0^\infty e^{-sr \cosh z} \cos \nu z dz. \tag{5}$$

Several results from reference [2] will be used in an analysis of the first-order response. They are listed as follows: Primary cylindrical SH-waves (created originally at the corners of the foundation) at corner 1 are given by

$$\bar{w}^{(r,0)*} = -A(\lambda) \bar{W}(s) \sinh(\pi\lambda/4) \operatorname{sech}(3\pi\lambda/4) \times \begin{cases} \sinh \lambda(\theta_1 - \pi/2) \\ \sinh \lambda(2\pi - \theta_1) \end{cases} \tag{6}$$

for $\pi/2 \le \theta_1 < \pi$ and $3\pi/2 < \theta_1 \le 2\pi$, respectively;

$$\bar{w}^{(d,0)*} = \bar{f}(s) A(\lambda) \sinh(\pi\lambda/2) \operatorname{csch}(3\pi\lambda/2) \times \begin{cases} \sinh \lambda(\alpha + \pi/2) \sinh \lambda(\theta_1 - \pi/2) \\ \sinh \lambda(\pi - \alpha) \sinh \lambda(2\pi - \theta_1) \end{cases} \tag{7}$$

for $\pi/2 \le \theta_1 < \pi - \alpha$ and $2\pi - \alpha < \theta_1 \le 2\pi$, respectively, where the superscripts r and d denote radiation and diffraction subproblems, respectively; θ_1 is specified in Fig. 2 and

$$A(\lambda) = \pi\lambda^{-1} \operatorname{csch}(\pi\lambda); \tag{8}$$

the force due to the primary cylindrical waves and the plane waves in radiation subproblem is given by

$$\bar{F}^{(r,0)} = (K - Cs)\bar{W}(s); \qquad C = 1 + 2h, \tag{9a}$$

where K is the known constant; furthermore, the force due to the primary cylindrical and plane waves in diffraction subproblem is given by

$$\bar{F}^{(d,0)} = \bar{f}(s)\,[g_1(\alpha) + e^{-s \cos \alpha} g_2(\alpha) + e^{-s2h \sin \alpha} g_3(\alpha)], \tag{10}$$

where $g_i(\alpha)$ are known.

To insure that no primary cylindrical wave from one corner arrives at an adjacent corner before time $t = 1$, it was assumed that embedment depth $h \ge 1/2$.

In order to find the first-order solution to the problem, one must consider a phenomenon of rediffraction of primary cylindrical waves at the corners of the foundation. Indeed, when a primary cylindrical wave from one corner strikes another corner, a new wave, which is denoted as a *secondary* one, is created. Therefore, one needs the relations between the primary and secondary cylindrical waves. It is sufficient to consider a primary cylindrical wave $w^{(0)}$ from corner 1 which strikes corner 2, thus causing a secondary cylindrical wave $w^{(1)}$ to emanate from corner 2. Within the interval of time $1 < t < 2$, the situation is presented in Fig. 2. Then, for a total wave field $w(r, \theta)$ the following conditions should be satisfied:

$$w(r, 0+) = w(r, 0-) \tag{11a}$$

$$\partial w(r, 0+)/\partial l = \partial w(r, 0-)/\partial l, \tag{11b}$$

where

$$w(r, 0+) = w^{(1)}(r, 0) \tag{12a}$$

$$w(r, 0-) = w^{(1)}(r, 0) + w^{(0)}(r_1, 2\pi). \tag{12b}$$

In the foregoing equations, (r_1, θ_1) and (r, θ) represent plane polar coordinate systems placed at corners 1 and 2, respectively, while $\mathbf{l}$ is a unit vector tangent to both primary and secondary cylindrical waves.

Radiation Subproblem

In the radiation subproblem, one encounters the phenomenon of diffraction of elastic waves as follows: Due to the motion of the foundation a primary cylindrical SH-wave, together with two plane SH-waves, will emanate from each corner of the foundation. At time $t = 1$, the primary cylindrical SH-wave will first strike an adjacent corner and rediffract a secondary cylindrical SH-wave. Owing to the symmetry of the radiation subproblem it is sufficient to consider the case shown in Fig. 2. Thus the objective is to determine the secondary cylindrical SH-wave at corner 2.

It can be seen from (6) that a primary cylindrical wave satisfies homogenous boundary conditions along the foundation sides $\theta_1 = \pi/2$ and $\theta_1 = 2\pi$. The nonhomogenous part of the boundary condition is taken care of by the plane waves. Therefore, the secondary cylindrical wave at corner 2 has to satisfy the homogenous boundary conditions $w^{(r,1)} = 0$ along $\theta = \pi/2$ and $\theta = -\pi$ (see Fig. 2). Furthermore, it satisfies the wave equation (2) which in the Laplace-K-L domain becomes $\bar{w}_{\theta\theta}{}^{(r,1)*} - \nu^2\bar{w}^{(r,1)*} = 0$, where $(\cdot)_\theta \equiv \partial(\cdot)/\partial\theta$. Thus one can assume the equation of the secondary cylindrical SH-wave at corner 2, due to the primary cylindrical SH-wave from corner 1, to be

$$\bar{w}^{(r,1)*} = A(\nu)\bar{W}(s) \begin{cases} E(\nu) \sinh \nu(\pi/2 - \theta) \sinh (\pi\nu) \\ F(\nu) \sinh \nu(\pi + \theta) \sinh (\pi\nu/2) \end{cases} \tag{13}$$

for $0 \leq \theta \leq \pi/2$ and $-\pi \leq \theta \leq 0$, respectively. Here $A(\cdot)$ is defined by (8), while E and F are functions yet to be determined. By substituting (13) and (6) into condition ($11a$) one obtains

$$E(\nu) = F(\nu). \tag{14}$$

From condition ($11b$) it follows then

$$\bar{w}_\theta{}^{(r,1)}(r, 0+) - \bar{w}_\theta{}^{(r,1)}(r, 0-) = r(r+1)^{-1}\,\bar{w}_{\theta_1}{}^{(r,0)}(r+1, 2\pi). \tag{15}$$

By employing the equations of primary and secondary cylindrical SH-waves (6) and (13), together with the result (14) and the inverse K-L transform, the last equation implies

$$-\int_0^\infty E(\nu)\sinh(3\pi\nu/2)\,K_{i\nu}(sr)d\nu = r(r+1)^{-1}\int_0^\infty \lambda \sinh(\pi\lambda/4) \times \operatorname{sech}(3\pi\lambda/4)\,K_{i\nu}(s(r+1))d\lambda. \tag{16}$$

Denoting the RHS of (16) as $f_1(r)$ and writing

$$E(\nu)\sinh(3\pi\nu/2) = 2\pi^{-2}\,\tilde{E}(\nu)\sinh(\pi\nu), \tag{17}$$

it follows from (16) that the unknown function $\tilde{E}(\nu)$ is given by

$$\tilde{E}(\nu) = -f_1{}^*(\nu). \tag{18}$$

Thus, by taking the K-L transform of the RHS of (16) and using (5), (17), and (18), it follows that the unknown function $E(\nu)$ is given by

$$E(\nu) = -2(3\pi i)^{-1}\operatorname{csch}(3\pi\nu/2)\int_0^\infty e^{-s\cosh z}\sin(\nu z)\operatorname{sech}(2z/3 - i\pi/6)dz. \tag{19}$$

Therefore, from the aforementioned equation, together with (13) and (14), the equation of the secondary cylindrical SH-wave at corner 2 caused by rediffraction of the primary cylindrical SH-wave (6) is known.

In order to obtain the force at corner 2, due to the secondary cylindrical waves, it is necessary to integrate the stress field, corresponding to the displacement field (13), along the sides of the foundation, Fig. 2. However, by introducing an infinite wedge instead of finite one at corner 2, one can see that the integration of the stress field would yield an exact value for the force only for the period of time up to $t = 2$. For time $t > 2$, the expression for the force would not be exact anymore, but, in any case, the solution which is under consideration is exact up to $t = 2$. Now, the force at corner 2 due to the secondary wave (9) can be expressed as

$$\bar{F}^{(r,1)} = \int_0^\infty r^{-1}\,[\bar{w}_\theta{}^{(r,1)}(r, -\pi) - \bar{w}_\theta{}^{(r,1)}(r, \pi/2)]dr. \tag{20}$$

By substituting (15) and (10) into (9) and then applying the inverse K-L transform, it follows from the last equation

$$\bar{F}^{(r,1)} = -8\bar{W}(9\pi)^{-1}\int_0^\infty e^{-s\cosh z}G_0(z)R(z)dz, \tag{21}$$

where G_0 and R are defined by

$$G_0(z) = \sinh(2z/3)/[1 + 2\cosh(4z/3)] \tag{22a}$$

$$R(z) = \tanh(z/3)\,[5 + 6\cosh(2z/3)]/[1 + 2\cosh(2z/3)]. \tag{22b}$$

However, in the process of calculating the force due to the primary radiated cylindrical waves, the same method of introducing infinite wedges for calculation of the force has been used in reference [2]. This means that the force due to the primary radiated cylindrical waves calculated in [2] is exact only up to $t = 1$. In order to use these results in the current analysis of the first-order response, a correction to the force due to the primary cylindrical SH-waves has to be introduced to make all quantities exact at least up to $t = 2$. To explain the process in more detail, it is sufficient to consider the situation of a primary radiated cylindrical wave at corner 1 that strikes corner 2, thus creating a new, secondary cylindrical wave, Fig. 2. The primary wave contributes to the force on the foundation, only while traveling the distance from corner 1 to corner 2, and so it does not contribute to the force exerted on the foundation any further. In calculating the force due to the primary cylindrical wave, an infinite wedge has been introduced in [2]. So, by following the example in Fig. 2, one can see that the integration of the stress along the base of the foundation yields the zeroth-order force in [2] for time $t > 1$ as if the base were of infinite width. Therefore, the correction along the base, which it is necessary to subtract from the zeroth-order force, is given by

$$\bar{F}_{\text{corr}}{}^{(r)} = -\int_I^\infty r_1{}^{-1}\,\bar{w}_{\theta_1}{}^{(r,0)}(r_1, 2\pi)dr_1. \tag{23a}$$

Using the equation of primary cylindrical waves (6), together with (5), and the inverse K-L transform ($4b$), the foregoing equation implies

$$\bar{F}_{\text{corr}}{}^{(r)} = -8\bar{W}(3\pi)^{-1}\int_0^\infty e^{-s\cosh z}\tanh z \cdot G_0(z)dz, \tag{23b}$$

where G_0 has been defined by ($22a$). It was found convenient to combine the correction force ($23b$) together with the force due to the secondary cylindrical wave (21) to obtain

$$\bar{F}_I{}^{(r,1)} = \bar{F}^{(r,1)} - \bar{F}_{\text{corr}}{}^{(r)} = 64\bar{W}(9\pi)^{-1}\int_0^\infty e^{-s\cosh z}G_0{}^2(z)dz. \tag{24a}$$

Table 1 Secondary cylindrical waves at corners 1 and 2

Corner	$w^{(d,0)}$ From corner	$w^{(d,1)}$ Caused by	At time	Event
1	1′	$w^{(i')}$	$2h$	A_1
	2	$w^{(i)}$	$\cos\alpha + 1$	B_1
	1′	$w^{(i)}$	$2h\ (\sin\alpha + 1)$	C_1
2	1	$w^{(i)}$	1	A_2
	1	$w^{(i')}$	$2h\ (\sin\alpha + 1)$	B_2
	2′	$w^{(i')}$	$\cos\alpha + 2h$	C_2

Similarly, the force due to the secondary cylindrical SH-wave at corner 2 caused by rediffraction of the primary cylindrical SH-wave from corner 2′ (mirror corner of 2), combined with the corresponding correction along the vertical wall, is given by

$$\bar{F}_{II}^{(r,1)} = 64\bar{W}(9\pi)^{-1}\int_0^\infty e^{-s2h\cosh z}G_0^2(z)dz. \qquad (24b)$$

By adding $\bar{F}_I^{(r,1)}$ to $\bar{F}_{II}^{(r,1)}$, one obtains the *total force at corner 2*, due to the secondary cylindrical SH-waves which is exact up to time $t = 2$. Owing to the symmetry involved in this subproblem, the same force would apply to the corners 1, 1′, and 2′. Therefore, the force due to the secondary cylindrical waves (with correction to the force due to the primary cylindrical waves) in the radiation subproblem for the actual model is given by

$$\bar{F}_{\text{tot}}^{(r,1)} = 2(\bar{F}_I^{(r,1)} + \bar{F}_{II}^{(r,1)}). \qquad (25)$$

Diffraction Subproblem

As defined previously, in the diffraction subproblem the incident wave field $w^{(i)}$ strikes at the immobile foundation. Due to the symmetry involved in the equivalent model, Fig. 1, the problem can be reduced to more fundamental ones. First of all, it is necessary to take into account all the secondary cylindrical SH-waves. This is presented in Table 1 for corners 1 and 2 only. It turns out that all problems for the secondary cylindrical waves in the problem under consideration can be solved if the solutions for events A_2 and B_2 in Table 1 are known. These will be denoted as the fundamental events. For fundamental event A_2, the equation of the primary cylindrical wave is given by (7). Therefore one can assume the equation of the secondary cylindrical wave at corner 2, Fig. 2, in the form

$$\bar{w}^{(d,1)*} = \bar{f}A(\nu)\begin{cases} E_1(\nu)\sinh\nu(\pi/2-\theta)\sinh(\pi\nu) \\ F_1(\nu)\sinh\nu(\pi+\theta)\sinh(\pi\nu/2)\end{cases} \qquad (26)$$

for $0 \le \theta \le \pi/2$ a and $-\pi \le \theta \le 0$, respectively, with $E_1(\nu)$ and $F_1(\nu)$ yet to be determined. It follows from condition (11*a*) that

$$E_1(\nu) = F_1(\nu). \qquad (27)$$

Condition (11*b*) implies formally the relation (15), with superscript (r) replaced by (d). Following the procedure outlined in the radiation subproblem, it follows that

$$E_1(\nu) = 2(3^{1/2}\pi)^{-1}\operatorname{csch}(3\pi\nu/2)$$
$$\times \int_0^\infty e^{-as\cosh z}G_1(z, \pi - \alpha)\sin(\nu z)dz, \qquad (28a)$$

where it is temporarily assumed that the width of the foundation is set to be equal to "a" (instead of unity) to avoid a repetition of calculations for different events of Table 1. In the foregoing equation, G_1 is defined by

$$G_{1,2}(z,\beta) = [\sinh(4z/3)\sin(2\beta/3) \mp \sinh(2z/3)\sin(4\beta/3)]$$
$$\times[\cosh(2z) \mp \cos(2\beta)]^{-1}. \qquad (28b)$$

Then the force due to the secondary cylindrical wave in event A_2 is given by

$$\bar{F}_{A_2}^{(d,1)}(a,\alpha) = 4\bar{f}(3^{3/2}\pi)^{-1}\int_0^\infty e^{-as\cosh z}G_1(z, \pi-\alpha)R_1(z), \qquad (29a)$$

where $R_1(z)$ is defined by

$$R_1(z) = [\cosh(2z) + \cosh^2 z\text{-}2\cosh(4z/3)]\operatorname{csch}(2z). \qquad (29b)$$

For the same reason as in the radiation subproblem, it is necessary to introduce a correction to the force due to the primary cylindrical wave for event A_2, which turns out to be

$$\bar{F}_{A_2}^{\text{corr}}(a,\alpha) = 2\bar{f}(3^{1/2}\pi)^{-1}$$
$$\times\int_0^\infty e^{-as\cosh z}\tanh z\cdot G_1(z,\pi-\alpha)dz. \qquad (30)$$

A similar procedure yields the force due to the secondary cylindrical wave for event B_2 and the corresponding correction force as

$$\bar{F}_{B_2}^{(d,1)}(a,\alpha) = 4\bar{f}(3^{3/2}\pi)^{-1}$$
$$\times\int_0^\infty e^{-as\cosh z}\cdot G_2(z,\pi/2-\alpha)R_1(z)dz \qquad (31a)$$

$$\bar{F}_{B_2}^{\text{corr}}(a,\alpha) = 2\bar{f}(3^{1/2}\pi)^{-1}$$
$$\times\int_0^\infty e^{-as\cosh z}\tanh z\cdot G_2(z,\pi/2-\alpha)dz, \qquad (31b)$$

where G_2 has been defined by (28*b*). Again, as in the radiation subproblem, it was found convenient to combine the force due to the secondary cylindrical wave with the correction force. Therefore for event A_1 one associates a pair of forces

$$\bar{F}_{A_1} = \bar{F}_{A_2}^{(d,1)}(2h, \pi/2-\alpha) - \bar{F}_{A_2}^{\text{corr}}(2h,\pi/2-\alpha). \qquad (32a)$$

Similarly, for the events B_1 and C_1, it follows that

$$\bar{F}_{B_1} = e^{-s\cos\alpha}[\bar{F}_{A_2}^{(d,1)}(1,\pi-\alpha) - \bar{F}_{A_2}^{\text{corr}}(1,\pi-\alpha)] \qquad (32b)$$

$$\bar{F}_{C_1} = e^{-s2h\sin\alpha}[\bar{F}_{A_2}^{(d,1)}(2h,\pi/2+\alpha)$$
$$- \bar{F}_{A_2}^{\text{corr}}(2h,\pi/2+\alpha)]. \qquad (32c)$$

Therefore, the total corrected force at corner 1 due to the secondary cylindrical SH-waves is given as a sum

$$\bar{F}_1^{(d,1)} = \bar{F}_{A_1} + \bar{F}_{B_1} + \bar{F}_{C_1}. \qquad (33)$$

Analogously, the total first-order force at corner 2 is given as

$$\bar{F}_2^{(d,1)} = \bar{F}_{A_2} + \bar{F}_{B_2} + \bar{F}_{C_2}, \qquad (34)$$

where

$$\bar{F}_{A_2} = \bar{F}_{A_2}^{(d,1)}(1,\alpha) - \bar{F}_{A_2}^{\text{corr}}(1,\alpha)$$
$$\bar{F}_{B_2} = e^{-s2h\sin\alpha}[\bar{F}_{B_2}^{(d,1)}(1,\alpha) - \bar{F}_{B_2}^{\text{corr}}(1,\alpha)]$$
$$\bar{F}_{C_2} = e^{-s\cos\alpha}[\bar{F}_{B_2}^{(d,1)}(2h,\pi/2-\alpha)$$
$$- \bar{F}_{B_2}^{\text{corr}}(2h,\pi/2-\alpha)]. \qquad (35)$$

Owing to the symmetry involved in the diffraction subproblem, it is sufficient to know $\bar{F}_1^{(d,1)}$ and $\bar{F}_2^{(d,1)}$ in order to find the *total force* due to the secondary cylindrical waves

$$\bar{F}^{(d,1)} = \bar{F}_1^{(d,1)} + \bar{F}_2^{(d,1)}. \qquad (36)$$

When all the forces in diffraction and radiation subproblems are known, one can proceed with the formulation of the equation of motion, which is obtained by equating the total force exerted on the foundation, given as a sum of (23), (9*a*), (36), and (10), to its intertia force. Therefore, in the Laplace transform domain

$$(ms^2 + Cs + K - \mathcal{F}_{\text{tot}}^{(r,1)})\bar{W} = (\mathcal{F}^{(d,0)} + \mathcal{F}^{(d,1)})\bar{f}, \qquad (37)$$

where $\mathcal{F}$ is normalized force (with respect $\bar{W}$ or $\bar{f}$). Therefore, the first-order response of the foundation is given by

$$\bar{W}^{(1)} = \bar{W}^{(0)} + [\mathcal{F}^{(d,1)}\bar{q}_1(s) + \mathcal{F}^{(d,0)}\mathcal{F}^{(r,1)}\bar{q}_2(s)]\bar{f}, \qquad (38)$$

where $\bar{q}_1 = (ms^2 + Cs + K)^{-1}$, $\bar{q}_2 = \bar{q}_1^2$, and $\bar{W}^{(0)} = \mathcal{F}^{(d,0)}\bar{q}_1\bar{f}$.

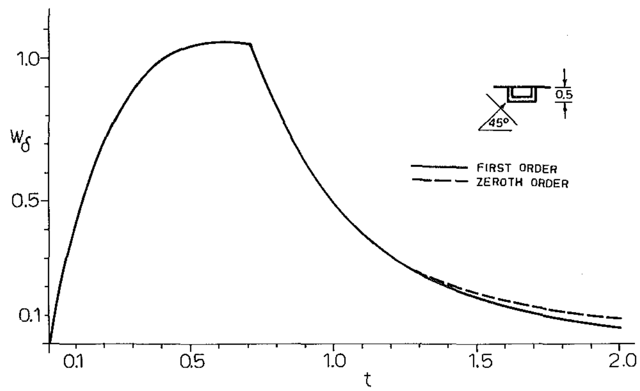

Fig. 3(a) Delta-function response

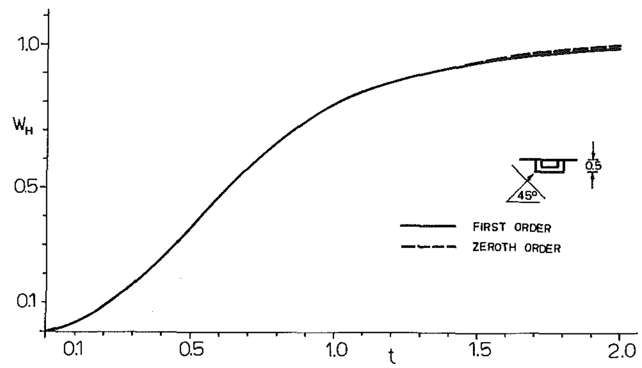

Fig. 3(b) Step-function response

Fig. 3 Responses of foundation

Hence, the first-order solution $\bar{W}^{(1)}$ is given as a sum of the zeroth-order solution $\bar{W}^{(0)}$ and the terms that account for the process of rediffraction at the corners of the foundation.

Numerical Results

Based on the inverse Laplace transform of (38), numerical results are evaluated for mass $m = 0.5$, $\alpha = \pi/2$ and $h = 0.5$. For a more extensive numerical evaluation see [1]. Fig. (3a) represents the delta-function response ($f(t) = \delta(t)$). The dashed line represents the zeroth-order solution, while the solid line denotes the first-order response. One can see that the difference between those two is relatively small. In addition, both solutions approach the physically predictable long time limit, i.e., zero. This suggests the zeroth-order solution as a very good approximation of the first-order response.

The step-function response ($f(t) = H(t)$) of the results of (38) are presented in Fig. 3(b). Again, the difference between the first and the zeroth-order response is very small. By application of the Tauberian theorem [4], one can compute the exact value of the long time limits for the zeroth and the first-order responses. It was found that $\lim_{t\to\infty} W_H^{(0)}(t) = 1.107$ and $\lim_{t\to\infty} \bar{W}_H^{(1)}(t) = 1.065$, where the exact value of the physically predictable long time limit should be equal to one. Thus, although the first-order solution yields a better approximation than the zeroth-order one, the much simpler analysis involved in the latter one should be taken into account.

Conclusions

The results obtained for the response of the buried foundation to an incident plane SH-wave are exact up to two units of time. It turns out that the solution is an approximate one beyond $t = 2$. Indeed, the first-order response, as well as the zeroth-order response, approaches the physically predictable long time limits. For example, in the case of the step function response, the incident wave field (1) will cause the entire half space, together with the foundation, to move with unit displacement as a rigid body as $t \to \infty$. Or if one associates the step-function response with velocity, then it follows that the whole half space will move with unit velocity as a rigid body as $t \to \infty$. This implies that the corresponding delta-function response, or acceleration, will tend to zero as $t \to \infty$. The small difference demonstrated between the zeroth and the first-order response suggests the zeroth-order response as an approximation of the exact solution, thus reducing the complexity of the analysis greatly. Once the delta-function response is known, one can use it as a Green's function to obtain the response for an arbitrary profile of the incident wave field (1).

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