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On I-convergent sequence spaces defined by a compact operator and a modulus function

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Abstract: In this article, we introduce and study I-convergent sequence spaces $S^I(f)$, $S_0^I(f)$, and $S_\infty^I(f)$ with the help of compact operator T on the real space $\mathbb{R}$ and a modulus function f. We study some topological and algebraic properties, and prove some inclusion relations on these spaces.

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1. Introduction and preliminaries
Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of all natural, real, and complex numbers, respectively. We denote

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \text{ or} \mathbb{C}\}$$

the space of all real or complex sequences.

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PUBLIC INTEREST STATEMENT
The term sequence has a great role in Analysis. Sequence spaces play an important role in various fields of Real Analysis, Complex Analysis, Functional Analysis, and Topology Convergence of sequences has always remained a subject of interest to the researchers. Several new types of convergence of sequences were studied by the researchers and named them as usual convergence, uniform convergence, strong convergence,week convergence, etc. Later, the idea of statistical convergence came into existence which is the generalization of usual convergence. Statistical convergence has several applications in different fields of Mathematics like Number Theory, Trigonometric Series, Summability Theory, Probability Theory, Measure Theory, Optimization, and Approximation Theory. The notion of Ideal convergence (I-convergence) is a generalization of the statistical convergence and equally considered by the researchers for their research purposes since its inception.

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Let ℓ_∞, c, and c_0 be denote the Banach spaces of bounded, convergent, and null sequences of reals, respectively with norm

$$\|x\| = \sup_k |x_k|$$

Any subspace λ of ω is called a sequence space. A sequence space λ with linear topology is called a *K*-space provided each of maps $p_i \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous, for all $i \in \mathbb{N}$. A space λ is called an *FK*-space provided λ is complete linear metric space. An *FK*-space whose topology is normable is called a *BK*-space.

Definition 1.1　Let X and Y be two normed linear spaces and $T : \mathcal{D}(T) \to Y$ be a linear operator, where $\mathcal{D}(T) \subset X$. Then, the operator T is said to be bounded, if there exists a positive real k such that

$$\| Tx \| \le k \| x \|, \quad \text{for all } x \in \mathcal{D}(T)$$

The set of all bounded linear operators $\mathcal{B}(X, Y)$ is a normed linear space normed by (see Kreyszig, 1978)

$$\| T \| = \sup_{x \in X, \|x\|=1} \| Tx \|$$

and $\mathcal{B}(X, Y)$ is a Banach space if Y is Banach space.

Definition 1.2　Let X and Y be two normed linear spaces. An operator $T : X \to Y$ is said to be a compact linear operator (or completely continuous linear operator), if

(1) T is linear,

(2) T maps every bounded sequence (x_k) in X onto a sequence $T(x_k)$ in Y which has a convergent subsequence.

The set of all compact linear operators $C(X, Y)$ is closed subspace of $\mathcal{B}(X, Y)$ and $C(X, Y)$ is a Banach space if Y is Banach space.

Following Basar and Altay (2003) and Sengönül (2009), we introduce the sequence spaces S and S_0 with the help of compact operator T on the real space $\mathbb{R}$ as follows.

$$S = \left\{ x = (x_k) \in \ell_\infty : T(x) \in c \right\}$$

and

$$S_0 = \left\{ x = (x_k) \in \ell_\infty : T(x) \in c_0 \right\}$$

Definition 1.3　A function $f : [0, \infty) \longrightarrow [0, \infty)$ is called a modulus if

(1) $f(t) = 0$ if and only if $t = 0$,

(2) $f(t + u) \le f(t) + f(u)$ for all $t, u \ge 0$,

(3) f is increasing, and

(4) f is continuous from the right at zero.

For any modulus function f, we have the inequalities

$$| f(x) - f(y) | \le f(| x - y |)$$

and

$$f(nx) \le nf(x), \quad \text{for all } x, y \in [0, \infty]$$

A modulus function f is said to satisfy $\Delta_2 -$ **Condition** for all values of u if there exists a constant $K > 0$ such that $f(Lu) \le KLf(u)$ for all values of $L > 1$.

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The idea of modulus was introduced by Nakano (1953).

Ruckle (1967, 1968, 1973) used the idea of a modulus function f to construct the sequence space

$$X(f) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty \right\}$$

This space is an *FK*-space and Ruckle (1967, 1968, 1973) proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences.

The space $X(f)$ is closely related to the space ℓ_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle (1967, 1968, 1973) proved that, for any modulus f.

$X(f) \subset \ell_1$ and $X(f)^\alpha = \ell_\infty$

where

$$X(f)^\alpha = \left\{ y = (y_k) \in \omega : \sum_{k=1}^{\infty} f(|y_k x_k|) < \infty \right\}$$

Spaces of the type $X(f)$ are a special case of the spaces structured by Gramsch (1967). From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Symmetric sequence spaces, which are locally convex have been frequently studied by Garling (1966), Köthe (1970), and Ruckle (1967, 1968, 1973).

The sequence spaces by the use of modulus function was further investigated by Maddox (1969, 1986), Khan (2005, 2006), Bhardwaj (2003), and many others.

As a generalization of usual convergence, the concept of statistical convergent was first introduced by Fast (1951) and also independently by Buck (1953) and Schoenberg (1959) for real and complex sequences. Later on, it was further investigated from sequence space point of view and linked with the Summability Theory by Fridy (1985), Šalát (1980), Tripathy (1998), Khan (2007), Khan and Sabiha (2012), Khan, Shafiq, and Rababah (2015), and many others.

Definition 1.4   A sequence $x = (x_k) \in \omega$ is said to be statistically convergent to a limit $L \in \mathbb{C}$ if for every $\varepsilon > 0$, we have

$$\lim_{k \to \infty} \frac{1}{k} | \{ n \in \mathbb{N} : |x_n - L| \geq \varepsilon, n \leq k \} | = 0$$

where vertical lines denote the cardinality of the enclosed set.

That is, if $\delta(A(\varepsilon)) = 0$, where

$A(\varepsilon) = \{ k \in \mathbb{N} : | x_k - L | \geq \varepsilon \}$

The notation of ideal convergence (I-convergence) was introduced and studied by Kostyrko, Mačaj, Salăt, and Wilczyński (2000). Later on, it was studied by Šalát, Tripathy, and Ziman (2004, 2005), Tripathy and Hazarika (2009, 2011), Khan and Ebadullah (2011), Khan, Ebadullah, Esi, and Shafiq (2013), and many others.

Now, we recall the following definitions:

Definition 1.5   Let $\mathbb{N}$ be a non-empty set. Then a family of sets $I \subseteq 2^{\mathbb{N}}$ (power set of $\mathbb{N}$) is said to be an ideal if

(1) I is additive i.e $\forall A, B \in I \Rightarrow A \cup B \in I$

(2) I is hereditary i.e $\forall A \in I$ and $B \subseteq A \Rightarrow B \in I$.

Definition 1.6   A non-empty family of sets $£(I) \subseteq 2^{\mathbb{N}}$ is said to be filter on $\mathbb{N}$ if and only if

(1) $\Phi \notin £(I)$,

(2) $\forall A, B \in £(I)$ we have $A \cap B \in £(I)$,

(3) $\forall A \in £(I)$ and $A \subseteq B \Rightarrow B \in £(I)$.

Definition 1.7   An Ideal $I \subseteq 2^{\mathbb{N}}$ is called non-trivial if $I \neq 2^{\mathbb{N}}$.

Definition 1.8   A non-trivial ideal $I \subseteq 2^{\mathbb{N}}$ is called admissible if

$\{\{x\} : x \in \mathbb{N}\} \subseteq I$

Definition 1.9   A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

Remark 1.10   For each ideal I, there is a filter $£(I)$ corresponding to I. i.e $£(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} \setminus K$.

Definition 1.11   A sequence $x = (x_k) \in \omega$ is said to be I-convergent to a number L if for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in I$.

In this case, we write $I - \lim x_k = L$.

Definition 1.12   A sequence $x = (x_k) \in \omega$ is said to be I-null if $L = 0$. In this case, we write $I - \lim x_k = 0$.

Definition 1.13   A sequence $x = (x_k) \in \omega$ is said to be I-Cauchy if for every $\varepsilon > 0$ there exists a number $m = m(\varepsilon)$ such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \varepsilon\} \in I$.

Definition 1.14   A sequence $x = (x_k) \in \omega$ is said to be I-bounded if there exists some $M > 0$ such that $\{k \in \mathbb{N} : |x_k| \geq M\} \in I$.

Definition 1.15   A sequence space E is said to be solid (normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for any sequence (α_k) of scalars with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$.

Definition 1.16   A sequence space E is said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $x_k \in E$. where π is a permutation on $\mathbb{N}$.

Definition 1.17   A sequence space E is said to be sequence algebra if $(x_k) * (y_k) = (x_k.y_k) \in E$ whenever $(x_k), (y_k) \in E$.

Definition 1.18   A sequence space E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$, for all k.

Definition 1.19   Let $K = \{k_1 < k_2 < k_3 < k_4 < k_5 \ldots\} \subset \mathbb{N}$ and E be a Sequence space. A K-step space of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in \omega : (x_k) \in E\}$.

Definition 1.20   A canonical pre-image of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_k) \in \omega$ defined by

$$y_k = \begin{cases} x_k, & \text{if } k \in K, \\ 0, & \text{otherwise} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of preimages all elements in λ_K^E. i.e. y is in the canonical preimage of λ_K^E iff y is the canonical preimage of some $x \in \lambda_K^E$.

Definition 1.21   A sequence space E is said to be monotone if it contains the canonical preimages of its step space.

Definition 1.22 (see, Khan et al., 2015; Kostyrko et al., 2000). If $I = I_f$, the class of all finite subsets of v. Then, I is an admissible ideal in v and I_f convergence coincides with the usual convergence.

Definition 1.23 (see, Khan et al., 2015; Kostyrko et al., 2000). If $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$. Then, I is an admissible ideal in $\mathbb{N}$ and we call the I_δ-convergence as the logarithmic statistical convergence.

Definition 1.24 (see, Khan et al., 2015; Kostyrko et al., 2000). If $I = I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$. Then, I is an admissible ideal in $\mathbb{N}$ and we call the I_d-convergence as the asymptotic statistical convergence.

Remark 1.25 If $I_\delta - \lim x_k = l$, then $I_d - \lim x_k = l$

Definition 1.26 A map $\hbar$ defined on a domain $D \subset X$ i.e $\hbar : D \subset X \to \mathbb{R}$ is said to satisfy Lipschitz condition if $|\hbar(x) - \hbar(y)| \leq K|x - y|$ where K is known as the Lipschitz constant. The class of K-Lipschitz functions defined on D is denoted by $\hbar \in (D, K)$.

Definition 1.27 A convergence field of I-covergence is a set

$$F(I) = \{x = (x_k) \in l_\infty : \text{ there exists } I - \lim x \in \mathbb{R}\}$$

The convergence field $F(I)$ is a closed linear subspace of l_∞ with respect to the supremum norm, $F(I) = l_\infty \cap c^I$ (see Šalát et al., 2004, 2005).

Definition 1.28 Let X be a linear space. A function $g : X \longrightarrow R$ is called paranorm, if for all $x, y \in X$,

$(P_1) \, g(x) = 0$ if $x = \theta$,
$(P_2) \, g(-x) = g(x)$,
$(P_3) \, g(x + y) \leq g(x) + g(y)$,
(P_4) If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and $x_n, a \in X$ with $x_n \to a$ $(n \to \infty)$ in the sense that $g(x_n - a) \to 0$ $(n \to \infty)$, then $g(\lambda_n x_n - \lambda a) \to 0$ $(n \to \infty)$.

The notation of paranorm sequence spaces was studied at the initial stage by Nakano (1953). Later on, it was further investigated by Maddox (1969), Tripathy and Hazarika (2009), Khan et al. (2013), and the references therein.

Throughout the article, we use the same techniques as used in Tripathy and Hazarika (2009, 2011).

We used the following lemmas for establishing some results of this article.

Lemma 1 (see, Tripathy & Hazarika, 2009, 2011). *Every solid space is monotone.*

Lemma 2 (see, Tripathy & Hazarika, 2009, 2011). *If $I \subseteq 2^N$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$.*

Lemma 3 (see, Tripathy & Hazarika, 2009, 2011). *Let $K \in \pounds(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.*

Throughout the article, S^I, S^I_0, S^I_∞, $\mathcal{M}^I_S$, and $\mathcal{M}^I_{S_0}$ represent the I-convergent, I-null, I-Bounded , bounded I-convergent, and bounded I-null Sequences spaces defined by a compact operator T on the real space $\mathbb{R}$, respectively.

2. Main results

In this article, we introduce the following classes of sequences.

$$S^I(f) = \left\{ x = (x_k) \in \ell_\infty : \{k \in \mathbb{N} : f\left(\, |\, T(x_k) - L\, |\, \right) \geq \epsilon\} \in I, \text{ for some } L \in \mathbb{C} \right\} \tag{2.1}$$

$$S^I_0(f) = \left\{ x = (x_k) \in \ell_\infty : \{k \in \mathbb{N} : f\left(\, |\, (Tx_k)\, |\, \right) \geq \epsilon\} \in I \right\} \tag{2.2}$$

$$S_\infty^I(f) = \left\{ x = (x_k) \in \ell_\infty : \{ k \in \mathbb{N} : \exists\, K < 0 \text{ such that } f\left(\mid T(x_k) \mid \right) \geq K \} \in I \right\} \tag{2.3}$$

$$S_\infty(f) = \left\{ x = (x_k) \in \ell_\infty : \sup_k f\left(\mid T(x_k) \mid \right) < \infty \right\} \tag{2.4}$$

where f is a modulus function.

We also denote

THEOREM 2.1 *Let f be a modulus function. Then, the classes of sequences $S^I(f)$, $S_0^I(f)$, $\mathcal{M}_s^I(f)$, and $\mathcal{M}_{S_\circ}^I(f)$ are linear spaces.*

Proof We shall prove the result for $S^I(f)$. The proof for the other spaces will follow similarly. For, let $x = (x_k)$, $y = (y_k) \in S^I(f)$ and α, β be scalars. Then, for a given $\epsilon > 0$, we have

$$\left\{ k \in \mathbb{N} : f\left(\mid T(x_k) - L_1 \mid \right) \geq \frac{\epsilon}{2}, \text{ for some } L_1 \in \mathbb{C} \right\} \in I \tag{2.5}$$

$$\left\{ k \in \mathbb{N} : f\left(\mid T(x_k) - L_2 \mid \right) \geq \frac{\epsilon}{2}, \text{ for some } L_1 \in \mathbb{C} \right\} \in I \tag{2.6}$$

Let

$$A_1 = \left\{ k \in \mathbb{N} : f\left(|T(x_k) - L_1| \right) < \frac{\epsilon}{2}, \text{ for some } L_1 \in \mathbb{C} \right\} \in \mathcal{L}(I) \tag{2.7}$$

$$A_2 = \left\{ k \in \mathbb{N} : f\left(|T(y_k) - L_2| \right) < \frac{\epsilon}{2}, \text{ for some } L_2 \in \mathbb{C} \right\} \in \mathcal{L}(I) \tag{2.8}$$

be such that $A_1^c, A_2^c \in I$.

Since f is a modulus function, we have

$$
\begin{aligned}
A_3 &= \left\{ k \in \mathbb{N} : f\left(|(\alpha T(x_k) + \beta T(y_k) - (\alpha L_1 + \beta L_2)|) < \epsilon \right\} \right. \\
&\supseteq \left[\left\{ k \in \mathbb{N} : f\left(|\alpha||T(x_k) - L_1| \right) < \frac{\epsilon}{2} \right\} \right. \\
&\qquad \left. \cap \left\{ k \in \mathbb{N} : f\left(|\beta||T(y_k) - L_2| \right) < \frac{\epsilon}{2} \right\} \right] \\
&\supseteq \left[\left\{ k \in \mathbb{N} : f\left(|T(x_k) - L_1| \right) < \frac{\epsilon}{2} \right\} \right. \\
&\qquad \left. \cap \left\{ k \in \mathbb{N} : f\left(|T(y_k) - L_2| \right) < \frac{\epsilon}{2} \right\} \right]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A_3 &= \left\{ k \in \mathbb{N} : f\left(|(\alpha T(x_k) + \beta T(y_k) - (\alpha L_1 + \beta L_2)|) < \epsilon \right\} \right. \\
&\supseteq \left[\left\{ k \in \mathbb{N} : f\left(|T(x_k) - L_1| \right) < \frac{\epsilon}{2} \right\} \right. \\
&\qquad \left. \cap \left\{ k \in \mathbb{N} : f\left(|T(y_k) - L_2| \right) < \frac{\epsilon}{2} \right\} \right]
\end{aligned}
\tag{2.9}
$$

implies that $A_3 \in \pounds(I)$. Thus, $A_3^c = A_1^c \cup A_2^c \in I$. Therefore, $\alpha x_k + \beta y_k \in S^I(f)$, for all scalars α, β, and (x_k), $(y_k) \in S^I(f)$.

Hence, $S^I(f)$ is a linear space.

THEOREM 2.2 *The classes of sequences $\mathcal{M}^I_S(f)$ and $\mathcal{M}^I_{S_\circ}(f)$ are paranormed spaces, paranormed by*

$$g(x) = g(x_k) = \sup_k f\left(|T(x_k)|\right)$$

Proof Let $x = (x_k)$, $y = (y_k) \in \mathcal{M}^I_S(f)$.

(P_1) It is clear that $g(x) = 0$ if $x = \theta$, a zero vector.
(P_2) $g(x) = g(-x)$ is obvious.
(P_3) For $x = (x_k)$, $y = (y_k) \in \mathcal{M}^I_S(f)$, we have

$$g(x + y) = g(x_k + y_k) = \sup_k f\left(|T(x_k + y_k)|\right)$$
$$= \sup_k f\left(|T(x_k) + T(y_k)|\right) \le \sup_k f\left(|T(x_k)|\right)$$
$$+ \sup_k f\left(|T(y_k)|\right) | = g(x) + g(y)$$

Therefore, $g(x + y) \le g(x) + g(y)$
(P_4) Let (λ_k) be a sequence of scalars with $(\lambda_k) \to \lambda\, (k \to \infty)$ and $(x_k), L \in \mathcal{M}^I_S(f)$ such that

$$x_k \to L\,(k \to \infty)$$

in the sense that

$$g(x_k - L) \to 0\,(k \to \infty)$$

Then, since the inequality

$$g(x_k) \le g(x_k - L) + g(L)$$

holds by subadditivity of g, the sequence $\{g(x_k)\}$ is bounded.

Therefore,

$$g\left[(\lambda_k x_k - \lambda L)\right] = g\left[(\lambda_k x_k - \lambda x_k + \lambda x_k - \lambda L)\right]$$
$$= g\left[(\lambda_k - \lambda)x_k + \lambda(x_k - L)\right]$$
$$\le g\left[(\lambda_k - \lambda)x_k\right] + g\left[\lambda(x_k - L)\right]$$
$$\le |\,(\lambda_k - \lambda)\,|\,g(x_k) + |\,\lambda\,|\,g(x_k - L) \to 0$$

as $(k \to \infty)$. That is to say that scalar multiplication is continuous. Hence, $\mathcal{M}^I_S(f)$ is a paranormed space.

For $\mathcal{M}^I_{S_\circ}(f)$, the result is similar.

THEOREM 2.3 *A sequence $x = (x_k) \in \ell_\infty$ I-converges if and only if for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that*

$$\left\{k \in \mathbb{N} : f\left(\,|\,T(x_k) - T(x_{N_\epsilon})\,|\,\right) < \epsilon\right\} \in £(I) \qquad (2.10)$$

Proof Let $x = (x_k) \in \ell_\infty$.

Suppose that $L = I - \lim x$. Then, the set

$$B_\epsilon = \left\{k \in \mathbb{N} : f\left(\,|\,T(x_k) - L\,|\,\right) < \frac{\epsilon}{2}\right\} \in £(I) \text{ for all } \epsilon < 0$$

Fix an $N_\epsilon \in B_\epsilon$. Then we have,

$$f\left(\left|T(x_k)-T(x_{N_\epsilon})\right|\right)\leq f\left(\left|T(x_k)-L\right|\right)+f\left(\left|T(x_{N_\epsilon})-L\right|\right)<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

which holds for all $k\in B_\epsilon$. Hence $\left\{k\in\mathbb{N}:f\left(\left|T(x_k)-T(x_{N_\epsilon})\right|\right)<\epsilon\right\}\in \pounds(I)$ Conversely, suppose that

$$\left\{k\in\mathbb{N}:f\left(\left|T(x_k)-T(x_{N_\epsilon})\right|\right)<\epsilon\right\}\in \pounds(I)$$

That is $\left\{k\in\mathbb{N}:\left|f\left(|T(x_k)|\right)-f\left(|T(x_{N_\epsilon})|\right)\right|<\epsilon\right\}\in \pounds(I)$, for all $\epsilon>0$. Then, the set

$$C_\epsilon=\left\{k\in\mathbb{N}:f\left(|T(x_k)|\right)\in[f\leq(|T(x_{N_\epsilon})|)-\epsilon,f\left(|T(x_{N_\epsilon})|\right)+\epsilon]\right\}\in \pounds(I)\text{ for all }\epsilon<0$$

Let $J_\epsilon=\left[f\left(|T(x_{N_\epsilon})|\right)-\epsilon,f\left(|T(x_{N_\epsilon})|\right)+\epsilon\right]$. If we fix an $\epsilon>0$ then we have $C_\epsilon\in \pounds(I)$ as well as $C_{\frac{\epsilon}{2}}\in \pounds(I)$.

Hence $C_\epsilon\cap C_{\frac{\epsilon}{2}}\in \pounds(I)$. This implies that

$$J=J_\epsilon\cap J_{\frac{\epsilon}{2}}\neq\phi$$

That is

$$\{k\in\mathbb{N}:f\left(|T(x_k)|\right)\in J\}\in \pounds(I)$$

That is

$$diamJ\leq diamJ_\epsilon$$

where the diam of J denotes the length of interval J.

In this way, by induction, we get the sequence of closed intervals

$$J_\epsilon=I_0\supseteq I_1\supseteq\ldots\supseteq I_k\supseteq\ldots$$

with the property that $diamI_k\leq\frac{1}{2}diamI_{k-1}$ for $(k=2,3,4,\ldots)$ and $\{k\in\mathbb{N}:f\left(|T(x_k)|\right)\in I_k\}\in \pounds(I)$ for $(k=1,2,3,4,\ldots)$. Then, there exists a $\xi\in\cap I_k$ where $k\in\mathbb{N}$ such that $\xi=I-\lim f\left(|T(x_k)|\right)$ showing that $x=(x_k)\in\ell_\infty$ is I-convergent. Hence the result.

THEOREM 2.4   *Let* f_1 *and* f_2 *be two modulus functions and satisfying* Δ_2- Condition, *then*

(a) $\mathcal{X}(f_2)\subseteq\mathcal{X}(f_1f_2)$,
(b) $\mathcal{X}(f_1)\cap(f_2)\subseteq\mathcal{X}(f_1+f_2)$
   for $\mathcal{X}=S^I,S_o^I,\mathcal{M}_S^I$ and $\mathcal{M}_{S_o}^I$.

Proof   (a) Let $x=(x_k)\in S_o^I(f_2)$ be any arbitrary element. Then, the set

$$\left\{k\in\mathbb{N}:f_2\left(\left|T(x_k)\right|\right)\geq\epsilon\right\}\in I \tag{2.11}$$

Let $\epsilon>0$ and choose δ with $0<\delta<1$ such that $f_1(t)<\epsilon$, $0\leq t\leq\delta$.

Let us denote

$$y_k=f_2\left(\left|T(x_k)\right|\right)$$

and consider

$$\lim_k f_1(y_k)=\lim_{y_k\leq\delta,k\in\mathbb{N}}f_1(y_k)+\lim_{y_k>\delta,k\in\mathbb{N}}f_1(y_k)$$

Now, since f_1 is an modulus function, we have

$$\lim_{y_k \leq \delta, k \in \mathbb{N}} f_1(y_k) \leq f_1(2) \lim_{y_k \leq \delta, k \in \mathbb{N}} (y_k) \tag{2.12}$$

For $y_k > \delta$, we have

$$y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}$$

Now, since f_1 is non-decreasing and modulus, it follows that

$$f_1(y_k) < f_1\left(1 + \frac{y_k}{\delta}\right) < \frac{1}{2}f_1(2) + \frac{1}{2}f_1\left(\frac{2y_k}{\delta}\right)$$

Again, since f_1 satisfies $\Delta_2 -$ Condition, we have

$$f_1(y_k) < \frac{1}{2}K\frac{(y_k)}{\delta}f_1(2) + \frac{1}{2}K\frac{(y_k)}{\delta}f_1(2)$$

Thus, $f_1(y_k) < K\frac{(y_k)}{\delta}f_1(2)$ Hence,

$$\lim_{y_k > \delta, k \in \mathbb{N}} f_1(y_k) \leq \max\{1, K\delta^{-1}f_1(2) \lim_{y_k > \delta, k \in \mathbb{N}} (y_k) \tag{2.13}$$

Therefore, from Equations 2.11–2.13, we have $(x_k) \in S_{\circ}^I(f_1 f_2)$ Thus, $S_{\circ}^I(f_2) \subseteq S_{\circ}^I(f_1 f_2)$. Hence, $\mathcal{X}(f_2) \subseteq \mathcal{X}(f_1 f_2)$ for $\mathcal{X} = S_{\circ}^I$. For $\mathcal{X} = S^I, \mathcal{M}_S^I,$ and $\mathcal{M}_{S_{\circ}}^I$ the inclusions can be established similarly.

(b) Let $x = (x_k) \in S_{\circ}^I(f_1) \cap S_{\circ}^I(f_2)$. Let $\epsilon > 0$ be given. Then, the sets

$$\left\{ k \in \mathbb{N} : f_1\left(|T(x_k)| \right) \geq \epsilon \right\} \in I \tag{2.14}$$

and

$$\left\{ k \in \mathbb{N} : f_2\left(|T(x_k)| \right) \geq \epsilon \right\} \in I \tag{2.15}$$

Therefore, from Equations 2.14 and 2.15 the set

$$\left\{ k \in \mathbb{N} : (f_1 + f_2)\left(|T(x_k)| \right) \geq \epsilon \right\} \in I$$

Thus, $x = (x_k) \in S_{\circ}^I(f_1 + f_2)$. Hence, $S_{\circ}^I(f_1) \cap S_{\circ}^I(f_2) \subseteq S_{\circ}^I(f_1 + f_2)$. For $\mathcal{X} = S^I, \mathcal{M}_S^I,$ and $\mathcal{M}_{S_{\circ}}^I$ the inclusions are similar.

For $f_2(x) = x$ and $f_1(x) = f(x), \forall x \in [0, \infty)$, we have the following corollary.

COROLLARY 2.5 $\quad \mathcal{X} \subseteq \mathcal{X}(f)$ for $\mathcal{X} = S^I, S_{\circ}^I, \mathcal{M}_S^I$ and $\mathcal{M}_{S_{\circ}}^I$

THEOREM 2.6 *For any modulus function f, the spaces $S_{\circ}^I(f)$ and $\mathcal{M}_{S_{\circ}}^I(f)$ are solid and monotone.*

Proof we prove the result for the space $S_{\circ}^I(f)$. For $\mathcal{M}_{S_{\circ}}^I(f)$, the proof can be obtained similarly.

For, let $(x_k) \in S_{\circ}^I(f)$ be any arbitrary element. Then, the set

$$\{k \in \mathbb{N} : f\left(|T(x_k)| \right) \geq \epsilon\} \in I \tag{2.16}$$

Let (α_k) be a sequence of scalars such that

$| \alpha_k | \leq 1$, for all $k \in \mathbb{N}$

Then the result follows from Equation 2.16 and the following inequality.

$$f\left(| T(\alpha_k x_k) | \right) = f\left(| \alpha_k T(x_k) | \right) \leq | \alpha_k | f\left(| T(x_k) | \right) \leq f\left(| T(x_k) | \right), \quad \text{for all } k \in \mathbb{N}$$

That the space $S_o^I(f)$ is monotone follows from the Lemma (I). Hence $S_o^I(f)$ is solid and monotone.

THEOREM 2.7   *The spaces $S^I(f)$ and $\mathcal{M}_S^I(f)$ are not neither solid nor monotone.*

Proof   Here we give a counter example for the proof of this result.

Counter example. Let $I = I_f$ and $f(x) = x$ for all $x \in [0, \infty)$. Consider the K-step $\mathcal{Z}_K$ of $\mathcal{Z}$ defined as follows.

Let $(x_k) \in \mathcal{Z}$ and let $(y_k) \in \mathcal{Z}_K$ be such that

$$y_k = \begin{cases} x_k, & \text{if } k \text{ is even,} \\ 0, & \text{otherwise} \end{cases}$$

Consider the sequence (x_k) defined as by $x_k = 1$ for all $k \in \mathbb{N}$. Then $(x_k) \in S^I(f)$ and $\mathcal{M}_S^I(f)$ but its K-step preimage does not belong to $S^I(f)$ and $\mathcal{M}_S^I(f)$. Thus, $S^I(f)$ and $\mathcal{M}_S^I(f)$ are not monotone. Hence, $S^I(f)$ and $\mathcal{M}_S^I(f)$ are not solid by Lemma(I).

THEOREM 2.8   *If $(x = x_k)$ and $(y = y_k)$ be two sequences with $T(x \cdot y) = T(x)T(y)$. Then, the spaces $S^I(f)$ and $S_o^I(f)$ are sequence algebra.*

Proof   Let $(x = x_k)$ and $(y = y_k)$ be two elements of $S_o^I(f)$ with $T(x \cdot y) = T(x)T(y)$.

Then, the sets

$$\left\{ k \in \mathbb{N} : f\left(| T(x_k) | \right) \geq \epsilon \right\} \in I \tag{2.17}$$

and

$$\left\{ k \in \mathbb{N} : f\left(| T(y_k) | \right) \geq \epsilon \right\} \in I \tag{2.18}$$

Therefore,

$$\left\{ k \in \mathbb{N} : f\left(| T(x_k) . T(y_k) | \right) \geq \epsilon \right\} \in I$$

Thus, $(x_k) . (y_k) \in S_o^I(f)$.

Hence, $S_o^I(f)$ is sequence algebra. For $S^I(f)$, the result can be proved similarly.

THEOREM 2.9   *Let f be a modulus function. Then, $S_o^I(f) \subset S^I(f) \subset S_\infty^I(f)$.*

Proof   The inclusion $S_o^I(f) \subset S^I(f)$ is obvious.

Next, let $(x_k) \in S^I(f)$. Then there exists some L such that

$$\{ k \in \mathbb{N} : f\left(| T(x_k) - L | \right) \geq \epsilon \} \in I$$

We have

$$f\left(\,|\,T(x_k)\,|\,\right) \leq \frac{1}{2}f\left(\,|\,T(x_k)-L\,|\,\right) + f\left(\frac{1}{2}\,|\,L\,|\,\right)$$

Taking supremum over k on both sides, we get $(x_k) \in S_\infty^I(f)$

Hence, $S_\circ^I(f) \subset S^I(f) \subset S_\infty^I(f)$

THEOREM 2.10 *If $f(x) = x$ for all $x \in [0, \infty]$. Then, the function $\hbar : \mathcal{M}_S^I(f) \to \mathbb{R}$ defined by $\hbar(x) = I - \lim f(|\,T(x_k)\,|)$, where $\mathcal{M}_S^I(f) = S_\infty(f) \cap S^I(f)$ is a Lipschitz function and hence uniformly continuous.*

Proof Clearly, the function $\hbar$ is well defined. Let $x = (x_k)$, $y = (y_k) \in \mathcal{M}_S^I(f)$, $x \neq y$.

Then, the sets

$$A_x = \left\{ k \in \mathbb{N} : f\left(\,|\,T(x) - \hbar(x)\,|\,\right) \geq \|\,x - y\,\|_* \right\} \in I$$

$$A_y = \left\{ k \in \mathbb{N} : f\left(\,|\,T(y) - \hbar(y)\,|\,\right) \geq \|\,x - y\,\|_* \right\} \in I$$

where

$$\|\,x - y\,\|_* = \sup_k f\left(\,|\,T(x_k) - T(y_k)\,|\,\right)$$

Thus, the sets

$$B_x = \left\{ k \in \mathbb{N} : |\,T(x) - \hbar(x)\,| < \|\,x - y\,\|_* \right\} \in \pounds(I)$$

$$B_y = \left\{ k \in \mathbb{N} : |\,T(y) - \hbar(y)\,| < \|\,x - y\,\|_* \right\} \in \pounds(I)$$

Hence, $B = B_x \cap B_y \in \pounds(I)$, so that $B \neq \emptyset$ Now, taking $k \in B$, we have

$$|\,\hbar(x) - \hbar(y)\,| \leq |\,\hbar(x) - T(x_k)\,| + |\,T(x_k) - T(y_k)\,| + |\,T(y_k) - \hbar(y)\,| \leq 3\,\|\,x - y\,\|_*$$

Therefore, $\hbar$ is Lipschitz function and hence uniformly continuous.

THEOREM 2.11 *If $f(x) = x$ for all $x \in [0, \infty]$ and if $x = (x_k)$, $y = (y_k) \in \mathcal{M}_S^I(f)$ with $T(x \cdot y) = T(x)T(y)$. Then $(x \cdot y) \in \mathcal{M}_S^I(f)$ and $\hbar(xy) = \hbar(x)\hbar(y)$ where $\hbar : \mathcal{M}_S^I(f) \to \mathbb{R}$ is defined by $\hbar(x) = I - \lim f(|\,T(x_k)\,|)$.*

Proof For $\epsilon > 0$, the sets
$$B_x = \{k \in \mathbb{N} : |T(x_k) - \hbar(x)| < \epsilon\} \in \pounds(I) \tag{2.19}$$

$$B_y = \{k \in \mathbb{N} : |T(y_k) - \hbar(y)| < \epsilon\} \in \pounds(I) \tag{2.20}$$

where $\|\,x - y\,\|_* = \epsilon$

Now,

$$\begin{aligned}|T(x_k y_k) - \hbar(x)\hbar(y)| &= |T(x_k)T(y_k) - T(x_k)\hbar(y) + T(x_k)\hbar(y) - \hbar(x)\hbar(y)| \\ &\leq |T(x_k)||y_k - \hbar(y)| + |\hbar(y)||x_k - \hbar(x)|\end{aligned} \tag{2.21}$$

As $\mathcal{M}_S^I(f) \subseteq S_\infty(f)$, there exists an $M \in \mathbb{R}$ such that $|T(x_k)| < M$ and $|\hbar(y)| < M$.

Therefore, from Equations 2.19–2.21, we have

$$|T(x_k y_k) - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon$$

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for all $k \in B_x \cap B_y \in \pounds(I)$.

Hence $(x \cdot y) \in \mathcal{M}_S^I(f)$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

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