7th International Symposium on Imprecise Probability: Theories and Applications, Innsbruck, Austria, 2011

Characterizing joint distributions of random sets with an application to set-valued stochastic processes

Bernhard Schmelzer
Unit of Engineering Mathematics,
University of Innsbruck, Austria
bernhard.schmelzer@uibk.ac.at

Abstract

By the Choquet theorem, distributions of random closed sets can be characterized by a certain class of set functions called capacity functionals. In this paper a generalization to the multivariate case is presented, that is, it is proved that the joint distribution of finitely many random sets can be characterized by a set function fulfilling certain properties. Furthermore, we use this result to formulate an existence theorem for set-valued stochastic processes.

Keywords. Random set, Choquet theorem, capacity functional, joint distribution, Daniell-Kolmogorov theorem.

1 Introduction

Random sets, or set-valued maps, can be used to model uncertainty. They can be interpreted as imprecise observations of random variables ([10]) which assign to each element of the underlying probability space a set instead of a single value. These sets (called focal sets) are supposed to contain the true value of the variable.

We will consider random closed sets, that is, random maps whose values are closed subsets of a topological space $\mathbb{E}$, since they have favorable properties. The family of all closed subsets of $\mathbb{E}$ will be denoted by $\mathcal{F}$ which can in turn be topologized by the so-called Fell topology ([1]). Random closed sets can then be seen as random elements with values in $\mathcal{F}$ and classical probability theory can be applied. As already mentioned, they can also be interpreted as imprecise observations of random variables ([10]). In this case, one is more interested in events from the Borel-σ-algebra $\mathcal{B}(\mathbb{E})$, than from $\mathcal{B}(\mathcal{F})$ and non-additive set functions (so-called lower and upper probabilities, see [4]) are introduced to measure if the focal elements hit or miss a certain set from $\mathcal{B}(\mathbb{E})$. The link between these two interpretations is given by the so-called Choquet theorem (also referred to as the Choquet-Matheron-Kendall theorem, see [13, 15, 17]), which states a one-to-one correspondence between probability distributions on $\mathcal{B}(\mathcal{F})$ and a certain class of non-additive set functions, called capacity functionals, on $\mathcal{B}(\mathbb{E})$.

The goal of this paper is to present characterizations of the joint distribution of finitely many random sets. More precisely, given n random sets we will link their joint distribution defined on the product-σ-algebra $\mathcal{B}(\mathcal{F})^{\otimes n}$ to set functions defined on the compacts of the co-product $\mathbb{E} \times \{1, \ldots, n\}$ or a certain class of subsets of $\mathbb{E}^n$.

The plan of the paper is as follows. In Section 2 we review the most important facts on random sets and their distributions including the classical Choquet theorem. The main part of the paper is Section 3 where joint distributions of random sets are considered and characterized by multivariate capacities. In Section 4 the latter is used to formulate a Daniell-Kolmogorov existence theorem ([5, 7]) for set-valued stochastic processes. Furthermore, we consider Brownian motion as an example.

2 Random closed sets and Choquet theorem

In this section we review the most important facts about random closed sets. As already mentioned in the introduction we consider maps whose values are closed subsets of some topological space $\mathbb{E}$. Throughout the paper, $\mathcal{G}$, $\mathcal{F}$, $\mathcal{K}$ will denote the families of open, closed, compact subsets of $\mathbb{E}$, respectively. Furthermore, we will use the following notation

$$\begin{aligned} \mathcal{F}_A &= \{F \in \mathcal{F} : F \cap A \neq \emptyset\} \\ \mathcal{F}^A &= \{F \in \mathcal{F} : F \cap A = \emptyset\} \\ \mathcal{F}^A_{A_1,\ldots,A_k} &= \mathcal{F}^A \cap \mathcal{F}_{A_1} \cap \cdots \cap \mathcal{F}_{A_k} \end{aligned}$$

for arbitrary subsets $A, A_1, \dots, A_k$ of $\mathbb{E}$. The family $\mathcal{F}$ is endowed with the Fell topology ([1]). Recall that the latter has as a sub-base $\{\mathcal{F}_G\}_{G\in\mathcal{G}} \cup \{\mathcal{F}^K\}_{K\in\mathcal{K}}$, that is, sets of the form $\mathcal{F}^K_{G_1,\dots,G_k}$ ($K \in \mathcal{K}$, $G_i \in \mathcal{G}$) constitute a base. We shall always assume that $\mathbb{E}$ is a locally compact Hausdorff second countable (LCHS) space. In this case, $\mathcal{F}$ together with the Fell topology becomes a compact Hausdorff second countable space ([1]). In addition, we introduce on $\mathcal{F}$ the so-called Effros-σ-algebra $\mathcal{B}(\mathcal{F})$ which is generated by the sets $\{\mathcal{F}_G\}_{G\in\mathcal{G}}$. By virtue of the LCHS property of $\mathbb{E}$, the Effros-σ-algebra is also generated by $\{\mathcal{F}_K\}_{K\in\mathcal{K}}$ and is the Borel-σ-algebra with respect to the Fell topology. For details and further information about topologies on $\mathcal{F}$ the reader is referred to the monograph [1].

A map $X : \Omega \to \mathcal{F}$ on a probability space (Ω, Σ, P) will be called Effros-measurable if

$$X^-(G) = \{\omega : X(\omega) \cap G \neq \emptyset\} = X^{-1}(\mathcal{F}_G) \in \Sigma$$

for all $G \in \mathcal{G}$ whereas X will be called random (closed) set if it is strongly measurable ([18]), i.e., $X^-(B) \in \Sigma$ for all $B \in \mathcal{B}(\mathbb{E})$. Note that in general the two conditions are not equivalent unless (Ω, Σ, P) is complete (see [2, 8]). The distribution of an Effros-measurable map X is then the image measure P_X of P on $\mathcal{B}(\mathcal{F})$. For the generating sets $\mathcal{F}_K$ ($K \in \mathcal{K}$) of $\mathcal{B}(\mathcal{F})$ the probabilities $P_X(\mathcal{F}_K) = P(X^-(K))$ can be expressed by a set function $\varphi : \mathcal{K} \to [0,1], K \mapsto P_X(\mathcal{F}_K)$. This set function corresponds to the upper probability of a random set introduced by Dempster and Shafer ([4]) and has (among others) the following properties:

(CF1) $0 \le \varphi \le 1$ and $\varphi(\emptyset) = 0$,

(CF2) For $K, K_1, \dots, K_n \in \mathcal{K}$, $n \ge 0$, the probabilities $P_X(\mathcal{F}^K_{K_1,\dots,K_n})$ can be written in terms of φ as

$$P_X(\mathcal{F}^K_{K_1,\dots,K_n}) = \Delta_n\varphi(K; K_1, \dots, K_n)$$

where $\Delta_0\varphi(K) = 1 - \varphi(K)$ and for $n \ge 1$

$$\begin{aligned}\Delta_n\varphi(K; K_1, \dots, K_n) &\\ = \Delta_{n-1}\varphi(K; K_1, \dots, K_{n-1}) &\\ - \Delta_{n-1}\varphi(K \cup K_n; K_1, \dots, K_{n-1}). &\end{aligned}$$

Thus, $\Delta_n\varphi \ge 0$ for $n \ge 0$.

(CF3) φ is continuous from above, that is, for a decreasing sequence $\{K_n\}_{n\in\mathbb{N}}$ with limit $K = \bigcap_{n\in\mathbb{N}} K_n$ it holds that $\varphi(K_n) \searrow \varphi(K)$.

Note that a set function fulfilling Condition (CF2) is called completely alternating. Furthermore, for $n \ge 1$ the successive differences can be expressed as follows:

$$\Delta_n\varphi(K; K_1, \dots, K_n) = -\sum_{I\subseteq\{1,\dots,n\}} (-1)^{|I|}\, \varphi\big(K \cup \bigcup_{i\in I} K_i\big) \quad (1)$$

where the union over $\emptyset$ is set to $\emptyset$. A set function on $\mathcal{K}$ fulfilling these three properties is called capacity functional. The following theorem known as the Choquet theorem (see [13, 15, 17]) says that there is a one-to-one correspondence between capacity functionals and probability measures on $\mathcal{B}(\mathcal{F})$.

Theorem 1. Let $\mathbb{E}$ be an LCHS space and let $\varphi : \mathcal{K} \to [0,1]$ be a capacity functional. Then there exists a unique probability measure Π on $\mathcal{B}(\mathcal{F})$ such that $\varphi(K) = \Pi(\mathcal{F}_K)$ for all $K \in \mathcal{K}$.

For later reference we give a sketch of the proof ([13]): First, note that a capacity functional φ can be extended to the power set $\mathcal{P}$ of $\mathbb{E}$ by setting

$$\begin{aligned}\varphi_*(G) &= \sup\{\varphi(K) : K \subseteq G, K \in \mathcal{K}\} \text{ if } G \in \mathcal{G},\\ \varphi^*(A) &= \inf\{\varphi_*(G) : G \supseteq A, G \in \mathcal{G}\} \text{ if } A \in \mathcal{P}.\end{aligned} \quad (2)$$

The extension φ^* is a completely alternating Choquet-$\mathcal{K}$-capacity, that is, φ^* is continuous from above on $\mathcal{K}$ and continuous from below on $\mathcal{P}$ ([3, 14]). Furthermore, the extension is consistent, i.e., on $\mathcal{K}$ the extension yields the same results as if φ is directly applied. To obtain the desired probability measure on $\mathcal{B}(\mathcal{F})$ the set function φ^* is considered on $\mathcal{V} = \{G \cup K : G \in \mathcal{G}, K \in \mathcal{K}\}$ and a set function Π is defined on $\mathcal{H} = \{\mathcal{F}^V_{V_1,\dots,V_k} : V, V_j \in \mathcal{V}, k \ge 0, 1 \le j \le k\}$ by $\Pi(\mathcal{F}^V_{V_1,\dots,V_k}) = \Delta_k\varphi^*(V; V_1, \dots, V_k)$. Π is proved to be (finitely) additive and extended to a measure on $\mathcal{B}(\mathcal{F})$ (which is generated by $\mathcal{H}$) by using [16, Prop. I.6.2] and continuity properties of φ^*. Moreover, one can show (cf. [6], Appendix, 2, Satz 2) that for all $B \in \mathcal{B}(\mathbb{E})$ it holds that $\mathcal{F}_B \in \mathcal{B}(\mathcal{F})^0$ and $\varphi^*(B) = \Pi^0(\mathcal{F}_B)$ where $(\mathcal{F}, \mathcal{B}(\mathcal{F})^0, \Pi^0)$ denotes the completed probability space with respect to Π.

3 The multivariate case

Let $n \ge 2$ and $\mathbb{E}_i$ be LCHS spaces with $\mathcal{G}_i$, $\mathcal{F}_i$, $\mathcal{K}_i$ denoting the families of open, closed, compact subsets of $\mathbb{E}_i$, respectively, $1 \le i \le n$. As already outlined in the introduction the goal is to characterize probability measures on the Borel sets of

$$\mathcal{F}^n = \mathcal{F}_1 \times \cdots \times \mathcal{F}_n = \{(F_1, \dots, F_n) : F_i \in \mathcal{F}_i\}$$

by set functions. $\mathcal{F}^n$ will be endowed with the product Fell topology which is generated by the cylindrical sets

$$\mathcal{F}^{K_1}_{G_{11},\dots,G_{1k_1}} \times \cdots \times \mathcal{F}^{K_n}_{G_{n1},\dots,G_{nk_n}}$$

where $G_{ij_i} \in \mathcal{G}_i$, $K_i \in \mathcal{K}_i$. From the one-dimensional case one can infer that the product-Effros-σ-algebra $\mathcal{B}(\mathcal{F}^n) = \mathcal{B}(\mathcal{F}_i)^{\otimes n} = \mathcal{B}(\mathcal{F}_1) \otimes \cdots \otimes \mathcal{B}(\mathcal{F}_n)$ is generated by the sets

$$\mathcal{F}_{K_1} \times \cdots \times \mathcal{F}_{K_n}$$

where $K_i \in \mathcal{K}_i$. For n Effros-measurable maps (random sets) $X_i : \Omega \to \mathcal{F}_i$ on a probability space (Ω, Σ, P) their joint distribution is then given by

$$\begin{aligned}
&P_{X_1,\ldots,X_n}(\mathcal{F}_{K_1} \times \cdots \times \mathcal{F}_{K_n}) \\
&= P(\{\omega : (X_1(\omega), \ldots, X_n(\omega)) \in \mathcal{F}_{K_1} \times \cdots \times \mathcal{F}_{K_n}\}) \\
&= P(\{\omega : X_1(\omega) \cap K_1 \neq \emptyset, \ldots, X_n(\omega) \cap K_n \neq \emptyset\}) \\
&= P\Big(\bigcap_{i=1}^{n} X_i^-(K_i)\Big). \quad (3)
\end{aligned}$$

The latter can be expressed by using $K_1 \times \cdots \times K_n$ which is a subset of $\mathbb{E}^n = \mathbb{E}_1 \times \cdots \times \mathbb{E}_n$:

$$\begin{aligned}
&P_{X_1,\ldots,X_n}(\mathcal{F}_{K_1} \times \cdots \times \mathcal{F}_{K_n}) \\
&= P(\{\omega : X_1(\omega) \times \cdots \times X_n(\omega) \cap K_1 \times \cdots \times K_n \neq \emptyset\})
\end{aligned} \quad (4)$$

Motivated by this, we use the following notation for arbitrary $V, V_1, \ldots, V_k \subseteq \mathbb{E}^n$

$$\begin{aligned}
{}^n\mathcal{F}_V &= \{(F_1, \ldots, F_n) \in \mathcal{F}^n : F_1 \times \cdots \times F_n \cap V \neq \emptyset\} \\
{}^n\mathcal{F}^V &= \{(F_1, \ldots, F_n) \in \mathcal{F}^n : F_1 \times \cdots \times F_n \cap V = \emptyset\} \\
{}^n\mathcal{F}^V_{V_1,\ldots,V_k} &= {}^n\mathcal{F}^V \cap {}^n\mathcal{F}_{V_1} \cap \cdots \cap {}^n\mathcal{F}_{V_k}
\end{aligned}$$

which implies $\mathcal{F}_{K_1} \times \cdots \times \mathcal{F}_{K_n} = {}^n\mathcal{F}_{K_1 \times \cdots \times K_n}$. The event $(X_1, \ldots, X_n)^{-1}({}^n\mathcal{F}_V)$ corresponds to the event that the set-valued map

$$X : \omega \mapsto X_1(\omega) \times \cdots \times X_n(\omega) \quad (5)$$

hits V. Note that the values of X are closed subsets of $\mathbb{E}^n$, more precisely closed cylindrical sets, and not elements of $\mathcal{F}^n$. One can prove ([19]) that X is Effros-measurable by using selections and the so-called Fundamental measurability theorem for multifunctions ([2, 8]). Consequently, the map

$$K \mapsto P(X^-(K))$$

is a capacity functional on the compact subsets of $\mathbb{E}^n$ denoted by $\mathcal{K}(\mathbb{E}^n)$. One could thus think of characterizing joint distributions of n random sets by capacity functionals on $\mathcal{K}(\mathbb{E}^n)$. But applying the Choquet theorem leads to a probability measure on the Borel sets of $\mathcal{F}(\mathbb{E}^n)$ denoting the family of closed subsets of $\mathbb{E}^n$. The latter is clearly different from $\mathcal{F}^n$ which can only be identified with the cylindrical closed subsets of $\mathbb{E}^n$, that is, $\{F_1 \times \cdots \times F_n : F_i \in \mathcal{F}_i\}$ which is a proper subset of $\mathcal{F}(\mathbb{E}^n)$.

Hence, there is the need for a different concept. In the following, we will consider the co-product of the spaces $\mathbb{E}_i$, that is,

$$\mathbb{E}^n_{\amalg} = \bigcup_{i=1}^{n} \mathbb{E}_i \times \{i\}$$

which is a union of n mutually disjoint sets. We endow $\mathbb{E}^n_{\amalg}$ with the sum topology, that is, we take

$$\mathcal{G}^n_{\amalg} = \bigcap_{i=1}^{n} \{G \subseteq \mathbb{E}^n_{\amalg} : \iota_i^{-1}(G) \in \mathcal{G}_i\}$$

as the family of open sets. The latter is the smallest topology on $\mathbb{E}^n_{\amalg}$ such that the canonical injections $\iota_i : \mathbb{E}_i \to \mathbb{E}^n_{\amalg}, x \mapsto (x, i)$ are continuous. Moreover, $\mathcal{G}^n_{\amalg} = \{\bigcup_{i=1}^n G_i \times \{i\} : G_i \in \mathcal{G}_i\}$ and the analogous relations hold for the families of closed, compact and Borel subsets of $\mathbb{E}^n_{\amalg}$, respectively. It easy is to see that all topological properties of the $\mathbb{E}_i$ carry over to the co-product and so $\mathbb{E}^n_{\amalg}$ is an LCHS space, too.

The question is how the co-product can be used to characterize probability distributions on $\mathcal{B}(\mathcal{F}^n)$. Obviously, each subset A of $\mathbb{E}^n_{\amalg}$ can be written in the form $A = \amalg A_i = \bigcup_{i=1}^n A_i \times \{i\}$ where the A_i are the sections of A, i.e. $A_i = \{x \in \mathbb{E}_i : (x, i) \in A\}$, and consequently A can be identified with the tuple $(A_1, \ldots, A_n)$. Hence, we have a one-to-one correspondence between subsets of the co-product $\mathbb{E}^n_{\amalg}$ and tuples of subsets of the $\mathbb{E}_i$. But this means that we have a one-to-one correspondence between $\mathcal{F}^n_{\amalg}$ and $\mathcal{F}^n$ and similarly between $\mathcal{K}^n_{\amalg}$ and $\mathcal{K}^n = \mathcal{K}_1 \times \cdots \times \mathcal{K}_n$.

Consequently, each set function φ on $\mathcal{K}^n_{\amalg}$ is related to a set function ψ on $\mathcal{K}^n$ by

$$\varphi(\amalg K_i) = \psi(K_1, \ldots, K_n). \quad (6)$$

The following lemma shows that φ is a capacity functional if and only if ψ is completely alternating and continuous from above in each component. From now on a set function on $\mathcal{K}^n$ fulfilling Conditions (MCF1) - (MCF3) of the following lemma shall be called multivariate capacity functional.

Lemma 1. Let $\varphi : \mathcal{K}^n_{\amalg} \to [0, 1]$ and $\psi : \mathcal{K}^n \to [0, 1]$ satisfying Equation (6) for all $(K_1, \ldots, K_n) \in \mathcal{K}^n$. Then φ is a capacity functional if and only if ψ fulfills the following conditions:

(MCF1) $\psi(\emptyset, \ldots, \emptyset) = 0$

(MCF2) For all $k \geq 0$, $1 \leq j \leq k$, $K = (K_1, \ldots, K_n)$, $K^j = (K^j_1, \ldots, K^j_n) \in \mathcal{K}^n$ it holds that

$$\Delta_k \psi(K; K^1, \ldots, K^k) \geq 0$$

where $\Delta_0\psi(K) = 1 - \psi(K_1,\dots,K_n)$,

$$\begin{aligned}
&\Delta_k\psi(K;K^1,\dots,K^k)\\
&\quad= \Delta_{k-1}\psi(K;K^1,\dots,K^{k-1})\\
&\qquad- \Delta_{k-1}\psi(K\cup K^k;K^1,\dots,K^{k-1})
\end{aligned}$$

and $K \cup K^k = (K_1 \cup K_1^k,\dots,K_n \cup K_n^k)$.

(MCF3) For all decreasing sequences $\{K_i^k\}_{k\in\mathbb{N}} \subseteq \mathcal{K}_i$, $1 \le i \le n$, it holds that $\psi(K_1^k,\dots,K_n^k) \searrow \psi(K_1,\dots,K_n)$ for $k\to\infty$ where $K_i = \bigcap_{k\in\mathbb{N}} K_i^k$.

Proof. The equivalence follows from the relation $\varphi(\bigcup_{i=1}^n K_i \times \{i\}) = \psi(K_1,\dots,K_n)$. Indeed, we get $\psi(\emptyset,\dots,\emptyset) = \varphi(\bigcup_{i=1}^n \emptyset\times\{i\}) = \varphi(\emptyset)$. Furthermore, by Formula (1) we have

$$\begin{aligned}
&\Delta_k\psi(K;K^1,\dots,K^k)\\
&\quad= -\sum_{J\subseteq\{1,\dots,k\}}(-1)^{|J|}\psi\Big(K\cup\bigcup_{j\in J}K^j\Big)\\
&= -\sum_{J\subseteq\{1,\dots,k\}}(-1)^{|J|}\psi\Big(K_1\cup\bigcup_{j\in J}K_1^j,\dots,K_n\cup\bigcup_{j\in J}K_n^j\Big)\\
&\ = -\sum_{J\subseteq\{1,\dots,k\}}(-1)^{|J|}\varphi\Big(\bigcup_{i=1}^n\big(K_i\cup\bigcup_{j\in J}K_i^j\big)\times\{i\}\Big)\\
&\quad = -\sum_{J\subseteq\{1,\dots,k\}}(-1)^{|J|}\varphi\Big(\big(\amalg K_i\big)\cup\bigcup_{j\in J}\big(\amalg K_i^j\big)\Big)\\
&\qquad = \Delta_k\varphi\Big(\amalg K_i;\amalg K_i^1,\dots,\amalg K_i^k\Big).
\end{aligned}$$

The equivalence of (MCF3) and (CF3) follows from the fact that $K_i^k \searrow K_i$ for all $1\le i\le n$ if and only if $\amalg K_i^k = \bigcup_{i=1}^n K_i^k\times\{i\} \searrow \bigcup_{i=1}^n K_i\times\{i\} = \amalg K_i$. $\square$

Given a multivariate set function $\psi : \mathcal{K}^n \to [0,1]$ fulfilling conditions (MCF1) - (MCF3) of the foregoing lemma, the Choquet theorem (Theorem 1) can be applied to the capacity functional $\varphi : \mathcal{K}_{\amalg}^n \to [0,1]$ defined by $\amalg K_i \mapsto \psi(K_1,\dots,K_n)$. This yields a probability measure $Q : \mathcal{B}(\mathcal{F}_{\amalg}^n) \to [0,1]$ such that for all $\amalg K_i \in \mathcal{K}_{\amalg}^n$ it holds that

$$\varphi(\amalg K_i) = Q(\{\amalg F_j \in \mathcal{F}_{\amalg}^n : \amalg F_j \cap \amalg K_i \neq \emptyset\}). \tag{7}$$

The right-hand side of Equation (7) can further be written in the following form:

$$\begin{aligned}
&Q(\{\amalg F_j \in \mathcal{F}_{\amalg}^n : \amalg F_j\cap\amalg K_i \neq\emptyset\})\\
&= Q\Big(\Big\{\amalg F_j\in\mathcal{F}_{\amalg}^n : \big(\bigcup_{j=1}^n F_j\times\{j\}\big)\cap\big(\bigcup_{i=1}^n K_i\times\{i\}\big)\neq\emptyset\Big\}\Big)\\
&= Q\Big(\bigcup_{i=1}^n\Big\{\amalg F_j\in\mathcal{F}_{\amalg}^n : \big(\bigcup_{j=1}^n F_j\times\{j\}\big)\cap(K_i\times\{i\})\neq\emptyset\Big\}\Big)\\
&\qquad = Q\Big(\bigcup_{i=1}^n\{\amalg F_j\in\mathcal{F}_{\amalg}^n : F_i\cap K_i\neq\emptyset\}\Big)\\
&\qquad = Q\Big(\bigcup_{i=1}^n\{\amalg F_j\in\mathcal{F}_{\amalg}^n : (F_1,\dots,F_n)\in\widehat{\mathcal{F}}_{K_i}\}\Big)\\
&\qquad = Q\Big(\Big\{\amalg F_j\in\mathcal{F}_{\amalg}^n : (F_1,\dots,F_n)\in\bigcup_{i=1}^n\widehat{\mathcal{F}}_{K_i}\Big\}\Big)
\end{aligned}\tag{8}$$

where $\widehat{\mathcal{F}}_{K_i} = \{(F_1,\dots,F_n)\in\mathcal{F}^n : F_i\cap K_i\neq\emptyset\}$.

As already mentioned we have a one-to-one correspondence between $\mathcal{F}_{\amalg}^n$ and $\mathcal{F}^n$. This can be used to define a probability measure Π on $\mathcal{B}(\mathcal{F}^n)$ from the probability measure Q on $\mathcal{B}(\mathcal{F}_{\amalg}^n)$ as the following lemma shows.

Lemma 2. It holds that

$$\mathcal{B}(\mathcal{F}_{\amalg}^n) = \big\{\{\amalg F_i\in\mathcal{F}_{\amalg}^n : (F_1,\dots,F_n)\in B\} : B\in\mathcal{B}(\mathcal{F}^n)\big\}.$$

Furthermore, if $Q : \mathcal{B}(\mathcal{F}_{\amalg}^n)\to[0,1]$ is a probability measure then $\Pi : \mathcal{B}(\mathcal{F}^n)\to[0,1]$ defined by

$$\Pi(B) = Q(\{\amalg F_i\in\mathcal{F}_{\amalg}^n : (F_1,\dots,F_n)\in B\}) \tag{9}$$

is a probability measure, too.

Proof. Let

$$\mathcal{A}_1 = \big\{\{\amalg F_i\in\mathcal{F}_{\amalg}^n : (F_1,\dots,F_n)\in B\} : B\in\mathcal{B}(\mathcal{F}^n)\big\}.$$

The σ-algebra $\mathcal{B}(\mathcal{F}_{\amalg}^n)$ is generated by sets of the form $\{\amalg F_i\in\mathcal{F}_{\amalg}^n : \amalg F_i\cap\amalg K_i\neq\emptyset\}$, $\amalg K_i\in\mathcal{K}_{\amalg}^n$. As in Equation (8) we obtain

$$\begin{aligned}
&\{\amalg F_j\in\mathcal{F}_{\amalg}^n : \amalg F_j\cap\amalg K_i\neq\emptyset\}\\
&\qquad = \Big\{\amalg F_j\in\mathcal{F}_{\amalg}^n : (F_1,\dots,F_n)\in\bigcup_{i=1}^n\widehat{\mathcal{F}}_{K_i}\Big\}
\end{aligned}$$

which lies in $\mathcal{A}_1$ since $\bigcup_{i=1}^n\widehat{\mathcal{F}}_{K_i}\in\mathcal{B}(\mathcal{F}^n)$. It is easy to see that $\mathcal{A}_1$ is a σ-algebra and thus $\mathcal{B}(\mathcal{F}_{\amalg}^n)\subseteq\mathcal{A}_1$. On the other hand, $\mathcal{B}(\mathcal{F}^n)$ is generated by sets of the form $\mathcal{F}_{K_1}\times\dots\times\mathcal{F}_{K_n}$, $K_i\in\mathcal{K}_i$. We obtain

$$\begin{aligned}
&\{\amalg F_j\in\mathcal{F}_{\amalg}^n : (F_1,\dots,F_n)\in\mathcal{F}_{K_1}\times\dots\times\mathcal{F}_{K_n}\}\\
&\quad = \bigcap_{i=1}^n\{\amalg F_j\in\mathcal{F}_{\amalg}^n : (F_1,\dots,F_n)\in\widehat{\mathcal{F}}_{K_i}\}\\
&\quad = \bigcap_{i=1}^n\{\amalg F_j\in\mathcal{F}_{\amalg}^n : \amalg F_j\cap(K_i\times\{i\})\neq\emptyset\}
\end{aligned}$$

which lies in $\mathcal{B}(\mathcal{F}_{\amalg}^n)$ since $K_i\times\{i\}\in\mathcal{K}_{\amalg}^n$. Furthermore, it is easy to see that

$$\mathcal{A}_2 = \{B\in\mathcal{B}(\mathcal{F}^n) : \{\amalg F_i : (F_1,\dots,F_n)\in B\}\in\mathcal{B}(\mathcal{F}_{\amalg}^n)\}$$

is a σ-algebra. Thus $\mathcal{B}(\mathcal{F}^n) = \mathcal{A}_2$ which further implies $\mathcal{A}_1\subseteq\mathcal{B}(\mathcal{F}_{\amalg}^n)$. It can be easily checked that Π is a probability measure. $\square$

From Equations (6), (7), (8) and (9) we obtain the following relation between the multivariate capacity functional ψ and the probability measure Π:

$$\psi(K_1,\ldots,K_n) = \varphi(\amalg K_i)$$
$$= Q(\{\amalg F_j \in \mathcal{F}^n_{\amalg} : \amalg F_j \cap \amalg K_i \neq \emptyset\} = \Pi\Big(\bigcup_{i=1}^{n} \widehat{\mathcal{F}}_{K_i}\Big)$$

We are now ready to formulate the following proposition which can be viewed as a multivariate version of the Choquet theorem.

Proposition 1. Let $\psi : \mathcal{K}^n \to [0,1]$ be a multivariate capacity functional (that is a set function fulfilling Conditions (MCF1) - (MCF3) of Lemma 1). Then there exists a unique probability measure $\Pi : \mathcal{B}(\mathcal{F}^n) \to [0,1]$ such that

$$\psi(K_1,\ldots,K_n) = \Pi\Big(\bigcup_{i=1}^{n} \widehat{\mathcal{F}}_{K_i}\Big)$$

for all $(K_1,\ldots,K_n) \in \mathcal{K}^n$.

This means that the probability of events of the form $\bigcup_{i=1}^{n} \widehat{\mathcal{F}}_{K_i}$ can be directly computed by ψ. Probabilities of other events like $\mathcal{F}_{K_1} \times \cdots \times \mathcal{F}_{K_n}$ can be computed by using the exclusion-inclusion principle and the complete alternation property:

$$\begin{aligned}
\Pi(\mathcal{F}_{K_1} \times \cdots \times \mathcal{F}_{K_n}) &= \Pi\Big(\bigcap_{i=1}^{n} \widehat{\mathcal{F}}_{K_i}\Big) \\
&= -\sum_{I\subseteq\{1,\ldots,n\}} (-1)^{|I|}\Pi\Big(\bigcup_{i\in I} \widehat{\mathcal{F}}_{K_i}\Big) \\
&= -\sum_{I\subseteq\{1,\ldots,n\}} (-1)^{|I|}\varphi\Big(\bigcup_{i\in I} K_i \times \{i\}\Big) \\
&= \Delta_n\varphi(\emptyset; K_1 \times \{1\},\ldots,K_n \times \{n\}) \\
&= \Delta_n\psi(\emptyset; \check{K}_1,\ldots,\check{K}_n) \quad (10)
\end{aligned}$$

where $\check{K}_i = (\emptyset,\ldots,\emptyset,K_i,\emptyset,\ldots,\emptyset) \in \mathcal{K}^n$. We can state an additional result concerning the probability of $\mathcal{F}'^n = \mathcal{F}'_1 \times \cdots \times \mathcal{F}'_n$, that is, the set of tuples of non-empty closed subsets.

Corollary 1. In the situation of Proposition 1, if ψ fulfills in addition for all $1 \le i \le n$

$$\sup\{\psi(\check{K}_i) : K_i \in \mathcal{K}_i\} = 1$$

then $\Pi(\mathcal{F}'^n) = 1$, that is, a tuple of closed sets almost surely consists of non-empty sets.

Proof. Let $\{L_i^k\}_{k\in\mathbb{N}} \in \mathcal{K}_i$ be increasing sequences such that $L_i^k \nearrow \mathbb{E}_i$ for all $1 \le i \le n$, let $\{M_i^k\}_{k\in\mathbb{N}} \subseteq \mathcal{K}_i$ be increasing sequences such that $\psi(\check{M}_i^k) \nearrow 1$ for all $1 \le i \le n$ and let $K_i^k = L_i^k \cup M_i^k$ for all $1 \le i \le n$ and $k \in \mathbb{N}$. Consequently,

$$\begin{aligned}
\mathcal{F}'^n &= \bigcup_{k\in\mathbb{N}} \mathcal{F}_{K_1^k} \times \cdots \times \mathcal{F}_{K_n^k} \\
&= \bigcup_{k\in\mathbb{N}} \bigcap_{i=1}^{n} \widehat{\mathcal{F}}_{K_i^k} = \bigcap_{i=1}^{n} \bigcup_{k\in\mathbb{N}} \widehat{\mathcal{F}}_{K_i^k}.
\end{aligned}$$

By the exclusion-inclusion principle we obtain

$$\begin{aligned}
\Pi(\mathcal{F}'^n) &= \Pi\Big(\bigcap_{i=1}^{n} \bigcup_{k\in\mathbb{N}} \widehat{\mathcal{F}}_{K_i^k}\Big) \\
&= -\sum_{\emptyset\neq I\subseteq\{1,\ldots,n\}} (-1)^{|I|}\Pi\Big(\bigcup_{i\in I} \bigcup_{k\in\mathbb{N}} \widehat{\mathcal{F}}_{K_i^k}\Big) \\
&= -\sum_{\emptyset\neq I\subseteq\{1,\ldots,n\}} (-1)^{|I|}\Pi\Big(\bigcup_{k\in\mathbb{N}} \bigcup_{i\in I} \widehat{\mathcal{F}}_{K_i^k}\Big) \\
&= -\sum_{\emptyset\neq I\subseteq\{1,\ldots,n\}} (-1)^{|I|} \sup_{k\in\mathbb{N}} \Pi\Big(\bigcup_{i\in I} \widehat{\mathcal{F}}_{K_i^k}\Big).
\end{aligned}$$

For all $I \neq \emptyset$, $i \in I$ and $k \in \mathbb{N}$ we have

$$\psi(\check{K}_i^k) = \Pi(\widehat{\mathcal{F}}_{K_i^k}) \le \Pi\Big(\bigcup_{i\in I} \widehat{\mathcal{F}}_{K_i^k}\Big) \le 1$$

and thus

$$\sup_{k\in\mathbb{N}} \psi(\check{K}_i^k) = \sup_{k\in\mathbb{N}} \Pi\Big(\bigcup_{i\in I} \widehat{\mathcal{F}}_{K_i^k}\Big) = 1.$$

Hence,

$$\Pi(\mathcal{F}'^n) = -\sum_{\emptyset\neq I\subseteq\{1,\ldots,n\}} (-1)^{|I|} = 1.$$

□

Note that if we consider n almost surely non-empty random sets $X_1,\ldots,X_n$ on a probability space (Ω,Σ,P) then the multivariate capacity functional of the product random set defined by Equation (5) is given by

$$\psi(K_1,\ldots,K_n) = P\Big(\bigcup_{i=1}^{n} X_i^-(K_i)\Big).$$

Hence, if the $\mathbb{E}_i$ are σ-compact spaces (which is the case if the $\mathbb{E}_i$ are LCHS spaces) and $\{K_i^k\}_{k\in\mathbb{N}} \subseteq \mathcal{K}_i$ are increasing sequences converging to $\mathbb{E}_i$, respectively, we obtain

$$\lim_{k\to\infty} \psi(\check{K}_i^k) = \lim_{k\to\infty} P(X_i^-(K_i^k)) = P(X_i \neq \emptyset) = 1$$

and thus the condition of Corollary 1 is fulfilled.

We will now relate multivariate capacity functionals to set functions on special classes of subsets of the product space $\mathbb{E}^n$. Up to now we have used the fact that a tuple $(A_1, \ldots, A_n)$ of subsets of the $\mathbb{E}_i$ can be identified with the set $\amalg A_i = \bigcup_{i=1}^n A_i \times \{i\}$ which is a subset of the co-product $\mathbb{E}^n_{\amalg}$. On the other hand, a tuple $(A_1, \ldots, A_n)$ can be identified with $\bigcup_{i=1}^n \hat{A}_i$ where

$$\hat{A}_i = \{(x_1, \ldots, x_n) \in \mathbb{E}^n : x_i \in A_i\}.$$

Consequently, we have a one-to-one correspondence between $\mathcal{K}^n$ and

$$\hat{\mathcal{K}}^n_{\cup} = \Big\{ \bigcup_{i=1}^n \hat{K}_i : K_i \in \mathcal{K}_i \Big\}$$

and each set function ψ on $\mathcal{K}^n$ is related to a set function ϕ on $\hat{\mathcal{K}}^n_{\cup}$ by

$$\psi(K_1, \ldots, K_n) = \phi\Big(\bigcup_{i=1}^n \hat{K}_i\Big). \tag{11}$$

Similar to Lemma 1 one has the following lemma.

Lemma 3. Let $\psi : \mathcal{K}^n \to [0,1]$ and $\phi : \hat{\mathcal{K}}^n_{\cup} \to [0,1]$ satisfying Equation (11) for all $(K_1, \ldots, K_n) \in \mathcal{K}^n$. Then ϕ is a capacity functional (that is, ϕ fulfills Conditions (CF1), (CF2) and (CF3) for sets from $\hat{\mathcal{K}}^n_{\cup}$) if and only if ψ is a multivariate capacity functional (that is, ψ fulfills Conditions (MCF1) - (MCF3) of Lemma 1).

Proof. The equivalence follows from the relation $\phi(\bigcup_{i=1}^n \hat{K}_i) = \psi(K_1, \ldots, K_n)$. Indeed, we have $\bigcup_{i=1}^n \hat{K}_i = \emptyset$ if and only if $K_i = \emptyset$ for all i and thus $\phi(\emptyset) = \psi(\emptyset, \ldots, \emptyset)$. Furthermore, by Formula (1) we have for all $K = (K_1, \ldots, K_n) \in \mathcal{K}^n$, $K^j = (K^j_1, \ldots, K^j_n) \in \mathcal{K}^n$

$$\begin{aligned}
&\Delta_k \psi(K; K^1, \ldots, K^k) \\
&\quad = -\sum_{J \subseteq \{1,\ldots,k\}} (-1)^{|J|} \psi\Big(K \cup \bigcup_{j \in J} K^j\Big) \\
&= -\sum_{J \subseteq \{1,\ldots,k\}} (-1)^{|J|} \psi\Big(K_1 \cup \bigcup_{j\in J} K^j_1, \ldots, K_n \cup \bigcup_{j \in J} K^j_n\Big) \\
&\quad = -\sum_{J \subseteq \{1,\ldots,k\}} (-1)^{|J|} \phi\Big(\bigcup_{i=1}^n \widehat{\big(K_i \cup \bigcup_{j\in J} K^j_i\big)}\Big) \\
&\quad = -\sum_{J \subseteq \{1,\ldots,k\}} (-1)^{|J|} \phi\Big(\bigcup_{i=1}^n \hat{K}_i \cup \bigcup_{j \in J} \bigcup_{i=1}^n \hat{K}^j_i\Big) \\
&\qquad = \Delta_k \phi\Big(\bigcup_{i=1}^n \hat{K}_i; \bigcup_{i=1}^n \hat{K}^1_i, \ldots, \bigcup_{i=1}^n \hat{K}^k_i\Big).
\end{aligned}$$

The equivalence of (MCF3) and (CF3) follows from the fact that $K^k_i \searrow K_i$ for all $1 \le i \le n$ if and only if $\bigcup_{i=1}^n \hat{K}^k_i \searrow \bigcup_{i=1}^n \hat{K}_i$. □

Together with Proposition 1 this implies the following proposition which gives a characterization of the joint distribution of n random sets by a set function on $\hat{\mathcal{K}}^n_{\cup}$.

Proposition 2. Let $\phi : \hat{\mathcal{K}}^n_{\cup} \to [0,1]$ be a capacity functional, that is, ϕ fulfills Conditions (CF1), (CF2) and (CF3) for sets from $\hat{\mathcal{K}}^n_{\cup}$. Then there exists a unique probability measure $\Pi : \mathcal{B}(\mathcal{F}^n) \to [0,1]$ such that

$$\phi\Big(\bigcup_{i=1}^n \hat{K}_i\Big) = \Pi\Big(\bigcup_{i=1}^n \widehat{\mathcal{F}}_{K_i}\Big)$$

for all $\bigcup_{i=1}^n \hat{K}_i \in \hat{\mathcal{K}}^n_{\cup}$. If, in addition, for all $1 \le i \le n$ it holds that $\sup\{\phi(\hat{K}_i) : K_i \in \mathcal{K}_i\} = 1$ then $\Pi(\mathcal{F}'^n) = 1$ and for all $L \in \hat{\mathcal{K}}^n_{\cup}$ it holds that

$$\phi(L) = \Pi({}^n\mathcal{F}_L).$$

Proof. The main assertion directly follows from applying Proposition 1 to $\psi : \mathcal{K}^n \to [0,1]$ defined by $\psi(K_1, \ldots, K_n) = \phi(\bigcup_{i=1}^n \hat{K}_i)$ which is a multivariate capacity functional by Lemma 3. The additional statement follows from the fact that $\psi(\check{K}_i) = \phi(\hat{K}_i)$. By virtue of Corollary 1 this implies $\Pi(\mathcal{F}'^n) = 1$ which further leads to

$$\Pi\Big(\bigcup_{i=1}^n \widehat{\mathcal{F}}_{K_i}\Big) = \Pi\Big(\mathcal{F}'^n \cap \bigcup_{i=1}^n \widehat{\mathcal{F}}_{K_i}\Big)$$

for all $K_i \in \mathcal{K}_i$. Furthermore, we obtain

$$\begin{aligned}
&\mathcal{F}'^n \cap \bigcup_{i=1}^n \widehat{\mathcal{F}}_{K_i} \\
&\quad = \bigcup_{i=1}^n \{(F_1, \ldots, F_n) \in \mathcal{F}'^n : F_i \cap K_i \neq \emptyset\} \\
&= \bigcup_{i=1}^n \{(F_1, \ldots, F_n) \in \mathcal{F}^n : F_1 \times \cdots \times F_n \cap \hat{K}_i \neq \emptyset\} \\
&\qquad = {}^n\mathcal{F}_{\bigcup_{i=1}^n \hat{K}_i}.
\end{aligned}$$

Hence, $\phi(L) = \Pi({}^n\mathcal{F}_L)$ for all $L \in \hat{\mathcal{K}}^n_{\cup}$. □

One can think of extending the various set functions to wider classes of sets. In case of a capacity functional $\varphi : \mathcal{K}^n_{\amalg} \to [0,1]$ the extensions from Equation (2) can be used to obtain a completely alternating Choquet-$\mathcal{K}^n_{\amalg}$-capacity $\varphi^* : \mathcal{P}^n_{\amalg} \to [0,1]$ on the power set of $\mathbb{E}^n_{\amalg}$. In case of a multivariate capacity functional $\psi : \mathcal{K}^n \to [0,1]$ or a capacity functional $\phi : \hat{\mathcal{K}}^n_{\cup} \to [0,1]$ one can define a corresponding capacity functional φ on $\mathcal{K}^n_{\amalg}$ by the relation $\varphi(\amalg K_i) = \psi(K_1, \ldots, K_n)$ or $\varphi(\amalg K_i) = \phi(\bigcup_{i=1}^n \hat{K}_i)$ and use φ^* to obtain ψ^* or ϕ^*. On the other hand, the extension procedure given by Equation (2) can be directly applied to ψ or ϕ which yields the same ψ^* or ϕ^* since $\amalg A_i \subseteq \amalg B_i$ if and only if $A_i \subseteq B_i$ for all i if and only if $\bigcup_{i=1}^n \hat{A}_i \subseteq \bigcup_{i=1}^n \hat{B}_i$.

We have seen how a capacity functional ϕ defined on $\hat{\mathcal{K}}_\cup^n$ can be extended to a Choquet-$\hat{\mathcal{K}}_\cup^n$-capacity on

$$\hat{\mathcal{P}}_\cup^n = \Big\{ \bigcup_{i=1}^n \hat{A}_i : A_i \in \mathcal{P}_i \Big\}.$$

We point out that a further extension to all subsets of $\mathbb{E}^n = \mathbb{E}_1 \times \cdots \times \mathbb{E}_n$ would not make much sense since this extension would not be unique. Indeed, consider the following two (deterministic) sets $X_1 = [0,1]^2$ and $X_2 = \{(x,y) \in [0,1]^2 : x + y \geq 1\}$. They can be seen as random compact sets in $\mathbb{R}^2$ on a one point probability space. The corresponding capacity functionals ϕ_1 and ϕ_2 are given by

$$\phi_i(A) = \begin{cases} 1 & \text{if } X_i \cap A \neq \emptyset \\ 0 & \text{if } X_i \cap A = \emptyset \end{cases}$$

for each $A \subseteq \mathbb{R}^2$. Obviously, ϕ_1 and ϕ_2 coincide on $\hat{\mathcal{K}}_\cup^2$ but they have different values on other sets, for example, $\phi_1(A) = 1$ and $\phi_2(A) = 0$ for $A = [0, 1/3]^2$.

4 Application to set-valued processes

Let T denote a time set, let $(\mathbb{M}, \mathcal{M})$ be a measurable space and let (Ω, Σ, P) be a probability space. Then a map $x : T \times \Omega \to \mathbb{M}$ is a stochastic process if for each $t \in T$ the partial map $x_t : \Omega \to \mathbb{M}$ is measurable, that is, $x_t^{-1}(B) \in \Sigma$ for all $B \in \mathcal{M}$. Denoting by $\mathcal{T}$ the set of all finite subsets of T, the process x induces a family $\{\mu_{\underline{t}}\}_{\underline{t} \in \mathcal{T}}$ of probability measures where

$$\begin{aligned} \mu_{\underline{t}} : \mathcal{M}^{\otimes n} &\to [0,1], \\ B &\mapsto P(\{\omega \in \Omega : (x_{t_1}(\omega), \ldots, x_{t_n}(\omega)) \in B\}), \end{aligned}$$

$\underline{t} = (t_1, \ldots, t_n)$, $\mathcal{M}^{\otimes n} = \mathcal{M} \otimes \cdots \otimes \mathcal{M}$. The latter is called the family of finite-dimensional distributions of x and obviously fulfills the following two conditions:

(i) For all $\underline{t} = (t_1, \ldots, t_n) \in \mathcal{T}$, $B_1, \ldots, B_n \in \mathcal{M}$ and each permutation σ of $\{1, \ldots, n\}$ it holds that

$$\mu_{\underline{t}}(B_1 \times \cdots \times B_n) = \mu_{\sigma(\underline{t})}(B_{\sigma(1)} \times \cdots \times B_{\sigma(n)})$$

where $\sigma(\underline{t}) = (t_{\sigma(1)}, \ldots, t_{\sigma(n)})$.

(ii) For all $\underline{t} = (t_1, \ldots, t_n) \in \mathcal{T}$, $t_{n+1} \in T$, $B \in \mathcal{M}^{\otimes n}$ it holds that

$$\mu_{t_1, \ldots, t_{n+1}}(B \times \mathbb{M}) = \mu_{\underline{t}}(B).$$

A family of finite-dimensional distributions is said to be consistent if these two conditions are fulfilled. Under the assumption that $\mathbb{M}$ is a complete separable metric space endowed with its Borel sets $\mathcal{B}(\mathbb{M})$, the well-known Daniell-Kolmogorov theorem [5, 7] says that for any consistent family of finite-dimensional distributions there exists a stochastic process whose finite dimensional distributions coincide with that family. More precisely, consider the set of maps from T to $\mathbb{M}$ denoted by $\mathbb{M}^T$ which is endowed with the σ-algebra $\mathcal{B}(\mathbb{M}^T)$ generated by sets of the form $\{\omega \in \mathbb{M}^T : (\omega(t_1), \ldots, \omega(t_n)) \in B\}$, $B \in \mathcal{B}(\mathbb{M}^n)$, $t_i \in T$, $n \geq 1$. Then there exists a probability measure μ on $\mathcal{B}(\mathbb{M}^T)$ such that for all $\underline{t} = (t_1, \ldots, t_n) \in \mathcal{T}$ $(n \geq 1)$ and $B \in \mathcal{B}(\mathbb{M}^n)$ it holds that

$$\mu_{\underline{t}}(B) = \mu(\{\omega \in \mathbb{M}^T : (\omega(t_1), \ldots, \omega(t_n)) \in B\}).$$

The desired process is then given by $(t, \omega) \to \omega(t)$.

By set-valued stochastic processes we mean stochastic processes where $\mathbb{M} = \mathcal{F}$, that is, maps of the form

$$X : T \times \Omega \to \mathcal{F}$$

where $X_t : \Omega \to \mathcal{F}$ is Effros-measurable for all $t \in T$. With the aid of Proposition 1 we can now formulate an existence theorem for set-valued processes by using multivariate capacity functionals.

Proposition 3. Let $\{\psi_{\underline{t}} : \underline{t} \in \mathcal{T}\}$ be a family of multivariate capacity functionals (i.e. set functions fulfilling Conditions (MCF1) - (MCF3) of Lemma 1). Assume that the following consistency conditions are fulfilled:

(i) For all $n \geq 1$, $\underline{t} = (t_1, \ldots, t_n) \in \mathcal{T}$, $K_1, \ldots, K_n \in \mathcal{K}$ and each permutation σ of $\{1, \ldots, n\}$ it holds that

$$\psi_{\underline{t}}(K_1, \ldots, K_n) = \psi_{\sigma(\underline{t})}(K_{\sigma(1)}, \ldots, K_{\sigma(n)})$$

where $\sigma(\underline{t}) = (t_{\sigma(1)}, \ldots, t_{\sigma(n)})$.

(ii) For all $n \geq 1$, $\underline{t} = (t_1, \ldots, t_n) \in \mathcal{T}$, $t_{n+1} \in T$, $K_1, \ldots, K_n \in \mathcal{K}$ it holds that

$$\psi_{t_1, \ldots, t_{n+1}}(K_1, \ldots, K_n, \emptyset) = \psi_{\underline{t}}(K_1, \ldots, K_n).$$

Then the family $\{\Pi_{\underline{t}} : \underline{t} \in \mathcal{T}\}$ obtained from Proposition 1 is a consistent family of probability measures and there exists a probability measure Π on $\mathcal{B}(\mathcal{F}^T)$ such that for all $\underline{t} = (t_1, \ldots, t_n) \in \mathcal{T}$ $(n \geq 1)$ and $(K_1, \ldots, K_n) \in \mathcal{K}^n$ it holds that

$$\begin{aligned} &\psi_{\underline{t}}(K_1, \ldots, K_n) \\ &= \Pi_{\underline{t}}\Big(\Big\{\omega \in \mathcal{F}^T : (\omega(t_1), \ldots, \omega(t_n)) \in \bigcup_{i=1}^n \widehat{\mathcal{F}}_{K_i}\Big\}\Big). \end{aligned} \tag{12}$$

In addition, the condition $\sup\{\psi_t(K) : K \in \mathcal{K}\} = 1$ implies $\Pi_t(\{\omega \in \mathcal{F}^T : \omega(t) \neq \emptyset\}) = 1$ for all $t \in T$.

Proof. Since $\mathbb{E}$ is an LCHS space, $\mathcal{F}$ is a compact Hausdorff second countable space. Thus, $\mathcal{F}$ is also a Polish space, that is, separable and completely metrizable. Hence, if we show that $\{\Pi_{\underline{t}} : \underline{t} \in \mathcal{T}\}$ is a consistent family of probability measures the classical Daniell-Kolmogorov theorem can be applied directly and Equation (12) is obtained from Proposition 1:

$$\psi_{\underline{t}}(K_1, \dots, K_n) = \Pi_{\underline{t}}\Big(\bigcup_{i=1}^{n} \widehat{\mathcal{F}}_{K_i}\Big)$$
$$= \Pi\Big(\Big\{\omega \in \mathcal{F}^T : (\omega(t_1), \dots, \omega(t_n)) \in \bigcup_{i=1}^{n} \widehat{\mathcal{F}}_{K_i}\Big\}\Big)$$

It is enough to prove that the consistency conditions for $\{\Pi_{\underline{t}} : \underline{t} \in \mathcal{T}\}$ are fulfilled for cylindrical sets of the form

$$\mathcal{F}_{K_{11},\dots,K_{1k_1}} \times \cdots \times \mathcal{F}_{K_{n1},\dots,K_{nk_n}},$$

$K_{ij_i} \in \mathcal{K}$, since they constitute a generating class of $\mathcal{B}(\mathcal{F}^n) = \mathcal{B}(\mathcal{F})^{\otimes n}$ which is closed under finite intersections. Similarly as in Equation (10) we obtain the following formula

$$\Pi_{\underline{t}}(\mathcal{F}_{K_{11},\dots,K_{1k_1}} \times \cdots \times \mathcal{F}_{K_{n1},\dots,K_{nk_n}})$$
$$= \Pi_{\underline{t}}\Big(\bigcap_{i=1}^{n} \widehat{\mathcal{F}}_{K_{i1},\dots,K_{ik_i}}\Big) = \Pi_{\underline{t}}\Big(\bigcap_{i=1}^{n}\bigcap_{j_i=1}^{k_i} \widehat{\mathcal{F}}_{K_{ij_i}}\Big)$$
$$= -\sum_{J \in \mathcal{J}} (-1)^{|J|} \Pi_{\underline{t}}\Big(\bigcup_{i=1}^{n}\bigcup_{j_i \in J_i} \widehat{\mathcal{F}}_{K_{ij_i}}\Big)$$
$$= -\sum_{J \in \mathcal{J}} (-1)^{|J|} \Pi_{\underline{t}}\Big(\bigcup_{i=1}^{n} \widehat{\mathcal{F}}_{\bigcup_{j_i \in J_i} K_{ij_i}}\Big)$$
$$= -\sum_{J \in \mathcal{J}} (-1)^{|J|} \psi_{\underline{t}}\Big(\bigcup_{j_1 \in J_1} K_{1j_1}, \dots, \bigcup_{j_n \in J_n} K_{nj_n}\Big)$$

where $\mathcal{J} = \{(J_1, \dots, J_n) : J_i \subseteq \{1, \dots, k_i\}\}$ and $|J| = \sum_{i=1}^{n} |J_i|$. Together with (i) this implies

$$\Pi_{\underline{t}}(\mathcal{F}_{K_{11},\dots,K_{1k_1}} \times \cdots \times \mathcal{F}_{K_{n1},\dots,K_{nk_n}})$$
$$= \Pi_{\sigma(\underline{t})}(\mathcal{F}_{K_{\sigma(1)1},\dots,K_{\sigma(1)k_{\sigma(1)}}} \times \cdots \times \mathcal{F}_{K_{\sigma(n)1},\dots,K_{\sigma(n)k_{\sigma(n)}}})$$

In a similar manner as before we obtain

$$\Pi_{t_1,\dots,t_{n+1}}(\mathcal{F}_{K_{11},\dots,K_{1k_1}} \times \cdots \times \mathcal{F}_{K_{n1},\dots,K_{nk_n}} \times \mathcal{F})$$
$$= -\sum_{J \in \mathcal{J}} (-1)^{|J|} \psi_{t_1,\dots,t_{n+1}}\Big(\bigcup_{j_1 \in J_1} K_{1j_1}, \dots, \bigcup_{j_n \in J_n} K_{nj_n}, \emptyset\Big).$$

and thus (ii) implies

$$\Pi_{t_1,\dots,t_{n+1}}(\mathcal{F}_{K_{11},\dots,K_{1k_1}} \times \cdots \times \mathcal{F}_{K_{n1},\dots,K_{nk_n}} \times \mathcal{F})$$
$$= \Pi_{\underline{t}}(\mathcal{F}_{K_{11},\dots,K_{1k_1}} \times \cdots \times \mathcal{F}_{K_{n1},\dots,K_{nk_n}}).$$

The additional statement that $\sup\{\psi_t(K) : K \in \mathcal{K}\} = 1$ implies $\Pi_t(\{\omega \in \mathcal{F}^T : \omega(t) \neq \emptyset\}) = 1$ directly follows from Corollary 1. □

It should be mentioned that in [9] a Daniell-Kolmogorov theorem for supremum preserving (also called maxitive) upper probabilities has been proved.

With the aid of the foregoing proposition we can now try to construct something like a set-valued Brownian motion. Brownian motion is a real-valued stochastic process in continuous time which is defined via a consistent family of Gaussian distributions. More precisely, it is a process with continuous sample functions starting at time 0 with value 0, and it has independent, Gaussian distributed increments with mean 0. We denote by $\{\beta_{\underline{t}}\}_{\underline{t} \in \mathcal{T}}$ ($T = [0, \infty)$) its family of finite dimensional distributions which is clearly consistent. According to Equation (11) and Lemma 3 we get a family of multivariate capacity functionals $\{\psi_{\underline{t}}\}_{\underline{t} \in \mathcal{T}}$ which can be easily seen to be consistent. In addition, we have for all $\underline{t} = (t_1, \dots, t_n) \in \mathcal{T}$ and for all $1 \leq i \leq n$ that

$$\sup\{\psi_{\underline{t}}(\check{K}_i) : K_i \in \mathcal{K}\} = \sup\{\beta_{\underline{t}}(\hat{K}_i) : K_i \in \mathcal{K}\}$$
$$= \sup\{\beta_{t_i}(K_i) : K_i \in \mathcal{K}\} = 1.$$

By applying Propositions 2 and 3 we get a probability measure Π on $\mathcal{B}(\mathcal{F}^{[0,\infty)})$ such that for each $\underline{t} = (t_1, \dots, t_n) \in \mathcal{T}$ it holds that $\Pi_{\underline{t}}(\mathcal{F}'^n) = 1$ and for each $(K_1, \dots, K_n) \in \mathcal{K}^n$ we get

$$\Pi\Big(\Big\{\omega \in \mathcal{F}^{[0,\infty)} : (\omega(t_1), \dots, \omega(t_n)) \in {}^n\mathcal{F}_{\bigcup_{i=1}^{n} \hat{K}_i}\Big\}\Big)$$
$$= \Pi_{\underline{t}}\Big(\bigcup_{i=1}^{n} \widehat{\mathcal{F}}_{K_i}\Big) = \psi_{\underline{t}}(K_1, \dots, K_n) = \beta_{\underline{t}}\Big(\bigcup_{i=1}^{n} \hat{K}_i\Big).$$

By defining

$$B : [0, \infty) \times \mathcal{F}^{[0,\infty)} \to \mathcal{F},\ (t, \omega) \mapsto B_t(\omega) = \omega(t)$$

we get a set-valued process with finite dimensional distributions $\{\Pi_{\underline{t}}\}_{\underline{t}}$ and finite dimensional capacity functionals $\{\psi_{\underline{t}}\}_{\underline{t}}$. For time $t \in [0, \infty)$ and $G \in \mathcal{G}$ we get

$$\Pi(\{\omega : B_t(\omega) \cap G \neq \emptyset\}) = \Pi_t(\mathcal{F}_G) = \Pi_t\Big(\bigcup_{n \in \mathbb{N}} \mathcal{F}_{K_n}\Big)$$
$$= \lim_{n \to \infty} \Pi_t(\mathcal{F}_{K_n}) = \lim_{n \to \infty} \beta_t(K_n) = \beta_t(G)$$

where $\{K_n\}_{n \in \mathbb{N}} \subseteq \mathcal{K}$ is an increasing sequence such that $\bigcup_{i=1}^{n} K_n = G$. On the other hand, if we approximate G^{c} by an increasing sequence $\{K_n\}_{n \in \mathbb{N}} \subseteq \mathcal{K}$ we obtain

$$\begin{aligned}\Pi(\{\omega : B_t(\omega) \subseteq G\}) &= \Pi_t(\mathcal{F}^{G^c}) = 1 - \Pi_t(\mathcal{F}_{G^c}) \\ &= 1 - \lim_{n\to\infty} \Pi_t(\mathcal{F}_{K_n}) = 1 - \lim_{n\to\infty} \beta_t(K_n) \\ &= 1 - \beta_t(G^c) = \beta_t(G).\end{aligned}$$

Consequently, the lower and the upper probability of B_t coincide and thus, B_t is almost surely a singleton. This means that although B has values in $\mathcal{F}$ it is actually not a set-valued process but a version of classical Brownian motion.

Note that there are other approaches to define a set-valued Brownian motion via support functions (see [11, 12]), but at least in the real-valued case they also lead to set-valued processes that almost surely consist of singletons.

5 Summary and conclusion

The goal of this paper was to give a characterization of probability measures on the Borel subsets of $\mathcal{F}^n$ ($n \geq 2$) by set functions. The first approach was to use a set function φ defined on the compact subsets of the co-product $\mathbb{E}^n_{\amalg}$ and to apply the (classical) Choquet theorem leading to a probability measure Q on the Borel-σ-algebra of the closed subsets of $\mathbb{E}^n_{\amalg}$. It has been shown that instead of φ one can equivalently use a set function ψ defined on the cartesian product $\mathcal{K}^n = \mathcal{K}_1 \times \cdots \times \mathcal{K}_n$ (Lemma 1). Moreover, it has been demonstrated how to obtain a probability measure Π on $\mathcal{B}(\mathcal{F}^n)$ from Q (Lemma 2). This resulted in a characterization of probability measures on $\mathcal{B}(\mathcal{F}^n)$ by set functions on $\mathcal{K}^n$ called multivariate capacity functionals (Proposition 1). In addition, Proposition 2 stated a characterization using set functions on $\hat{\mathcal{K}}^n_{\cup}$ which is a special class of subsets of the product space $\mathbb{E}^n = \mathbb{E}_1 \times \cdots \times \mathbb{E}_n$. Figure 1 gives an overview of the proposed characterizations.

$$\begin{tikzcd}
\varphi : \mathcal{K}^n_{\amalg} \to [0,1] \arrow[r, "\text{Thm. 1}"] \arrow[d, shift left, "\text{Lemma 1}"] & Q : \mathcal{B}(\mathcal{F}^n_{\amalg}) \to [0,1] \arrow[d, "\text{Lemma 2}"] \\
\psi : \mathcal{K}^n \to [0,1] \arrow[u, shift left] \arrow[r, "\text{Prop. 1}"] \arrow[d, shift left, "\text{Lemma 3}"] & \Pi : \mathcal{B}(\mathcal{F}^n) \to [0,1] \\
\phi : \hat{\mathcal{K}}^n_{\cup} \to [0,1] \arrow[u, shift left] \arrow[ur, "\text{Prop. 2}"'] &
\end{tikzcd}$$

Figure 1: Overview over characterizations of probability measures by set functions.

In Section 4, we have stated a Daniell-Kolmogorov theorem for set-valued stochastic processes, that is, we have demonstrated that for a consistent family of multivariate capacity functionals there exists a set-valued process whose finite dimensional upper probabilities coincide with these multivariate capacity functionals.

Acknowledgements

I would like to thank the reviewers for their useful comments and suggestions. Especially, one of the remarks led to a substantial improvement of the paper.

References

[1] G. Beer. Topologies on Closed and Closed Convex Sets. Kluwer Academic Publishers, Dordrecht, 1993.

[2] C. Castaing, M. Valadier. Convex analysis and measurable multifunctions. Lecture notes in mathematics 580, Springer, 1977.

[3] G. Choquet. Theory of capacities. Annales de l'Institut Fourier, 5:131–295, 1953.

[4] A. P. Dempster. Upper and lower probabilities induced by a multivalued mapping. Annals of Mathematical Statistics 38:325–339, 1967.

[5] J. L. Doob. Stochastic Processes. Wiley, 1990.

[6] E. B. Dynkin. Die Grundlagen der Theorie der Markoffschen Prozesse. Springer, 1961.

[7] I. I. Gikhman, A. V. Skorokhod. Introduction to the theory of random processes. Saunders Company, 1969.

[8] C. J. Himmelberg. Measurable relations. Fundamenta Mathematicae, 87:53–72, 1975.

[9] H. Janssen, G. de Cooman, E. E. Kerre. A Daniell-Kolmogorov theorem for supremum preserving upper probabilities. Fuzzy Sets and Systems 102:429–444, 1999.

[10] R. Kruse, K. D. Meyer. Statistics with vague data. D. Reidel Publishing Company, Dordrecht, 1987.

[11] Shoumei Li, Li Guan. Fuzzy set-valued Gaussian processes and Brownian motions. Information Sciences 177:3251–3259, 2007.

[12] C. Y. Liu, Xu. Han, M. Feng, S. K. Li. Itô integral for bounded closed convex set valued Wiener stochastic processes. In Y. Liu, G. Chen, M. Ying, editors, Fuzzy Logic, Soft Computing and Computational Intelligence, Proceedings of the Eleventh International Fuzzy Systems Association World Congress, Beijing, China, July 28–31, 2005, pages 184–187.

[13] G. Matheron. Random Sets and Integral Geometry. Wiley, 1975.

[14] P. A. Meyer. Probability and Potentials. Waltman, London, 1966.

[15] I. Molchanov. Theory of random sets. Springer, 2005.

[16] J. Neveu. Mathematical Foundations of the Calculus of Probability. Holden-Day, Inc., 1965.

[17] H. T. Nguyen. An Introduction to Random Sets. Chapman & Hall, 2006.

[18] H. T. Nguyen. On random sets and belief functions. J. Math. Anal. Appl., 65:531-542, 1978.

[19] B. Schmelzer. On solutions of stochastic differential equations with parameters modelled by random sets. PhD thesis, 2011.