Transient Heat Transfer between a Plate and a Fluid whose Temperature Varies Periodically with Time

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Using the method of complex temperature in conjunction with the Laplace transformation, an exact analytical solution is found for the transient, conjugate, forced convection problem consisting of a plate, whose base is insulated, interacting with a fluid, moving in a steady slug fashion, whose temperature, at points far from the plate, varies sinusoidally with time. Simple quasi-steady results are derived for comparison. Also presented is a method for determining the qualitative conditions under which one might expect a quasi-steady analysis to be valid in a general problem.

Introduction

The problem of predicting the surface heat flux and surface temperature, as a function of position and time, of a body subjected to a flowing fluid whose temperature, far away from the body, varies periodically with time, has a number of possible applications. For instance, gas turbine blades and vanes must interact with a periodically varying gas temperature as a consequence of processes in the combustor. The thermal response of coatings to this periodic thermal loading may be a factor in coating life. In addition, the storage material in a parallel flow regenerative heat exchanger is subject to a periodically varying fluid inlet temperature, both during the transient start-up and in the ultimate cyclic, or periodic, unsteady state.

Willmott and Burns [1] use finite difference methods to solve the classical regenerator equations for the transient response of a regenerator operating initially in the periodic unsteady state when, suddenly, a step change in either inlet gas temperature or mass flow rate initiates a transient that causes the regenerator to go toward a new periodic unsteady state. Their analysis, in common with most analyses performed specifically for regenerators, employs the quasi-steady assumption which uses, in this case, a constant, both in space and time, surface coefficient of heat transfer. In [2], Kardas solves the case of the unidirectional regenerator with a sinusoidal or cosinusoidal inlet temperature. However, the quasi-steady approximation is employed and the results are only for the ultimate periodic unsteady state and not the transient leading up to this state. Chase, et al. [3] relax the quasi-steady assumption and use the slug flow approximation in a Laplace transform solution for the regenerator wall and bulk mean temperature after a step change in the inlet temperature to a value which is held constant in time thereafter. Sparrow and DeFarias [4] solve, by the method of complex temperature, the problem of slug flow in a parallel plate duct, with both outer plate surfaces insulated, when the fluid inlet temperature varies in a sinusoidal fashion. Their problem, like those of [1–3] is a conjugate one, that is, the temperature distribution within the moving fluid and in the solid wall are mutually coupled. The results of [4] are, however, limited to the periodic unsteady state. In [5] the author presents analytical results for the case of slug flow over a plate, cooled from below, when the fluid flows over it in a slug fashion and the inlet temperature is either a step or ramp function of time. Also presented in [5] is an approximate method for arbitrary fluid inlet temperature variation with time. (Discussion of other works in transient forced convection, both of the conjugated and non-conjugated type, which bear a peripheral relationship to the present work, is given in [5].)

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The present work considers a flat plate which is insulated on its lower surface and has a fluid flowing in a steady, laminar, slug fashion over its top surface. Initially both the plate and the fluid are at a constant temperature when, suddenly, at time $t = 0$, the fluid temperature at the leading edge varies sinusoidally with time. It is required to predict both the fluid and wall temperature distributions during the initial transient as well as in the ultimate periodic unsteady state. Use of the slug velocity profile allows an analytic solution to be obtained which, it is felt, displays the essential features to be expected in the actual case in which there is a two-dimensional velocity distribution. Numerical values will differ between the slug and non-slug flow cases, but, as evidenced by the work in [12], the major trends and insights will remain basically unchanged. With this in mind, the physical situation considered here can be used as a first model of an uncooled turbine vane or a thermal entrance region of a parallel flow regenerator. Response functions are presented for the local wall temperature and the local surface heat flux as well as for the local bulk mean temperature of the fluid in the case of the thermal entrance region. Also presented is a method whereby one can determine the qualitative conditions under which one would reasonably expect any quasi-steady solution to yield good results in a general transient forced convection problem.

Theoretically, one should be able to solve the problem at hand by use of Duhamel's Theorem coupled with the results presented in [5]. An attempt to do this, however, did not lead to a solution in terms of elementary (tabulated) functions. Thus it was decided to attempt a solution by the method of complex temperature [6] usually employed, insofar as the author is aware, to find only the eventual periodic unsteady state. In this work it is also employed successfully to find the transient leading up to the periodic unsteady state and to solve for the transient between one periodic unsteady state and a new periodic unsteady state due to a sudden change in the amplitude of the sinusoidal driving function. The linearity of the governing equations for the problem allows the results to be used, via harmonic analysis as in [2], to generate the response to more general periodic fluid temperature variations.

Analysis

Consider steady, laminar, constant property, low speed, two-dimensional planar, thin boundary layer type flow over a flat plate of thickness b units with a perfectly insulated lower surface. Both the plate and the moving fluid have an initial temperature excess of zero when a transient is initiated by a sudden change in the temperature excess of the fluid to $\Delta T \sin\omega t$ at the plate leading edge. Axial conduction in the fluid is neglected, by virtue of a large enough Peclet number, and is also neglected in the plate, as well, due to its thinness. It is assumed that the thermophysical properties of the plate are constant and that the local Biot number is small enough to allow one to lump the plate's temperature in the direction perpendicular to the plate. Lastly, a slug flow idealization of the actual velocity field will be utilized.

An energy balance on a volume of the plate, differential in extent to the x direction, yields, after invoking the conjugation condition, that is, equality of the local fluid and plate temperature at their common interface, and using the lumped condition for the plate,

$$t > 0, y = 0, x > 0 \quad \frac{\partial \theta}{\partial y} = r \frac{\partial \theta}{\partial t}. \tag{1}$$

The remainder of the mathematical description of the problem to be solved is as follows.

$$\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} = \alpha \frac{\partial^2 \theta}{\partial y^2} \tag{2}$$

$$t = 0, x > 0, y > 0 \quad \theta = 0 \tag{3}$$

$$t > 0, x = 0, y > 0 \quad \theta = \Delta T \sin\omega t \tag{4}$$

$$t > 0, x > 0, y \rightarrow \infty \quad \theta \text{ remains finite} \tag{5}$$

It is proposed to solve equations (1–5) by use of the method of complex temperature [6]. Thus, one defines

$$\phi = \theta^* + i\theta \tag{6}$$

where θ^* is the solution to equations (1–5) for a cosine thermal disturbance function at the inlet. θ itself is eventually found from

$$\theta = Im\,\{\phi\} \tag{7}$$

Since both the transient response and the ultimate periodic unsteady state are being sought, proper cognizance must be given to the initial condition (3) in the formulation of the problem in terms of ϕ. Using (6) and Euler's expansion formula, one arrives at the problem statement in terms of ϕ.

$$\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} = \alpha \frac{\partial^2 \phi}{\partial y^2} \tag{8}$$

$$t = 0, x > 0, y > 0 \quad \phi = 0 \tag{9}$$

$$t > 0, x = 0, y > 0 \quad \phi = \Delta T\, e^{i\omega t} \tag{10}$$

$$t > 0, x > 0, y \rightarrow \infty \quad \phi \text{ is finite} \tag{11}$$

$$t > 0, x > 0, y = 0 \quad \frac{\partial \phi}{\partial y} = r \frac{\partial \phi}{\partial t} \tag{12}$$

Next, a solution is effected to equations (8–12) by successive application of the Laplace transform to the independent variables x and t.

Defining,

$$\hat{\phi} = \int_0^\infty \phi e^{-sx}\, dx \tag{13}$$

and

$$\hat{\hat{\phi}} = \int_0^\infty \hat{\phi} e^{-pt}\, dt \tag{14}$$

equations (8–12) become, in the second transformed plane,

$$\frac{d^2\hat{\hat{\phi}}}{dy^2} - \frac{(su + p)}{\alpha} \hat{\hat{\phi}} = - \frac{u\Delta T}{\alpha(p - i\omega)} \tag{15}$$

$$y \rightarrow \infty \quad \hat{\hat{\phi}} \text{ is finite} \tag{16}$$

$$y = 0 \quad \frac{d\hat{\hat{\phi}}}{dy} = rp\hat{\hat{\phi}} \tag{17}$$

After solving (15), subject to (16) and (17), one has

$$\frac{\hat{\hat{\phi}}}{\Delta T} = \frac{u}{(su + p)\,(p - i\omega)} - \frac{rp\sqrt{\alpha/u}\; e^{-y\sqrt{u/\alpha}\,\sqrt{s+p/u}}}{(p - i\omega)(s + p/u)(\sqrt{s+p/u} + rp\sqrt{\alpha/u})} \tag{18}$$

With the aid of tables of transforms [7], as well as the substitution and translation properties, (18) is inverted and its imaginary part is found yielding the solution below.

$$\frac{\theta}{\Delta T} = 0 \quad \text{for} \quad \tau < 0$$

$$\frac{\theta(X, Y, \tau)}{\Delta T} = \text{erf } Y \sin\omega\tau + Im \left\{ e^{i(\omega\tau + 2YX) - X^2} \left[\text{erfc}\,(Y + iX) - \text{erfc}\left(\frac{\omega\tau}{2X} + Y + iX\right)\right]\right\} \text{ for } \tau > 0 \tag{19}$$

The new independent variables, τ, X, and Y, are given in the Nomenclature. Next, in order to put equation (19) into a more recognizable form, we define σ and γ as follows.

$$\sigma = -X + i\left(\frac{\omega\tau}{2X} + Y\right) \tag{20}$$

$$\gamma = -X + i\,Y \tag{21}$$

Now, from [8], a tabulated function $W(z)$, with z being a complex number, is defined in terms of the error function of complex argument as,

$$W(z) = e^{-z^2} \text{erfc}\,(-iz) = e^{-z^2} \left[1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{t^2}\, dt\right] \tag{22}$$

Nomenclature

$b^* = \omega\rho_w C_{pw} bR/k$
b = thickness of plate
C_{pw} = specific heat of plate
h_x = local surface coefficient of heat transfer
$i = \sqrt{-1}$
Im = imaginary part of
j = index
k = thermal conductivity of fluid
p = Laplace transform parameter
q_w = local wall heat flux
Q_w = nondimensional local wall heat flux
$r = \rho_w C_{pw} b/k$
$\mathbf{r}$ = position vector
R = half thickness of channel
Re = real part of
s = Laplace transform parameter
t = time
T = local temperature
T_0 = time averaged fluid inlet temperature
u = fluid velocity in x direction
$W(z)$ = function related to error function of complex argument and defined by equation (22)
x = space coordinate along plate
$X = r\omega\sqrt{\alpha x/u}$ nondimensional x coordinate
y = space coordinate perpendicular to plate
$Y = \frac{y}{2}\sqrt{u/\alpha x}$
α = thermal diffusivity of fluid
β = dummy variable of integration
γ = defined by equation (21)
ΔT = amplitude of fluid inlet temperature variation
$\theta = T - T_0$ temperature excess
λ = dummy variable of integration
ρ_w = mass density of wall material
σ = defined by equation (20)
$\chi = \alpha x/uR^2$
$\tau = t - x/u$
ω = angular frequency
ϕ = complex temperature defined by equation (6)

Subscripts

B = bulk mean condition
qs = quasi-steady condition
w = wall condition

Superscripts

$(\overline{\ })$ = complex conjugate of
$(\wedge)$ = Laplace transform of

Combining equations (19–22) yields,

$$\frac{\theta}{\Delta T} = 0 \quad \text{for} \quad \tau < 0$$

$$\frac{\theta}{\Delta T} = \text{erf } Y \sin\omega\tau + e^{-Y^2}\{\sin\omega\tau \, Re[W(-X+iY)] + \cos\omega\tau \, Im[W(-X+iY)]\} - e^{-(\omega\tau/2X+Y)^2} Im\left\{W\left[-X+i\left(\frac{\omega\tau}{2X}+Y\right)\right]\right\} \quad \text{for} \quad \tau > 0 \quad (23)$$[1]

The last term of equation (23) represents the transient response function which causes the fluid temperature excess to change from its initial value, zero, to the eventual periodic unsteady state which is given by the first two terms of equation (23).

Equation (23), being the exact solution function, allows the determination of all other quantities of interest, such as the wall temperature, surface heat flux and, for the case of a thermal entrance region in a channel, the local bulk mean temperature excess. Equation (23) also serves as a fundamental solution for any form of periodic inlet fluid temperature by applying (23) to each separate harmonic and adding.

Local Wall Temperature. Setting $Y = 0$ and making use of the fact [8] that,

$$W(-x+iy) = \overline{W(x+iy)}$$

allows one to find the wall temperature excess as,

$$\frac{\theta_w}{\Delta T} = e^{-X^2}\sin\omega\tau - \frac{2}{\sqrt{\pi}}\cos\omega\tau\left[e^{-X^2}\int_0^X e^{\lambda^2}d\lambda\right] - e^{-\omega^2\tau^2/4X^2} Im\{W(-X+i\omega\tau/2X)\} \quad \text{for} \quad \tau > 0 \quad (24)$$

and

$$\frac{\theta_w}{\Delta T} = 0 \quad \text{for} \quad \tau < 0$$

The first bracketed quantity in equation (24) is Dawson's integral for which numerical values can be found in [8].

Local Surface Heat Flux. By Fourier's law of conduction, the local surface heat flux is given by

$$q_w = -k\left(\frac{\partial\theta}{\partial y}\right)_{y=0}$$

Combining this equation with the energy balance on the wall, equation (1), gives the expression which follows for the nondimensional surface heat flux.

$$Q_w = \frac{q_w}{\rho_w C_{pw} b \Delta T \omega} = -\frac{\partial\theta_w/\partial\tau}{\omega\Delta T} \quad (25)$$

Performing the appropriate differentiation of equation (24) leads to the final expression for the nondimensional surface flux as,

$$Q_w = 0 \quad \tau < 0$$

$$Q_w = -\left[e^{-X^2}\cos\omega\tau + 2/\sqrt{\pi}\sin\omega\tau \, e^{-X^2}\int_0^X e^{\lambda^2}d\lambda - e^{-\omega^2\tau^2/4X^2}\text{Re}\left\{W\left(-X+i\frac{\omega\tau}{2X}\right)\right\}\right] \quad \text{for} \quad \tau > 0 \quad (26)$$

Local Bulk Mean Temperature Excess. Consider the thermal entrance length between two parallel plates spaced a distance $2R$ units apart. In this region of the parallel plate duct, the solution functions (23, 24) and (26) are applicable. An energy balance on a control volume of fluid R by dx by one unit is made, using the bulk mean temperature excess θ_B to characterize the average enthalpy per unit mass of the fluid across the duct at any x and t, and the following equation results.

[1] The generalization of the solution, equation (23), to a non-zero constant initial temperature excess θ_1 can be shown to be as follows: In equation (23), $\theta/\Delta T = \theta_1/\Delta T$ for $\tau < 0$ and, for $\tau > 0$, one must add $\theta_1/\Delta T$ erfc $[\omega\tau/2x + Y]$ to the result in equation (23).

$$\frac{\partial\theta_B}{\partial t} + u\frac{\partial\theta_B}{\partial x} = \frac{q_w(x,t)}{\rho C_p R} \quad (27)$$

Changing to τ, X coordinates in equation (27), inserting equation (26) for Q_w, and solving, subject to the side condition that at

$$X = 0, \tau > 0 \quad \theta_B = \Delta T \sin\omega\tau,$$

yields the response function for the local bulk mean temperature excess, namely,

$$\frac{\theta_B}{\Delta T} = 0 \quad \text{for} \quad \tau < 0$$

$$\frac{\theta_B}{\Delta T} = \left[1 - \frac{2}{\omega rR\sqrt{\pi}}\left(X - e^{-X^2}\int_0^X e^{\lambda^2}d\lambda\right)\right] \times \sin\omega\tau - \frac{(1-e^{-X^2})}{\omega rR}\cos\omega\tau + \frac{2}{\omega rR}\int_0^X e^{-\omega^2\tau^2/4\beta^2}\beta \, Re\left\{W\left(-\beta + \frac{i\omega\tau}{2\beta}\right)\right\}d\beta \quad \text{for} \quad \tau > 0 \quad (28)$$

Transient Change from One Periodic Unsteady State to Another. Suppose there exists initially a periodic unsteady state due to the inlet temperature $\Delta T_1 \sin\omega\tau$ and then, at $t = 0$, the inlet temperature's amplitude changes from ΔT_1 to $\Delta T_1 + \Delta T_2$. It is desired to find the response of the wall to this change. By a decomposition of the temperature excess into a periodic component driven by ΔT_1 $\sin\omega\tau$ before $t = 0$ and a transient component that satisfies equations (1–5) with ΔT set equal to ΔT_2, one finds the following solution.

$$\frac{\theta_w}{\Delta T_1} = e^{-X^2}\sin\omega\tau - 2/\sqrt{\pi}\cos\omega\tau\left[e^{-X^2}\int_0^X e^{\lambda^2}d\lambda\right] \quad \text{for} \quad \tau < 0 \quad (29)$$

$$\frac{\theta_w}{\Delta T_1} = \left[1 + \frac{\Delta T_2}{\Delta T_1}\right]\left[e^{-X^2}\sin\omega\tau - 2/\sqrt{\pi}\cos\omega\tau \, e^{-X^2}\int_0^X e^{\lambda^2}d\lambda\right] - \frac{\Delta T_2}{\Delta T_1}e^{-\omega^2\tau^2/4X^2} Im\{W(-X+i\omega\tau/2X)\} \quad \text{for} \quad \tau > 0 \quad (30)$$

Equation (29) is the original periodic wall response due to the sinusoidal inlet temperature excess variation of amplitude ΔT_1, while equation (30) provides the transient response of the wall, due to a sudden change in inlet temperature amplitude to $\Delta T_1 + \Delta T_2$, including the new ultimate periodic unsteady state.

Quasi-Steady Solution. The x dependent surface coefficient for steady slug flow over an isothermal plate was employed in the quasi-steady solution. For these conditions, one has, from [9],

$$h_x = k\sqrt{u/\pi\alpha x} \quad (31)$$

An energy balance on a segment of wall dx long gives,

$$\rho_w C_{pw} b \frac{\partial\theta_{wqs}}{\partial t} + h_x(\theta_{wqs} - \Delta T \sin\omega t) = 0 \quad (32)$$

Combining equations (31) and (32), solving the result subject to the condition that $\theta_{wqs} = 0$ at $t = 0$, then changing to X, $\omega\tau$ variables gives,

$$\frac{\theta_{wqs}}{\Delta T} = \frac{\sqrt{\pi}Xe^{-[\omega\tau/\sqrt{\pi}X + \omega x/u]\sqrt{\pi}X}}{1+\pi X^2} + \frac{\sin[\omega\tau + \omega x/u]}{1+\pi X^2} - \frac{\sqrt{\pi}X\cos[\omega\tau + \omega x/u]}{1+\pi X^2} \quad (33)$$

for all $\tau > -x/u$.

Conditions for Validity of Quasi-Steady Analyses. Often the line of reasoning advanced for the validity of quasi-steady approaches is to say that good results will be obtained if the term containing the time derivative of the fluid temperature, in the thermal energy equation, is negligible. A common way to assess the size of this term is by comparing characteristic diffusion times in both the fluid and the solid (at least for conjugated problems) and if the diffusion time for the fluid is much smaller than that of the solid it is asserted that the quasi-steady assumption will be a good one. This technique is demonstrated, in some detail, by Dorfman [10]. Sparrow and Gregg [11] show a more quantitative way of investigating conditions for a valid quasi-steady analysis which involves expansions of the instantaneous dependent variables about the quasi-steady state and then seeing what must be true for the quasi-steady solution to be close to the true solution. This procedure is complicated by the necessity to solve ordinary differential equations numerically.

Apparently, conditions for quasi-steadiness based on a small time derivative of the fluid temperature do not give the entire picture. As an example, consider the case of periodically varying fluid temperature as in [1, 2, 4] and the present work. In this case, the fluid temperature is always changing with time, perhaps at a fairly rapid rate, yet there are, even in this case, conditions under which the quasi-steady solution is close to the true solution.

Consider a transient forced convection problem which is governed by the following linear homogeneous partial differential equation.

$$L\,\theta(\mathbf{r}, t) = 0 \tag{34}$$

A quasi-steady solution to the problem, $\theta_{qs} = \theta_{q.s.}\,(\mathbf{r}, t)$, is available which satisfies the linear side conditions on the $\theta(\mathbf{r}, t)$ of equation (34). Next, a residual function θ_d is defined by the following decomposition.

$$\theta = \theta_{q.s.} + \theta_d \tag{35}$$

To find the conditions under which $\theta_d \to 0$, when θ is not known, equation (35) is inserted into (34) and gives,

$$L\,\theta_d = -L\,\theta_{q.s.} \tag{36}$$

Since $\theta_{q.s.}$ satisfies the side conditions on θ, it follows that θ_d satisfies homogeneous side conditions and therefore θ_d is expected to approach zero under the conditions that lead to the right side of (36) approaching zero. These will then be the conditions for the quasi-steady solution to approach the true solution.

The procedure will now be illustrated by application to the problem of the present work where a quasi-steady solution for the wall temperature is given by equation (33). First, one needs the quasi-steady solution for all x, y, and t. To find this, equation (2) is solved, with the time derivative set equal to zero, subject to the side conditions (4) and (5) and, at $y = 0$, $\theta_{q.s.} = \theta_w(x, t)$ from equation (33). The result for $\theta_{q.s.}$ $(\mathbf{r}, t)$ is as follows.

$$\theta_{q.s.} = [\theta_w(X, t) - \Delta T \sin\omega t]\,\text{erfc}\,Y + \Delta T \sin\omega t \tag{37}$$

For this problem, the equation which corresponds to (34) is equation (2) with the variables changed to X, Y, t, namely.

$$\frac{\partial^2\theta}{\partial Y^2} + 2Y\frac{\partial\theta}{\partial Y} - 2X\frac{\partial\theta}{\partial X} - \frac{4X^2}{\omega\alpha r^2}\frac{\partial\theta}{\partial(\omega t)} = 0 \tag{38}$$

Replacing θ, in equation (38), by $\theta_{q.s.} + \theta_d$ and operating on the known function $\theta_{q.s.}$ leads to the present problem's equivalent of equation (36), namely,

$$\frac{4X^2}{\omega\alpha r^2}\frac{\partial\theta_d}{\partial(\omega t)} + 2X\frac{\partial\theta_d}{\partial X} - 2Y\frac{\partial\theta_d}{\partial Y} - \frac{\partial^2\theta_d}{\partial Y^2} = -2X\frac{\partial\theta_{q.s.}}{\partial X} - \frac{4X^2}{\omega\alpha r^2}\frac{\partial\theta_{q.s.}}{\partial(\omega t)} \tag{39}$$

Inserting (33) into (37) and using the result in the right side of (39) gives,

$$R.H.S. = -2\left\{\left[-2\pi X^2 \sin\omega t + \sqrt{\pi}\,(X - \pi X^3)(e^{-\omega t/\sqrt{\pi}X} - \cos\omega t) + \frac{\omega t e^{-\omega t/\sqrt{\pi}X}}{1 + \pi X^2}\right]\text{erfc}\,Y + \frac{2X^2}{\omega\alpha r^2}\left[\left(\frac{\sqrt{\pi}X \sin\omega t - e^{-\omega t/\sqrt{\pi}X} - \pi X^2 \cos\omega t}{1 + \pi X^2}\right)\text{erfc}\,Y + \cos\omega t\right]\right\} \tag{40}$$

Upon study of the right side of (40), one observes that $R.H.S. \to 0$ as $X \to 0$ for all Y and t, therefore it is expected that the quasi-steady solution will approach the true solution for small values of X even though $\partial\theta/\partial t$ is not necessarily small. If one next considers X becoming large due to $r\sqrt{\omega\alpha}$ increasing at finite $\omega x/u$, (40) reduces to

$$R.H.S. = -r\left(\frac{\omega x}{u}\right)\cos\omega t\ \text{erf}\,Y \tag{41}$$

From (41), it is seen that the additional condition of $\omega x/u$ being small is needed, along with X becoming large, for the quasi-steady solution to approach the true solution. The physical reason for this requirement will be dealt with in the discussion section.

Results and Discussion

As mentioned before, the exact analytical solution function (23) and functions found from (23), such as (24, 26) and (28), are also expected to apply in the thermal entrance region of a duct with sinusoidally time varying fluid inlet temperature. The extent, in x, of the thermal entrance region of a parallel plate duct can be roughly assessed by a consideration of the exact solution for the local Nusselt number in steady, laminar, constant property, slug flow through an isothermal duct. Upon examination of the solution to this steady state case as given in Kays [9], it was decided that the thermal entrance region's length, l_e, can be given approximately as,

$$\frac{\alpha l_e}{uR^2} \approx 0.20$$

This nondimensional entrance length estimate does not translate into a single value of X because X also depends upon the thermal capacity of the wall material which is not a factor in a steady state response.

To partially check the integrity of the present results, selected comparisons are made for wall temperature and local bulk mean temperature with the results of Sparrow and DeFarias [4] whose results are for the ultimate periodic unsteady state only. The comparison is made for the following values of the parameters of [4].

$$\chi = \frac{\alpha x}{uR^2} = .10 \text{ and } b^* = 1.0,$$

which correspond to $X = .31622777$ of the present work. Equations (24) and (28) provide θ_w and θ_B, with values for Dawson's integral and the imaginary part of $W(z)$ provided by [8]. The last integral in equation (28) does not appear to be a tabulated function and it was evaluated by graphical integration. The results are presented in Fig. 1 for the transient start-up from a zero temperature excess to the eventual periodic unsteady state which, as can be seen, is achieved within 25 percent of the first cycle. The response of the bulk mean temperature is faster (transient portion over within 15 percent of the first cycle) than that at the wall because the bulk mean temperature is, in effect, a spatial average between $y = 0$ and $y = R$ and its response is expected to be closer to that of the inlet temperature. The actual transient response is given by the dashed curves which eventually, as noted above, coalesce into the solid curves which represent the periodic unsteady state. The solid curves are displayed even at low times to show the approach of the fluid and wall to the periodic unsteady state and also because all cycles beyond the first one shown are exactly like the solid curves of the first cycle. The circled points are for the periodic unsteady state and were extracted from Figs. 1 and 3 of Sparrow and DeFarias [4]. The agreement is good, as it should be in the case of two equivalent exact solutions, with the slight differences between two of the points and the solid curve probably attributable to slight inaccuracies in extracting data from the figures of [4].

Wall Temperature Results. In Fig. 2 is shown, as solid lines, the transient wall temperature, from equation (24), response curves for three representative values of X as well as one cycle of the fluid inlet temperature variation with time which is also the response at $X = 0$. One notes, especially for $X = 2.0$, that the wall temperatures are higher during the initial portion of the transient than they are at the corresponding time in the periodic unsteady state. This is due to the material, initially, being at a higher temperature than it will be in the periodic unsteady condition at the corresponding time. In evidence, also, is the lag and attenuation of the response as X increases. Physically, this is as expected since the time mean thermal boundary layer thickness increases with increasing X and this increased thickness can be viewed as being roughly analogous to different depths from the surface of a stationary solid with a periodic thermal disturbance function on the surface. The trends of Fig. 2 are also in qualitative agreement, in the periodic unsteady state, with those for duct flow in [4]. The wall is brought to its eventual periodic unsteady state in about 6 and 50 percent of cycle one for $X = .10$ and 1.0, respectively, while it requires about one and one third cycles for the transient to disappear for $X = 2.0$. Thus all cycles beyond the second are exactly like cycle two for $X = .1$ and 1.0 and this is even the case for $X = 2.0$ if an error of a few percent can be tolerated in the first third of the cycle.

Fig. 3 isolates the transient portion of the wall response function in a normalized manner convenient for study. TR is defined as the ratio of the absolute value of the transient term in (24) to the absolute value of $\theta_w/\Delta T$ when $\omega\tau = 0$ in the periodic unsteady state. That is,

$$TR = \frac{\left| e^{-\omega^2\tau^2/4X^2}\, Im\left\{W\left(-X + i\,\frac{\omega\tau}{2X}\right)\right\}\right|}{\frac{2}{\sqrt{\pi}}\, e^{-X^2} \int_0^X e^{\lambda^2}\, d\lambda} \qquad (42)$$

From Fig. 3, one sees that the transient response is over, for all intents and purposes, by the time the abscissa value reaches 2.0 to 2.2. Calling τ_c the value of τ at which $TR \leq .01$, a curve fit of the results yields the following relation, for $.1 \leq X \leq 4$,

$$\frac{\omega\tau_c}{2X} = .0468\, X^2 + .0596X + 1.5936 \qquad (43)$$

Equation (43) is useful in estimating the time from start-up to the eventual periodic unsteady state.

Heat Flux and Surface Coefficient of Heat Transfer. Equation (26) is used to plot the nondimensional surface heat flux for two values of X in Fig. 4. One observes the same overall pattern of phase lag and decay of amplitude with X as for θ_w in Fig. 2. The surface heat flux is slightly lower at short times than it is at the equivalent time in the cyclic unsteady state and this is caused by the higher wall temperatures, during the transient startup, noticed in connection with Fig. 2. The local instantaneous surface coefficient of heat transfer, based on the difference between the wall temperature and the fluid temperature far from the wall, was found by combining equations (26) and (24). The overall pattern of the Nusselt number with time is similar to that given in [4], Figs. 5 and 6.

Changes from One Periodic Unsteady State to Another. Considered next is the case where the plate of thickness b is initially in the periodic unsteady state in response to the sinusoidally varying fluid inlet temperature of amplitude ΔT_1, when at a time t taken to be zero, for convenience, the fluid inlet temperature's amplitude is suddenly doubled. The appropriate solution functions here are equations (29) and (30) with $\Delta T_2 = \Delta T_1$. The plot of the solution, for $X = 2.0$, is given in Fig. 5. Cycle 0 represents the wall temperature in the original cyclic state. In cycle 1, the first transient cycle, wall temperatures higher than those in the new cyclic state are reached for the same reason as in Fig. 2. The end of the transient portion of the response occurs in the first third of cycle 2 and equation (43) yields a good estimate of the duration of the transient.

Quasi-Steady Results. As can be seen from the response function, equation (33), the quasi-steady solution will always be out of phase with the exact solution because of the presence of the group $\omega x/u$ that is always adding to $\omega\tau$. Because of the fact that conditions sometimes may be such that $\omega x/u \ll 2\pi$, (see [2] and [4]) it was decided to, intentionally, favorably bias the quasi-steady result by neglecting $\omega x/u$ in the plot of the response. The circles, triangles, and squares in Fig. 2 are the quasi-steady wall temperatures for $X = .1$,

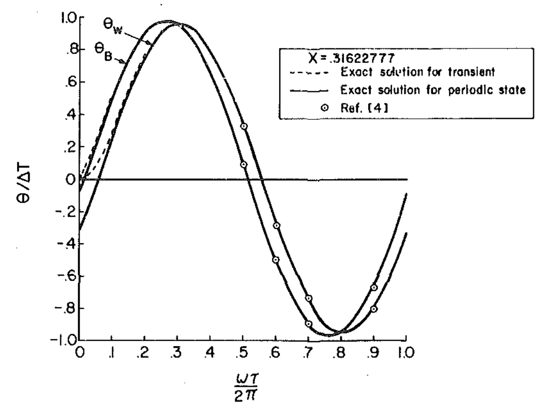

Fig. 1 Bulk mean and wall temperature response to sinusoidal time varying fluid temperature at $x = 0$

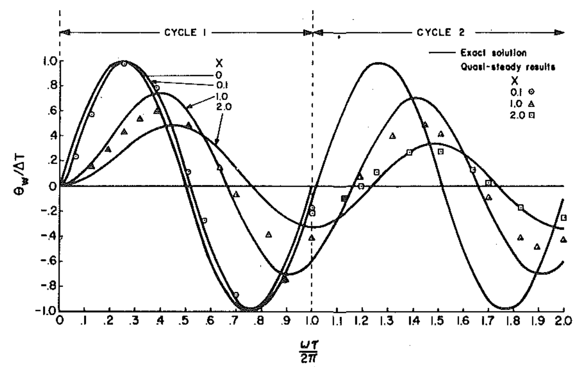

Fig. 2 Wall temperature response and comparison with quasi-steady solution

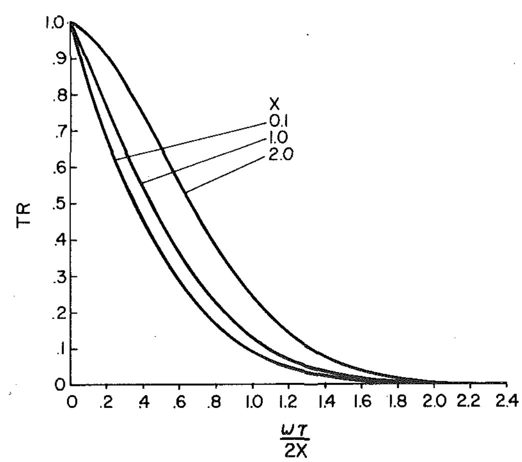

Fig. 3 Transient portion of the wall temperature response function

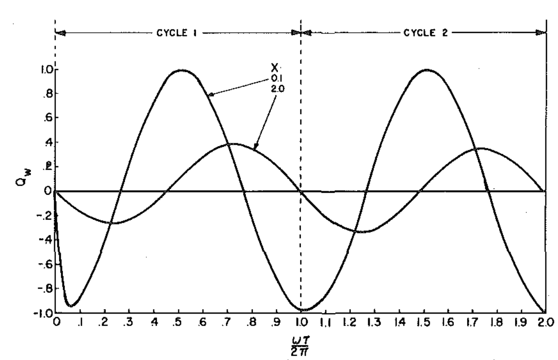

Fig. 4 Surface heat flux response to sinusoidal time varying fluid temperature at $x = 0$

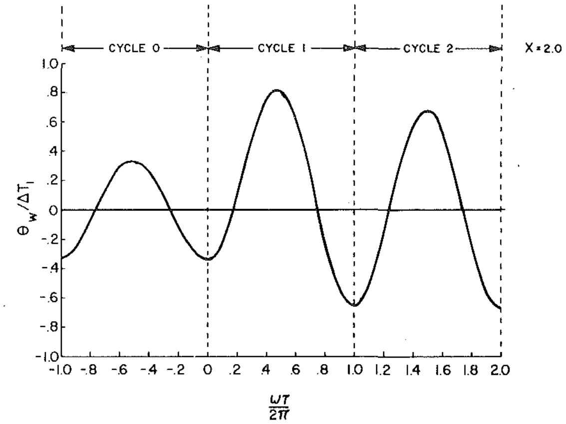

Fig. 5 Wall temperature response when the amplitude of the sinusoidal time varying fluid temperature at $x = 0$ is suddenly doubled at $t = 0$

1.0, and 2.0, respectively. To make the figure easier to read, quasi-steady results are given only for the first cycle at $X = .10$ (results are virtually identical for all succeeding cycles when $X = .10$) and only for the second cycle at $X = 2.0$. It is seen from Fig. 2 that the quasi-steady results provide a reasonably good description at $X = .10$, but display a rather large, unacceptable deviation from the exact wall temperatures at $X = 1.0$. For $X = 2.0$, the quasi-steady results are tending, once again, toward the exact solution but the error, while considerably less than for $X = 1.0$, is still significant.

Turning now to the procedure advanced earlier, in the analysis section, for finding the conditions under which the quasi-steady solution will approach the exact solution, we see, from equation (40) and the comments which follow it, that the two solutions should merge as $X \rightarrow 0$ and as X becomes large for small $\omega x/u$. Recalling that $\omega x/u$ was neglected in the quasi-steady results, it is seen that Fig. 2 confirms these conditions. Additional support for this conclusion is found in Figs. 6 and 8 of [4] after noting that the X of this work is equal to $b^*\sqrt{\chi}$ of [4]. The physical reason for $\omega x/u$ needing to be small, as X gets large, stems from the fact that x/u is the lag, or dwell, time before which the material cannot sense a change that occurred at the leading edge. Yet the quasi-steady solution (33) does not predict this and, hence, would be in error, even at long times, (the periodic nature of the disturbance, in effect, makes $\omega\tau = 2\pi$ the "largest" value of nondimensional time that is ever achieved in the ultimate periodic unsteady solution) unless $\omega x/u$ is a very small quantity.

Conclusions

An exact solution has been found for the transient temperature field within a moving fluid which communicates thermally with a plate of finite thermal storage capacity when the fluid temperature at the leading edge varies sinusoidally with time. The solution functions presented can also be used for more general periodic driving temperatures via the vehicle of decomposition into harmonics.

One key element of the analysis was the use of the method of complex temperature not only for the eventual periodic unsteady state, but also for the transient response leading to it. Hopefully this will also be of aid in other transient convection problems where the thermal loading is of a periodic nature.

Comparison of the exact solution with a quasi-steady approach that uses the x dependent surface coefficient of heat transfer for an isothermal plate indicates acceptable agreement at low t, at low X, and at large X provided $\omega x/u$ is small. In general, however, the quasi-steady solution does not provide adequate accuracy in predicting time varying wall temperatures.

A method, which combines the quasi-steady solution and the differential equation of the exact solution, is advanced for finding the circumstances in which the quasi-steady solution is close to the exact solution and is found to work well for the present problem.

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